ELLIPTIC ISLES IN FAMILIES OF AREA PRESERVING MAPS.

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Abstract: *We prove that every one parameter family of area preserving maps unfolding a homoclinic tangency has a sequence of parameter intervals, approaching to the bifurcation parameter, where the dynamics exhibits wild hyperbolic sets accumulated by elliptic isles. This is a parametric conservative analogue of a famous theorem of Newhouse on the abundance of wild hyperbolic sets.*

1. Introduction

This paper is about the dynamics of area-preserving surface diffeomorphisms. We assume the reader to be familiar with hyperbolic theory concepts such as 'hyperbolic periodic orbit', and 'hyperbolic basic set' of a diffeomorphism, as well as the bifurcation theory concepts of 'homoclinic' and 'heteroclinic tangencies'. A tangency between stable and unstable leaves of a hyperbolic set Λ is said to be a homoclinic tangency of Λ. A hyperbolic basic set is said to be wild if it has homoclinic tangencies which are persistent, in the sense that they can not be avoided with small perturbations of the underlying diffeomorphism. The concept of wild hyperbolic set was introduced by Newhouse [13] to disprove the density of Ω-stable diffeomorphisms on the sphere S^2. Later, in [14], he showed that, for dissipative dynamics, this phenomenon implies the co-existence of infinitely many sinks. Finally, in [15], he established the abundance of infinitely many sinks around a wild hyperbolic set. For surface dissipative diffeomorphisms, this phenomenon appears at the unfolding of every homoclinic tangency. A parametric version of this theorem appeared a couple of years later through the work of Robinson [17].

The techniques used in [15] and [17] do not apply to the conservative case. In [4] we have proved a conservative analogue to Newhouse theorem on the abundance of wild hyperbolic sets. Here we prove a parametric version for that theorem, which is the conservative analogue of Robinson's theorem. This result depends crucially on an asymptotic formula for the splitting angle of the Henón mapping separatrices, which was obtained by V. Gelfreich in [7].

This paper was essentially written some six years ago to be part of a larger article on Newhouse phenomenon for higher dimensional symplectic dynamics. As this broader project didn't come trough, with the consent of the other co-authors, I have decided to come forward with this contribution on the two dimensional dynamics. The arguments here rely heavily on a previous work [3], which the reader may find helpful to read in parallel.

2. Statement of results

Let M^2 denote a two dimensional symplectic manifold, i.e., an orientable surface together with some area form ω. A symplectic, or area-preserving, map is any diffeomorphism $f : M^2 \to M^2$ which preserves the area form ω. We denote by $\mathrm{Diff}^r(M^2)$, respectively $\mathrm{Diff}^r_\omega(M^2)$, the group of class C^r diffeomorphisms, respectively of class C^r symplectic maps $f : M^2 \to M^2$.

We assume the reader is familiar with hyperbolic theory concepts, namely those of hyperbolic periodic orbit, homoclinic orbit, hyperbolic invariant set, hyperbolic basic set, stable and unstable manifolds. As usual, $W^s(P) = W^s(P, f)$ and $W^u(P) = W^u(P, f)$ will respectively denote the stable and unstable manifolds of a point P, for a map f. A similar notation $W^s(\Lambda) = W^s(\Lambda, f)$ and $W^u(\Lambda) = W^u(\Lambda, f)$ is used to denote, respectively, the stable and unstable sets of a given a hyperbolic set Λ, for a map f. See [18] for a good introduction on hyperbolic theory.

Given a hyperbolic f-invariant set Λ, and two points $x, y \in \Lambda$, an intersection point in $W^s(x, f) \cap W^u(y, f) - \Lambda$ is called a homoclinic point of Λ. This homoclinic point is called a homoclinic tangency point if the corresponding intersection is not transversal.

Let Λ be a basic set for a map f. Recall that the analytic continuation of Λ is the maximal invariant set in a neighbourhood U of Λ, which is known to be another hyperbolic basic set, conjugated to Λ, for all maps in some neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^r(M^2)$. Following Newhouse, we say that Λ is a wild basic set over an open set $\mathcal{U} \subseteq \mathrm{Diff}^r(M^2)$, containing the map f, if for all maps $g \in \mathcal{U}$,

(1) the analytic continuation Λ_g is a hyperbolic basic set conjugated to Λ, and
(2) there is at least one orbit of homoclinic tangencies of Λ_g.

We shall refer to the open set $\mathcal{U}$ as a Newhouse region for the wild hyperbolic set Λ. The proof of the following proposition is quite standard. See [16], or [2] for a conservative argument.

Proposition 1. *Let Λ be a wild hyperbolic set over an open set of maps $\mathcal{U} \subseteq \mathrm{Diff}^r_\omega(M^2)$ with $r \geq 4$. Then*

(1) *Given any periodic point $P \in \Lambda$, there is a dense subset $\mathcal{D} \subseteq \mathcal{U}$ such that for every $g \in \mathcal{D}$, the periodic point P_g has an orbit of homoclinic tangencies.*
(2) *There is a residual subset $\mathcal{R} \subseteq \mathcal{U}$, i.e., a countable intersection of open subsets dense in $\mathcal{U}$, such that for every $g \in \mathcal{R}$, the basic set Λ_g is contained in the closure of all generic elliptic periodic points of g.*

A periodic point P, with period n, of $f \in \mathrm{Diff}^r_\omega(M^2)$, with $r \geq 4$, is said to be a generic elliptic point if both eigenvalues λ and λ^{-1} of Df^n_P sit in the unit circle without resonances of order ≤ 3, that is $|\lambda| = 1$ with $\lambda^2 \neq 1$ and $\lambda^3 \neq 1$, and the first coefficient of f^n's Birkhoff normal form at point P is nonzero. Under the non-resonance conditions above the Birkhoff normal form theorem says that after some smooth symplectic change of coordinates, mapping point P to origin, the diffeomorphism f^n takes the form

$$f^n(r\cos\theta, r\sin\theta) = \big(r\cos(\theta + \alpha + \beta r^2),\, r\sin(\theta + \alpha + \beta r^2)\big) + O(r^4)$$

where $\lambda = e^{i\alpha}$, and β is a symplectic invariant of f^n at the fixed point P, the so called Birkhoff normal form first coefficient. If $\beta \neq 0$ and $r \geq 5$, Moser theorem applies saying there is an invariant set Σ, with full Lebesgue density at P, which is a union of invariant curves. In each of these curves the map f^n is conjugated to an irrational rotation of the circle. This structure around P is usually described in the literature as an "elliptic isle".

A class C^r function $(\mu, x) \mapsto f_\mu(x)$ defined on $I \times M^2$, with values in M^2, where I is an interval of real numbers, and such that $f_\mu \in \mathrm{Diff}^r(M^2)$ for all $\mu \in I$, is called a class C^r one-parameter family of diffeomorphisms. If $f_\mu \in \mathrm{Diff}^r_\omega(M^2)$ for all $\mu \in I$, we say that $\{f_\mu\}_\mu$ is one-parameter family of symplectic maps.

We say that a family f_μ unfolds generically an orbit of homoclinic quadratic tangencies at $(\mu_0, Q_0) \in I \times M^2$, associated with some hyperbolic periodic point P if, denoting by P_μ the analytic continuation of P for the map f_μ,

(1) $W^s(P, f_{\mu_0})$ and $W^u(P, f_{\mu_0})$ have a quadratic tangency at Q_0.
(2) If ℓ is any smooth curve transversal to $W^s(P, f_{\mu_0})$ and $W^u(P, f_{\mu_0})$ at Q_0, then the local intersections of $W^s(P_\mu, f_\mu)$ and $W^u(P_\mu, f_\mu)$ with ℓ cross each other with relative non zero velocity at (μ_0, Q_0).

Let f_μ be a one-parameter family of maps in $\mathrm{Diff}^r(M^2)$. Take a parameter interval $\Delta \subseteq \mathbb{R}$, and let $\{\Lambda_\mu\}_{\mu\in\Delta}$ be a continuous family of basic sets. This means for each $\mu \in \Delta$, Λ_μ is a hyperbolic basic set of f_μ, and, furthermore, the correspondence $\mu \mapsto \Lambda_\mu$ is continuous with respect to Hausdorff distance. It follows that all basic sets Λ_μ are conjugated to each other. We say that Λ_μ are wild basic sets over Δ if for all $\mu \in \Delta$, there is at least one orbit of homoclinic quadratic tangencies of Λ_μ, which unfolds generically with μ. We shall also say, with the same meaning, that Δ is a Newhouse interval for the basic sets Λ_μ. More strongly, we will say that the basic sets Λ_μ are C^r-stably-wild basic sets over Δ if they are wild basic sets over Δ for all class C^r one-parameter families uniformly close to f_μ. Uniform proximity of one-parameter families refers to the following distance. The topology of the group $\mathrm{Diff}^r(M^2)$ is clearly metrizable. Taking any metric d_{C^r} inducing the topology of $\mathrm{Diff}^r(M^2)$, we define the following distance between one-parameter families of maps in $\mathrm{Diff}^r(M^2)$.

$$d\left(\{f_\mu\}_\mu, \{g_\mu\}_\mu\right) = \sup_{\mu\in I} d_{C^r}(f_\mu, g_\mu) .$$

The parametric version of proposition 1 is obtained in a similar way.

Proposition 2. *Let Λ be a wild hyperbolic set over an interval Δ, for a one-parameter family of maps $f_\mu \in \mathrm{Diff}^r_\omega(M^2)$, where $r \geq 4$. Then*

(1) *Given any periodic point $P \in \Lambda$, there is a dense subset $D \subseteq \Delta$ such that for every $\mu \in D$, the periodic point P_μ has an orbit of homoclinic tangencies.*
(2) *There is a residual subset $R \subseteq \Delta$, i.e., a countable intersection of open subsets dense in Δ, such that for every $\mu \in R$, the basic set Λ_μ is contained in the closure of all generic elliptic periodic points of f_μ.*

The *conservative Henón* is the family of area preserving maps $H_a : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$H_a(x, y) = (y, -x + a - y^2). \tag{1}$$

Note that at $a = -1$ the Hénon map has a parabolic fixed point at $x = y = -1$ which breaks into two fixed points, a saddle $O_s = \left(-1 - \sqrt{1+a}\,,\, -1 - \sqrt{1+a}\right)$ and an elliptic point $O_e = \left(-1 + \sqrt{1+a}\,,\, -1 + \sqrt{1+a}\right)$, for $a > -1$. It has been proved in [7] that for all $a > -1$ sufficiently close to the bifurcation parameter $a = -1$ the saddle fixed point O_s has a transversal homoclinic orbit. This transversality implies the existence of a hyperbolic set for all values of a just after the bifurcation moment. We examine the structure of this set, following the lines of [3] and using asymptotics for the homoclinic angle from [7]. Namely, we prove that for all $a > -1$ close to the bifurcation moment, the Hénon map has a C^2-stably-wild binary horseshoe which includes the saddle point O_s. This theorem is the main result of the paper.

Theorem A. *The Hénon map family (1) has a sequence of Newhouse intervals Δ_k associated with C^2-stably-wild horseshoes containing the saddle fixed point O_s. The sequence Δ_k converges to the bifurcation parameter $a = -1$ as $k \to +\infty$.*

Theorem B. *Let f_μ be a class C^r one-parameter family of symplectic maps in $\mathrm{Diff}^r_\omega(M^2)$ ($r \geq 6$). Let O be a periodic hyperbolic orbit, and Γ an orbit of quadratic homoclinic tangencies of f_0, which unfolds generically at $\mu = 0$. Take any small neighborhood U of $O \cup \Gamma$. Then there is a sequence of Newhouse intervals Δ_k converging to $\mu = 0$. Each Newhouse interval Δ_k is associated with a C^2-stably-wild hyperbolic basic set Λ_k such that $O \subseteq \Lambda_k \subseteq U$.*

Theorem B follows from theorem A. As explained in section 4 of [4], the argument uses a standard technique for renormalizing the dynamics at the unfolding of a homoclinic tangency, the Hénon map showing up in the limit process. The conservative two dimensional case of this renormalization process is done in [9], based on Shil'nikov co-ordinates. For simplicity, in [4] we have assumed all maps to be of class C^∞, but class C^r with $r \geq 6$ is enough. If f_μ is a class C^r one-parameter family of maps unfolding a homoclinic tangency, the renormalized maps converge to the Hénon map family in the C^{r-4} topology, as shown in [9]. Letting $r \geq 6$, this guarantees at least C^2 convergence to the Hénon family, which ensures that the C^2-stably wild basic sets of the Hénon map still persist in the renormalized dynamics at the homoclinic tangency unfolding.

Corollary. *Under the same assumptions there is a non meager set of parameters R, i.e., a set which is not a countable union of nowhere dense subsets of $\mathbb{R}$, having the homoclinic bifurcation parameter $\mu = 0$ as an accumulation point, such that for every $\mu \in R$, the closure of f_μ's generic elliptic periodic points contains a wild basic set Λ_k including the periodic orbit O.*

This corollary follows from proposition 2 and theorem B.

Let us now precise the construction of the wild set for the Hénon map. Stable-wildness comes from a large 'thickness' condition. The notion of thickness of a hyperbolic basic set Λ, of a two-dimensional C^2-diffeomorphism, denoted by $\tau(\Lambda)$, was introduced by Newhouse who proved the following:

Theorem (Newhouse). *Let Λ be a hyperbolic basic set of a diffeomorphism $f \in \mathit{Diff}^2(M^2)$. Assume $\tau(\Lambda) = \tau^s(\Lambda)\,\tau^u(\Lambda) > 1$ and that some periodic point $P \in \Lambda$ has an orbit Γ of quadratic homoclinic tangencies. Finally, let f_μ be a one-parameter family of maps in $\mathit{Diff}^2(M^2)$ with $f_0 = f$, that unfolds generically the orbit of homoclinic tangencies Γ. Then there are parameter intervals over which Λ is a C^2-stably-wild basic set.*

Next we describe the mechanism introduced in [13] to prove the existence of C^2-stably-wild hyperbolic sets. Let Λ be a hyperbolic basic set such that at some point H there is a tangency between stable and unstable leaves of Λ. Consider the Cantor like foliations $\mathcal{F}^s = W^s_{\text{loc}}(\Lambda)$ and $\mathcal{F}^u = W^u_{\text{loc}}(\Lambda)$ and iterate them, respectively backward and forward, until they meet at H. Extend C^1- smoothly these iterated foliations to a neighborhood of H. Then there is a C^1 curve ℓ through H consisting of tangencies between these extended foliations. Consider the Cantor sets K^s and K^u formed by the points where the first backward iterations of $\mathcal{F}^s$, respectively forward iteration of $\mathcal{F}^u$, intersect the curve ℓ. By definition of ℓ, Λ has a "homoclinic" tangency at each point in $K^s \cap K^u$. With this construction, persistent homoclinic tangencies of Λ is equivalent to persistent intersections between the Cantor sets K^u and K^s. The device used to guarantee the "persistent intersections" is the concept of *thickness* $\tau(K)$ of a one-dimensional Cantor set K, i.e. lying inside some curve ℓ, which we will define bellow. Let us call *gap* of K to every connected component of the complement $I - K$ where I is the *interval spanned by* K, i.e. the smallest closed connected subset of ℓ containing K. Roughly, the thickness of a Cantor set measures the relative size of its gaps, large thickness corresponding to small gaps. The following intersection criterion holds:

Gap lemma. *Let K^s, K^u be two Cantor sets in the same open curve ℓ such that the intervals spanned by K^s and K^u do intersect, but nor K^s is contained inside a gap of K^u, neither K^u is contained inside a gap of K^s. If*

$$\tau(K^s)\,\tau(K^u) > 1 \tag{2}$$

then both Cantor sets intersect, $K^s \cap K^u \neq \emptyset$.

Of course (2) is a stable condition only if we have the continuity of thickness, and, in fact, it was proved in [15] that for dynamically defined Cantor sets, as K^s and K^u in the previous context, their thicknesses depend continuously on the map, for the C^2- topology. Later in [10] the new concepts of *left thickness*, $\tau_L(K)$, and *right thickness*, $\tau_R(K)$, of a Cantor set K were introduced together with the remark that the hypothesis (2) in the gap lemma could be replaced by the weaker condition

$$\tau_L(K^s)\,\tau_R(K^u) > 1 \text{ and } \tau_R(K^s)\,\tau_L(K^u) > 1\ . \tag{3}$$

The usual definition of thickness, or lateral thicknesses, is strictly geometric and can be applied to any compact set lying on a curve. Of course, to have continuity we must restrict to dynamically defined Cantor sets. Here, as in [3], we will adopt a more dynamical definition of thickness, which only applies to dynamically defined Cantor sets. This slightly different definition is not equivalent to the usual geometric one. Nevertheless the same results, the continuity of thicknesses and the gap lemma still hold. Finally, and because this will be enough for our purposes, we will restrict the scope of definitions to *binary* Cantor sets and horse-shoes, although they can be easily generalized to arbitrary combinatorics.

Let us call *binary Cantor set* to any pair (K,ψ) such that K is a Cantor subset of an open curve I, $\psi : I_0 \cup I_1 \to I$ is a C^1 expanding map defined on the union, $I_0 \cup I_1$, of two subintervals of I, and such that the restriction of ψ to $K = \bigcap_{n\geq 0} \psi^{-n}(I_0 \cup I_1)$ is topologically conjugated to the Bernoulli shift $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$. Of course we may assume that I is the interval spanned by K, and that for each $i = 0,\,1$, I_i is the interval spanned by $K \cap I_i$. Then $\{I_0,\, I_1\}$ is a Markov partition for (K,ψ). The gaps of (K,ψ) are ordered in the following way. Let us call *covering intervals of order* n to the intervals spanned by the Cantor set components

$$K(a_0,\cdots,a_n) = \bigcap_{i=0}^{n} \psi^{-i}(K \cap I_{a_i})\ ,$$

where $(a_0,\cdots,a_n) \in \{0,1\}^{n+1}$. Then I_0 and I_1 are the covering intervals of order zero. $U_0 = I - (I_0 \cup I_1)$ is said to be the *gap of order* 0. In general the components of the complement in I of the union of all covering intervals of order $\leq n$, which are not gaps of order $\leq n-1$, are called *gaps of order* n. It is easy to check that every gap is obtained by this procedure and, therefore, has some definite order.

The definitions bellow, of left and right thickness, require the curve I to be oriented. Given a gap U of K, we denote by L_U, respectively R_U, the unique covering interval with the same order of U that is left, resp. right, adjacent to U. The greatest lower bounds

$$\tau_L(K,\psi) = \inf\left\{ \frac{|L_U|}{|U|}\ :\ U \text{ is a gap of } K \right\}$$

$$\tau_R(K,\psi) = \inf\left\{ \frac{|R_U|}{|U|}\ :\ U \text{ is a gap of } K \right\}$$

$$\tau(K,\psi) = \min\{\tau_L(K,\psi)\,,\ \tau_R(K,\psi)\}$$

are respectively called the *left thickness*, the *right thickness*, and the *thickness* of (K,ψ). $|U|$ denotes the length of an interval $U \subseteq I$. These three thicknesses are continuous functions of (K,ψ) over the space of all $C^{1+\alpha}$ binary Cantor sets ($\alpha > 0$) with its natural $C^{1+\alpha}$ topology. It was remarked in [10] that the lateral thicknesses may be discontinuous for non binary Cantor sets. However with our "dynamical" definition the lateral thicknesses are always continuous. The same argument, as for usual thickness, applies, see for instance [16]. To prove that the Gap lemma, with condition (3) replacing (2), still holds just follow the proof in [10], arguing that one could obtain pairs of linked gaps with ever higher order, instead of ever smaller lengths. The conclusion is then the same because as we consider gaps with strictly increasing order their lengths converge to zero.

Let us say that the binary Cantor set (K,ψ) is *positive* when the restriction of ψ to each interval I_i ($i=0,1$) preserves orientation. In order to estimate lateral thicknesses notice that, by the (orientation preserving) self-similarity property of positive binary Cantor sets, the ratio $|L_U|\,/\,|U|$, respectively $|R_U|\,/\,|U|$, is, up to a distortion factor, equal to $\tilde\tau_L(K,\psi) := |I_0|\,/\,|U_0|$, respectively $\tilde\tau_R(K,\psi) := |I_1|\,/\,|U_0|$. We shall refer to $\tilde\tau_L(K,\psi)$ and $\tilde\tau_R(K,\psi)$ as *top scale thicknesses* of (K,ψ). For affine Cantor sets, where the distortion factor is one, we have $\tilde\tau_L=\tau_L$ and $\tilde\tau_R=\tau_R$. In general, if distortion is small then top scale thicknesses $\tilde\tau_L$ and $\tilde\tau_R$ are good approximations of τ_L and τ_R, respectively. Lateral thicknesses are useless for *non-positive* binary Cantor sets, because in this case both left and right thicknesses equal the usual thickness.

We call *binary horse shoe* to any pair (Λ,T) such that $T\colon S_0\cup S_1\to\mathbb{R}^2$ is a one to one local diffeomorphism of class C^2, where S_0 and S_1 are disjoint compact rectangles (up to diffeomorphism), and such that $\Lambda=\bigcap_{n\in\mathbb{Z}}T^{-n}(S_0\cup S_1)$ is a hyperbolic basic set conjugated to the Bernoulli shift $\sigma:\{0,1\}^{\mathbb{Z}}\to\{0,1\}^{\mathbb{Z}}$. For each $i=0,1$ there is a unique fixed point $P_i\in S_i$ and we assume $\{S_0,S_1\}$ to form a Markov partition bounded by pieces of stable and unstable manifolds of the fixed points P_0 and P_1. When both fixed points have positive eigenvalues we will say that (Λ,T) is a *positive binary horse shoe*.

Let $\mathcal{F}^s=W^s_{\rm loc}(\Lambda)\cap(S_0\cup S_1)$ and $\mathcal{F}^u=W^u_{\rm loc}(\Lambda)\cap(T(S_0)\cup T(S_1))$. These sets may be seen as Cantor like foliations where the leaves are just the connected components of the sets $\mathcal{F}^s$ and $\mathcal{F}^u$. They both extend to C^1- foliations over $S_0\cup S_1$, respectively over $T(S_0)\cup T(S_1)$. The two foliations are transverse to each other. Pick the leaves I^s_* in $\mathcal{F}^s$ and I^u_* in $\mathcal{F}^u$ containing the fixed point P_0. Then the Cantor sets $\Lambda^s=\Lambda\cap I^u_*$ and $\Lambda^u=\Lambda\cap I^s_*$ can be identified with the foliation $\mathcal{F}^s$, respectively $\mathcal{F}^u$, via projections $\pi_s\colon\mathcal{F}^s\to\Lambda^s$ and $\pi_u\colon\mathcal{F}^u\to\Lambda^u$ whose fibers are precisely the leaves of the respective Cantor like foliations. The maps $\psi^s\colon\Lambda^s\to\Lambda^s$, $\psi^s=\pi_s\circ T$, and $\psi^u\colon\Lambda^u\to\Lambda^u$, $\psi^u=\pi_u\circ T^{-1}$, describe the action of T, respectively T^{-1}, on the foliation $\mathcal{F}^u$, respectively $\mathcal{F}^s$. Moreover ψ^u and ψ^s extend as C^1 expanding maps to $I^s_*\cap T(S_0)\cup I^s_*\cap T(S_1)$, and $I^u_*\cap S_0\cup I^u_*\cap S_1$, respectively.

Given a positive binary horse shoe (Λ,T) we orient the invariant local separatrices of P_0, I^s_* and I^u_*, so that orbits flow in the positive direction. These orientations in I^s_* and I^u_* induce orientations in all leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$, and also induce transverse orientations to these foliations. Remark that if we had chosen the other fixed point P_1 then and all these orientations would be reversed. Finally, notice that (Λ^s,ψ^s) and (Λ^u,ψ^u) are positive binary Cantor sets. We define the *left-right thickness* of (Λ,T) as

$$\tau_{LR}\,(\Lambda,T)=\min\left\{\tau_L(\Lambda^s,\psi^s)\,\tau_R(\Lambda^u,\psi^u),\ \tau_L(\Lambda^u,\psi^u)\,\tau_R(\Lambda^s,\psi^s)\right\}\ .$$

Once again, if we fix orientations with respect to the second fixed point P_1 then the left and right thicknesses of both Cantor sets (Λ^s, ψ^s) and (Λ^u, ψ^u) are exchanged, but the left-right thickness of (Λ, T) stays unchanged. We define the *top scale left-right thickness* of (Λ, T) to be

$$\tilde{\tau}_{LR}\,(\Lambda, T) = \min\left\{\tilde{\tau}_L(\Lambda^s, \psi^s)\,\tilde{\tau}_R(\Lambda^u, \psi^u),\ \tilde{\tau}_L(\Lambda^u, \psi^u)\,\tilde{\tau}_R(\Lambda^s, \psi^s)\right\}\ .$$

As before, when the stable and unstable distortions of (Λ, T) are both small then $\tilde{\tau}_{LR}(\Lambda, T)$ approximates well $\tau_{LR}(\Lambda, T)$.

From standard distortion estimates, see [15, 16], it can proved that this thickness depends continuously on (Λ, T), in the C^2 topology. We can now prove the following key

Proposition 3. *Let $f_\mu : M^2 \to M^2$ be a one-parameter family of class C^2-symplectic maps and (Λ_μ, T_μ) a family of positive binary horse shoe maps defined on the union of two smooth rectangles $S_0(\mu) \cup S_1(\mu) \subseteq M^2$ as*

$$T_\mu(x) = \begin{cases} f_\mu(x) & \text{if} \quad x \in S_0(\mu) \\ (f_\mu)^N(x) & \text{if} \quad x \in S_1(\mu) \end{cases} \quad , \qquad N \geq 1$$

Suppose that, at $\mu = 0$, $\tau_{LR}\,(\Lambda_0, T_0) > 1$ and the invariant manifolds of a fixed point $O = f_\mu(O) \in \Lambda_\mu$ unfold generically an orbit of quadratic homoclinic tangencies. Then there is a sequence of parameter intervals Δ_k, accumulating at $\mu = 0$, such that Λ_μ is C^2-stably-wild over each Δ_k.

Proof. We orient the stable and unstable branches of $W^s(O) - O$ and $W^u(O) - O$ so that orbits flow in the positive direction. Let us say that a homoclinic tangency of O is *positive* if both the orientations, on the stable and unstable branches, agree at the point of tangency.

Assume first that the homoclinic tangency of O, which by hypothesis unfolds generically at $\mu = 0$, is a positive one. Let, as before, H denote one homoclinic point in this orbit of tangencies, and let ℓ be the curve, through H, of tangencies between the C^1 extensions of the backward and forward iterations of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively. Again, let K^s and K^u be the Cantor sets formed by the points where the first backward and forward iteration of $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively, intersect the curve ℓ. By hypothesis, the condition (3) is fulfilled for the Cantor sets Λ^s and Λ^u. But locally K^s and K^u are the images, by the holonomies along the stable and unstable foliations, of the Cantor sets Λ^s and Λ^u, respectively. Both holonomies take the point O to H. Consider now the components $\Lambda^s_n = \Lambda^s(0, \cdots, 0)$ and $\Lambda^u_n = \Lambda^u(0, \cdots, 0)$, of order n, in the binary Cantor sets Λ^s and Λ^u, respectively. These are small neighbourhoods of O, respectively in Λ^s and Λ^u. They are both binary Cantor sets which obviously satisfy $\tau_L\,(\Lambda^t_n) \geq \tau_L\,(\Lambda^t)$ and $\tau_R\,(\Lambda^t_n) \geq \tau_R\,(\Lambda^t)$, for both $t = s, u$. Thus these small Cantor sets also satisfy condition (3).

For each n, let K^s_n and K^u_n be the full images, by the holonomies, of the Cantor sets Λ^s_n and Λ^u_n. These images are also binary Cantor sets. To estimate their thicknesses remark that these holonomies are maps of class C^1 and, therefore, they are almost linear, with very small distortion, near O. Since the map φ_μ preserves orientation the iterations of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ around H inherit the orientations from $\mathcal{F}^s$ and $\mathcal{F}^u$, and since the tangency at H is positive, their transversal orientations agree along the curve ℓ. But this means that the holonomy maps transforming Λ_n^s onto K_n^s, and Λ_n^u onto K_n^u, preserve orientation. Therefore, taking n large enough, the open condition (3) will still be satisfied by the Cantor sets K_n^s and K_n^u. This means that for some parameter interval Δ with $0 \in \partial\Delta$, and for all $\mu \in \Delta$, both intervals spanned by K_n^s and K_n^u have a boundary point interior to the other and, furthermore, the Cantor sets K_n^s and K_n^u fulfil condition (3). Since these are open conditions, for all maps C^2 close to some φ_μ, with $\mu \in \Delta$, we may apply the Gap lemma to show that the horse-shoe corresponding to Λ_μ has some "homoclinic" tangency due to an intersection in $K^s \cap K^u$. Therefore Λ_μ is C^2-stably-wild over Δ.

This completes the proof in case the homoclinic tangency H is positive. If not, arguing as in theorem 1 of section 3.1 in [16], one can easily prove that the bifurcation parameter $\mu = 0$ is accumulated by two alternating sequences μ_k^+ and μ_k^- where positive and negative quadratic homoclinic tangencies are unfolded near H. Applying the previous case to parameters μ_k^+, there is a sequence of small intervals Δ_k, with $\mu_k^+ \in \partial\Delta_k$, such that Λ_μ is C^2-stably-wild over Δ_k. $\square$

Theorem A follows by applying this abstract proposition to basic set families whose existence is stated in the next lemma.

Lemma A. *For each $n \geq 4$ there is a continuous family of hyperbolic basic sets $\Lambda_n = \Lambda_n(a)$ for the Hénon map (1), defined in a small parameter interval Δ_n, such that*

(1) *the sequence of intervals Δ_n converges to $a = -1$,*
(2) *the union of all intervals Δ_n covers $]-1, -1/2]$,*
(3) *each basic set Λ_n contains the fixed point O_s,*
(4) $\lim_{n\to+\infty} \tau_{LR}(\Lambda_n) = +\infty$.

Next we make a rough sketch of the basic set construction and give the heuristics behind the thickness asymptotics.

The Hénon maps is reversible with respect to the canonical involution $I(x,y) = (y,x)$. Recall that a symplectic diffeomorphism $f : M^2 \to M^2$ is called *reversible* if there is a smooth map $I : M^2 \to M^2$ such that $I \circ I = \mathrm{Id}_{M^2}$ and $I^*\omega = -\omega$ (where ω denotes the area form in M^2), which conjugates f with its inverse, $f \circ I = I \circ f^{-1}$. The map I is called an *involution.* A set which is invariant by f and I is called a *symmetric f-invariant set.* A periodic orbit is called symmetric if, as a set, it is a symmetric invariant set. In particular symmetric fixed points are common fixed points of f and I.

At $a = -1$ the Hénon family goes through a "saddle-centre" bifurcation where a pair of symmetric fixed points is created: a saddle O_s and an elliptic point O_e. It was proved in [1] that for $a > -1$ the unstable manifold of the saddle O_s has a transversal intersection, at some point Ω, with the symmetry line $\mathrm{Fix}(I) = \{(x,y) : x = y\}$. By reversibility, the stable manifold also intersects this symmetry line at Ω. The transversality of these intersections with $\mathrm{Fix}(I)$ implies that the symmetric homoclinic point Ω depends analytically in the parameter $a > -1$, but this is not enough to guarantee the transversality of the intersection between the invariant manifolds at Ω. It follows from [6] that the splitting angle at this intersection must be an exponentially small function of $a+1$, as $a \to -1$. The transversality was established in [7] where the authors give an asymptotic expression for the Lazutkin splitting invariant at Ω. See section 7 for the definition of Lazutkin invariant.

From this transversal homoclinic intersection at Ω we can argue, as in the classical Birkhoff's theorem, that for each $a > -1$ the saddle O_s is accumulated by two sequences of symmetric periodic points Q_n and Q'_n with even period $2\,n$. The points Q_n and Q'_n, as well as their $n-$th iterates, sit in the symmetry line $x = y$, respectively close to O_s and Ω. Both periodic points are hyperbolic. The eigenvalues of Q_n are both positive, while those of Q'_n are negative. For n large enough it is clear that the stable and unstable manifolds of Q_n, and Q'_n, intersect transversally the unstable and stable manifolds of O_s, respectively. Let S^n be the square bounded by the local invariant manifolds of O_s and Q_n; let S^n_0 be the rectangle formed by points in S^n whose first iteration stays inside S^n; and finally let S^n_1 be the rectangle of points in S^n which return to S^n after $2\,n$ iterations. For each $(x_0, y_0) \in S^n_0 \cup S^n_1$ denote by $\{(x_i, y_i)\}$ the forward orbit of the Hénon map with this initial state and define the map $T_n : S^n_0 \cup S^n_1 \to S^n$ setting

$$T_n(x_0, y_0) = \begin{cases} (x_1, y_1) & \text{if} \quad (x_0, y_0) \in S^n_0 \\ (x_{2n}, y_{2n}) & \text{if} \quad (x_0, y_0) \in S^n_1 \end{cases} .$$

Remark that S^n_0 contains the fixed point O_s while S^n_1 contains the periodic point Q_n. These two rectangles are bounded by the invariant manifolds of O_s and Q_n and together they form a Markov partition for the binary horse-shoe

$$\Lambda_n = \bigcap_{k\in\mathbb{Z}} (T_n)^{-k}(S^n_0 \cup S^n_1) .$$

See figure 1. Since both eigenvalues of O_s and Q_n are positive, (Λ_n, T_n) is a positive binary horse-shoe.

We now want to estimate the left-right thickness of Λ_n. Notice that as n tends to infinity each branch of the map T_n becomes more "linear" while its distortion tends to zero. Consider the *vertical* rectangles S^n, S^n_0 and S^n_1 and let w^n, w^n_0 and w^n_1 be their respective widths, measured along the unstable direction. Let δ be the logarithm of the (larger) eigenvalue of the saddle O_s and denote by λ_n the larger eigenvalue of Q_n. If n is large enough $\lambda_n^{-1} = o(\delta)$, and since T_n becomes almost linear in each branch, $w^n_0 \sim w^n\, e^{-\delta}$ and $w^n_1 \sim w^n\, \lambda_n^{-1}$. Therefore we get the following asymptotics on the left and right stable thickness of Λ_n:

$$\begin{aligned} \tau_L(\Lambda^s_n) &\sim \frac{w^n_0}{w^n - w^n_0 - w^n_1} = \frac{1}{e^{\delta} - 1 - \lambda_n^{-1}} = \mathcal{O}(\delta^{-1}) \\ \tau_R(\Lambda^s_n) &\sim \frac{w^n_1}{w^n - w^n_0 - w^n_1} = \frac{\lambda_n^{-1}}{1 - e^{-\delta} - \lambda_n^{-1}} = \mathcal{O}(\delta^{-1}\lambda_n^{-1}) \end{aligned}$$

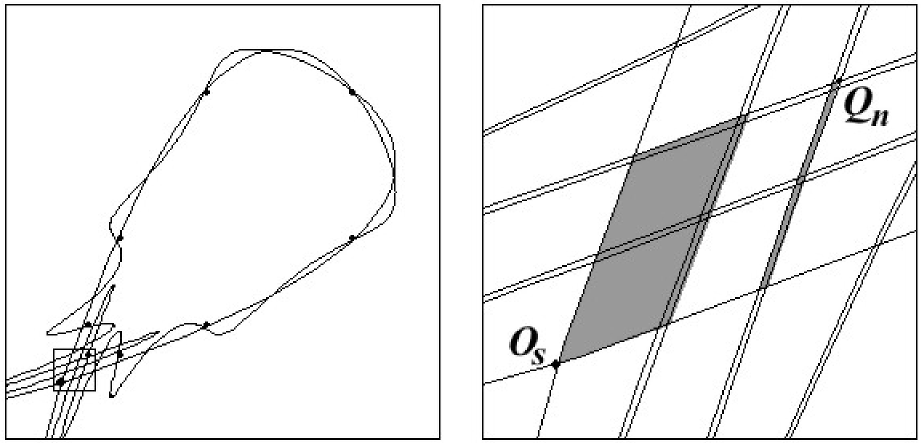

FIGURE 1. The binary horse shoe $T_4 : \Lambda_4 \to \Lambda_4$ for $\delta \approx 1.09$

Since by reversibility the unstable thicknesses have the same values we also get an asymptotic expression for left-right thickness of Λ_n:

$$\tau_{LR}(\Lambda_n, T_n) \sim \mathcal{O}(\delta^{-2}\lambda_n^{-1}) \tag{4}$$

Thus, as $n \to \infty$ the map T_n, becomes more linear with smaller distortion but, along with this, the left-right thickness decreases to zero. So we need to compromise choosing carefully the number of iterations n, which has to be large if we want small distortion, but not too large if we also want to keep left-right thickness large. Denoting by $\theta = \theta_\delta$ the splitting angle at the symmetric homoclinic point Ω we choose n so that $e^{2\delta n}\theta \sim \delta^{-3/2}$, which, one can easily check to be the asymptotic value of λ_n. Thus, replacing λ_n by $\delta^{-3/2}$ in (4) we obtain $\tau_{LR}(\Lambda_n, T_n) \sim \delta^{-1/2}$ which tends to infinity as $a \to -1$, or $\delta \to 0$. Of course now we have to prove that for this particular value of n (depending on a, or δ) the symmetric periodic saddle Q_n, and the corresponding horse-shoe Λ_n, already exist. Moreover we need to show that the distortion of (Λ_n, T_n) tends to zero when $a \to -1$.

Let us now outline the real proof of lemma A. The construction of Λ_n is carried out in Birkhoff co-ordinates. We re-scale the Hénon maps, for $a > -1$, in order to make the distance between the fixed points O_s and O_e constant. This is done in the first part of section 4. Define δ to be the logarithm of the eigenvalue at O_s. Parameterising the rescaled mappings in δ, we obtain a family of maps close to the *identity*, $F_\delta = \mathrm{Id}_{\mathbb{R}^2} + \delta F_0 + \mathcal{O}(\delta^2)$, where F_0 is a quadratic Hamiltonian vector field with two fixed points: a saddle O_s and an elliptic point O_e. At this point we follow closely the construction in [3] for maps near the identity. The main assumption, of theorem 1 there, is a bounding condition on the intersection geometry between the stable and unstable separatrices of the saddle point O_s. In our setting, this condition comes essentially from the asymptotics in [7] for the Lazutkin invariant at Ω, but the analytic dependence of Birkhoff co-ordinates on the parameter is also necessary. For that purpose, in section 3, we show that for analytic families of symplectic maps near the identity, as above, co-ordinates exist, depending analytically in $\delta > 0$, reducing each map F_δ to its Birkhoff normal form over a fixed size neighbourhood of O_s. Then in the second part of section 4, working in Birkhoff co-ordinates we translate the asymptotics on the splitting angle in [7] into a condition on the C^2 geometry of the unfolding of the separatrices at $\delta = 0$. As we said above, this condition is the main assumption of theorem 1 in [3]. The rest of the construction follows closely that work. Distortion estimates, needed to estimate thickness, follow from theorem 2 there. Unfortunately the construction of Λ_n is not a logical consequence of results there. Some adaptations must be done. In section 5 we provide some technical details on these adjustments. To finish, we just give a short description, a kind of road map to help the reader going through section 5 and [3].

In section 5 we associate an integer $n = n(\delta)$, called the *half return time*, to each $\delta > 0$, such that $e^{2\,\delta\,n}\,\theta \sim \delta^{-3/2}$. For this value of n, the periodic point Q_n will be exponentially close to $O_s = (0,0)$. More precisely $Q_n \sim \left(\delta^{3/4}\sqrt{\theta_\delta}\,,\,\delta^{3/4}\sqrt{\theta_\delta}\right)$. Since the scale of $\Lambda_{n(\delta)}$ shrinks, as $\delta \to 0$, we perform one last rescaling which brings Q_n close to $(1,1)$ and S^n close to a unit size square $[0,1]^2$. Because the second derivatives of T_n, which are needed to estimate distortion, are scale-dependent this "normalisation" is required in Theorem 2 of [3]. Then in these final co-ordinates we compute estimates for derivatives of T_n :

(1) in the first branch S_0,

$$DT_n = \begin{pmatrix} e^{\delta} & 0 \\ 0 & e^{-\delta} \end{pmatrix} + \mathcal{O}(\delta^{3/2}\,\theta_\delta)\ ,$$

and all the second derivatives of T_n on this branch are of exponentially small order $\mathcal{O}(\delta^{3/2}\,\theta_\delta)$. Similar bounds hold for T_n^{-1} on $T_n(S_0)$.

(2) in the second branch S_1,

$$DT_n = \begin{pmatrix} -\delta^{-3/2} & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\delta^{-1}) & o(1) \\ o(1) & o(1) \end{pmatrix} \quad ,$$

and the second derivatives of T_n on this branch are all uniformly bounded, except for the second derivative, in the variable x, of the first component of T_n, which is unbounded of order $\mathcal{O}(\delta^{-5/2})$. Again, similar bounds hold for T_n^{-1} on $T_n(S_1)$.

From these asymptotics the construction of Λ_n follows easily. It is clear that the left-right thickness of Λ_n has, up to a distortion factor, order $\delta^{-1/2}$. Applying theorem 2 in [3] we obtain, from 1. and 2. above, that distortion is small of order $\mathcal{O}(\delta^{1/2})$. This shows that distortion factors are close to 1. The half return time $n(\delta)$ is asymptotically equivalent to $-\frac{\log\left(\delta^{3/2}\,\theta\right)}{2\,\delta}$, and tends to $+\infty$, as $\delta \to 0$. Thus, when $a \to -1$, we have $\delta \to 0$, and so $\tau_{LR}\,(\Lambda_n, T_n) \to +\infty$.

Finally, lemma bellow is proved in section 6.

Lemma B. *There is a sequence of values of a accumulating at $a = (-1)^+$ for which the saddle point O_s of the Hénon map has an orbit of quadratic homoclinic tangencies which unfolds generically with parameter a.*

Lemmas A and B, in view of proposition 3, imply Theorem A.

3. Analytic Birkhoff co-ordinates

For area preserving maps we have Siegel-Moser's theorem on the convergence of the Birkhoff normal form around a hyperbolic fixed point. See [19]. This theorem says that given an analytic area preserving map $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, defined in a neighbourhood U of $(0,0)$, with a hyperbolic fixed saddle sitting at the origin, $F(x,y) = (\lambda\, x + \cdots\,,\ \lambda^{-1}\, y + \cdots\,)$, where $|\lambda| \neq 1$, then there is an analytic change of co-ordinates $\zeta_F(x,y) = (x + \cdots\,,\ y + \cdots\,)$, defined in a neighbourhood of $(0,0)$, and there is an analytic function $\alpha_F(\omega) = \log(\lambda) + \cdots$, of one variable ω, defined in another neighbourhood of 0 such that for all (x,y) close to $(0,0)$,

$$(F \circ \zeta_F)\ (x,y) = \zeta_F\left(e^{\alpha_F(x\,y)}\, x\,,\ e^{-\alpha_F(x\,y)}\, y\right)\ . \tag{5}$$

The map $L_F : \mathbb{R}^2 \to \mathbb{R}^2$, $L_F(x,y) = \left(e^{\alpha_F(x\,y)}\, x\,,\ e^{-\alpha_F(x\,y)}\, y\right)$, is called "a Birkhoff normal form" for F.

Following Birkhoff, the maps $\zeta_F(x,y)$ and $\alpha_F(\omega)$, $\omega = x\,y$, in this theorem are found as formal power series

$$\zeta_F(x,y) = \left(x + \sum_{n+m\geq 2} a_{n\,m}(F)\ x^n\, y^m\,,\ y + \sum_{n+m\geq 2} b_{n\,m}(F)\ x^n\, y^m \right) \tag{6}$$

$$\exp\left(\alpha_F(\omega)\right) = \lambda + \sum_{n=1}^{\infty} c_{2\,n}(F)\ \omega^n \tag{7}$$

which solve the conjugacy relation (5). The uniqueness of the formal solutions (6) and (7) of equation (5) is obtained by adding the following "normalising" condition for the formal solution (6)

$$a_{n+1\,n}(F) = b_{n\,n+1}(F) = 0 \qquad \text{for all} \quad n \geq 1\ . \tag{8}$$

These coefficients are obtained by recursive relations which involve the coefficients of F's Taylor series. Therefore each $a_{k\,m}(F)$, $b_{k\,m}(F)$, or $c_{2\,m}(F)$ is a polynomial in F's Taylor coefficients. This proves, taking the weak topology (of point wise convergence of coefficients) in the space of formal series, that the formal solutions $\zeta_F(x,y)$ and $\alpha_F(\omega)$ depend continuously on F.

Unfortunately the formal transformation $\zeta_F(x,y)$ does not, formally speaking, preserve area. (7) is not the appropriate normalising condition. The reason to consider this condition is because the convergence proof is much easier in this case. Anyway the normal form thus obtained (although not unique) is, formally, a reversible area preserving map.

We now outline Siegel-Moser's convergence proof in order to justify why the maps ζ_F and α_F depend continuously, and even analytically, on F.

Let U be an open neighbourhood of the origin in $\mathbb{C}^2$ and consider the set of systems $\mathcal{S} = \mathcal{S}(U)$ formed by all holomorphic maps $F \in \mathcal{H}(U, \mathbb{C}^2)$ of the form

$F(x,y) = (\lambda\, x + \cdots, \lambda^{-1}\, y + \cdots)$, for some $\lambda \in \mathbb{C}$ with $|\lambda| \neq 1$, and endow $\mathcal{S}$ with the topology of $\mathcal{H}(U, \mathbb{C}^2)$ [1].

Some definitions are required. Let $t(x,y) = \sum_{n=0}^{\infty} \alpha_{n\,k}\, x^n\, y^k$ and $s(x,y) = \sum_{n=0}^{\infty} \beta_{n\,k}\, x^n\, y^k$ be formal power series in the variables x and y. We say that $s(x,y)$ dominates $t(x,y)$, and write $t(x,y) \prec s(x,y)$, if for all $n, k \in \mathbb{N}$ $|\alpha_{n\,k}| \leq \beta_{n\,k}$. The relation $\prec$ partially orders the set $\mathbb{R}^+(x,y)$ of all formal series with non negative coefficients, which we will call *positive* formal series. A similar definition is given for formal power series in one single variable.

Let $F_0 \in \mathcal{S}$ be given. Choose $c > 0$ large so that the closed polydisc $\Delta_{c^{-1}} = \overline{D}(0, c^{-1}) \times \overline{D}(0, c^{-1})$ is contained inside the domain U. Take a neighbourhood $\mathcal{U}$ of F_0 in $\mathcal{S}$ such that all maps $F \in \mathcal{U}$ are uniformly bounded on $\Delta_{c^{-1}}$ by some constant $M > 0$. Then the Taylor coefficients of all $F \in \mathcal{U}$ are bounded by the sequence $\{M\, c^n\}$ and we may assume, taking a larger c, that $M = 1$.

Consider now the positive, convergent, formal series

$$G_c(x,y) = \frac{c\,(x+y)^2}{1 - c\,(x+y)} = \sum_{n=2}^{\infty} c^{n-1}\,(x+y)^n\ .$$

Then the components of the second order Taylor remainder, of any $F \in \mathcal{U}$, are both dominated by $G_c(x,y)$.

Write $\zeta_F(x,y) = (\varphi_F(x,y), \psi_F(x,y))$ and $\lambda_F(\omega) = \exp\{\alpha_F(\omega)\}$. Then a positive formal series $W_F(\omega) = \sum_{n=1}^{\infty} w_n(F)\ \omega^n$, with zero constant term, is constructed form the data $\varphi_F(x,y)$, $\psi_F(x,y)$ and $\lambda_F(\omega)$ satisfying the dominance relations

$$\begin{aligned} \varphi_F(x,y) - x \ &\prec\ (x+y)\, W_F(x+y) \\ \psi_F(x,y) - y \ &\prec\ (x+y)\, W_F(x+y) \\ \lambda_F(x\,y)^{-1} - \lambda^{-1} \ &\prec\ W_F(x+y) \end{aligned}$$

and the following "dominance equation"

$$\omega\, W_F(\omega) \ \prec\ \frac{c_1}{1 - c_2\, W_F(\omega)}\, G_c\,(\,\omega\,(W_F(\omega)+1)\,,\, \omega\,(W_F(\omega)+1)\,)\ . \tag{9}$$

Proving this fact is of course the main step in Siegel-Moser's theorem, but we will skip its proof here. Moving on, equation (9) easily implies

$$W_F(\omega) \ \prec\ \frac{c_3\,\omega\,(1+W_F)^2}{1 - c_2\, W_F - 2\,c\,\omega\,(1+W_F)} \tag{10}$$

where $W_F = W_F(\omega)$ and c_1, c_2 and c_3 are positive constants defined by open conditions depending only on λ and c. Consider now the *quadratic* fixed point equation in U

$$U(\omega) \ =\ \frac{c_3\,\omega\,(1+U)^2}{1 - c_2\, U - 2\,c\,\omega\,(1+U)} \tag{11}$$

which has two solutions, respectively satisfying $U(0) = 1$ and $U(0) = 0$, both being analytic in a neighbourhood of the origin. Let $U(\omega)$ be the second one,

[1]this is the space of holomorphic functions on the open set U with the natural topology of uniform convergence on compact subsets of U.

which defines a formal series around the origin with zero constant term, and having a positive radius of convergence $r > 0$. This radius r may be computed explicitly from the values c, c_1, c_2 and c_3.

The coefficients u_n of $U(\omega) = \sum_{n=1} u_n\, \omega^n$ may be recursively computed from (11). Similarly, the coefficients $w_n(F)$ of $W_F(\omega) = \sum_{n=1} w_n(F)\ \omega^n$ can be recursively estimated from (10). Comparing both sequences of coefficients, starting at $w_0(F) = 0 = u_0$, it is proved inductively that $w_n(F) \leq u_n$ for all $n \in \mathbb{N}$. The cause of this fact is that the right hand side of (11), the same as in (10), expands in a positive power series in the variables ω and U.

Therefore $W_F(\omega) \prec U(\omega)$, for all $F \in \mathcal{U}$, which proves that the family of formal series $\{\zeta_F(x,y)\}_{F\in\mathcal{U}}$ converges uniformly in $(F,x,y) \in \mathcal{U} \times \overline{D}(0,r/2)^2$. Analogously the family $\{\lambda_F(\omega)^{-1}\}_{F\in\mathcal{U}}$ converges uniformly in $(F,\omega) \in \mathcal{U}\times\overline{D}(0,r)$.

Given a family $\{F_h\ :\ h \in \Omega\}$ of maps in $\mathcal{U}$, which is holomorphic in $(h,x,y) \in \Omega\times U$, the coefficients $a_{k\,m}(h) := a_{k\,m}(F_h)$, $b_{k\,m}(h) := b_{k\,m}(F_h)$, and $c_{2\,m}(h) := c_{2\,m}(F_h)$ are polynomials in F_h Taylors' coefficients at $(x,y) = (0,0)$, and so depend analytically in h. Therefore, the partial sums of the formal power series (6) and (7), with $F = F_h$, are holomorphic functions in $(h,x,y) \in \Omega \times \mathbb{C}^2$ and in $(h,\omega) \in \Omega\times\mathbb{C}$, respectively. Thus the mappings $(h,x,y) \mapsto \zeta_{F_h}(x,y)$ and $(h,\omega) \mapsto \alpha_{F_h}(\omega)$, which are uniform limits of their partial sums over the domains $\Omega\times\overline{D}(0,r/2)^2$ and $\Omega\times\overline{D}(0,r)$, respectively, are also holomorphic in these domains.

Thus the proof in [19] shows that

Theorem 1 (Siegel-Moser). *Given $F_0 \in \mathcal{S}(U)$ there is some $r > 0$, and an open set $\mathcal{U} \subseteq \mathcal{S}(U)$ containing F_0 such that:*

(1) *for every $F \in \mathcal{U}$ the formal series $\zeta_F(x,y)$ and $\alpha_F(\omega)$, given in (6) and (7), which are uniquely determined by (5) and the normalising condition (8), converge uniformly to holomorphic functions defined on $D(0,r/2)^2$ and $D(0,r)$, respectively,*
(2) *both the transformations $\zeta : \mathcal{U} \to \mathcal{H}(D(0,r/2)^2, \mathbb{C}^2)$, $F \mapsto \zeta_F$, and $\alpha : \mathcal{U} \to \mathcal{H}(D(0,r), \mathbb{C})$, $F \mapsto \alpha_F$ are continuous.*
(3) *given a holomorphic function $F : \Omega\times U \to \mathbb{C}^2$, $F(h,x,y) = F_h(x,y)$, defining a family of mappings $F_h \in \mathcal{U} \subseteq \mathcal{S}(U)$ in the complex parameter $h \in \Omega \subseteq \mathbb{C}$, the functions $\zeta(h;\, x,y) := \zeta_{F_h}(x,y)$ and $\alpha(h;\, \omega) := \alpha_{F_h}(\omega)$ are holomorphic respectively in the domains $\Omega\times D(0,r/2)^2$ and $\Omega\times D(0,r)$.*

Remark 1. *Although $\zeta_F(x,y)$ is not area preserving, for all $|x|$, $|y| < r/2$,*

$$\det D\zeta_F(x,0) = 1 = \det D\zeta_F(0,y)\,.$$

In order to simplify the notation we will omit the subscript F in ζ_F. Since there is a "symmetry" in these two relations it is enough to prove the first one. Differentiating relation (5) w.r.t. x and y at the point $(x,0)$ one obtains the following relations

$$\begin{aligned} DF_{\zeta(x,0)} \cdot \zeta_x(x,0) &= \lambda\, \zeta_x(\lambda\, x, 0) \\ DF_{\zeta(x,0)} \cdot \zeta_y(x,0) &= \lambda\, x^2\, \alpha'(0)\, \zeta_x(\lambda\, x,0) \,+\, \lambda^{-1}\, \zeta_y(\lambda\, x, 0)\,. \end{aligned}$$

Therefore

$$\omega\,(\zeta_x(\lambda\,x,0)\,,\,\zeta_y(\lambda\,x,0)\,) = \omega\,(\zeta_x(x,0)\,,\,\zeta_y(x,0)\,)\ ,$$

and this relation holds for any power of λ. Taking negative powers λ^{-n} with $n \to +\infty$ we obtain

$$\det D\zeta_{(x,0)} = \omega\,(\zeta_x(x,0)\,,\,\zeta_y(x,0)\,) = \omega\,(\zeta_x(0,0)\,,\,\zeta_y(0,0)\,) = 1\ .$$

Remark 2. *Given $F \in \mathcal{S}$ then for all $n \in \mathbb{Z} - \{0\}$*

$$\zeta_{F^n}(x,y) = \zeta_F(x,y) \qquad \textit{and} \qquad \alpha_{F^n}(\omega) = n\,\alpha_F(\omega)\ .$$

In particular all these formal series converge on the same domain.

Remark 3. *Given a Hamiltonian vector field X with flow $\phi^t \in \mathcal{S}$ ($t \neq 0$), define $\zeta_X := \zeta_{\phi^1}$ and $\alpha_X := \alpha_{\phi^1}$. Then for all $t \in \mathbb{R} - \{0\}$,*

$$\zeta_{\phi^t}(x,y) = \zeta_X(x,y) \qquad \textit{and} \qquad \alpha_{\phi^t}(\omega) = t\,\alpha_X(\omega)\ .$$

Remark 4. *Given $F \in \mathcal{S}$ reversible w.r.t. the canonical involution $I(x,y) = (y,x)$, i.e. $F \circ I = I \circ F^{-1}$, then $\zeta_F \circ I = I \circ \zeta_F$. In particular the time reversing symmetries of F and L_F are conjugated by ζ_F.*

Remarks 2 and 3 are quite obvious. Let us prove 4 which follows from the uniqueness condition (8). Since

$$\begin{aligned} F^{-1} \circ (I \circ \zeta_F \circ I) &= F^{-1} \circ I \circ \zeta_F \circ I = I \circ F \circ \zeta_F \circ I = I \circ \zeta_F \circ L_F \circ I \\ &= I \circ \zeta_F \circ I \circ (L_F)^{-1} = (I \circ \zeta_F \circ I) \circ L_{F^{-1}} \end{aligned}$$

and $I \circ \zeta_F \circ I$ trivially fulfils condition (8), by remark 2 $I \circ \zeta_F \circ I = \zeta_{F^{-1}} = \zeta_F$. Therefore $\zeta_F \circ I = I \circ \zeta_F$.

Remark 5. *Given any $\alpha \in \mathcal{H}(D(0,r),\mathbb{C})$, defining $W_r = \{(x,y) \in \mathbb{C}^2 \,:\, |x\,y| < r\,\}$, the normal form*

$$L(x,y) = \left(\,e^{\alpha(x\,y)}\,x\,,\ e^{-\alpha(x\,y)}\,y\,\right)\ ,$$

defines a global holomorphic diffeomorphism of W_r onto W_r, with inverse

$$L^{-1}(x,y) = \left(\,e^{-\alpha(x\,y)}\,x\,,\ e^{\alpha(x\,y)}\,y\,\right)\ .$$

Thus, if $\mathcal{U}$ and $r > 0$ are as in theorem 1, then the maps $L^n : \mathcal{U} \to \mathcal{H}(W_r, \mathbb{C}^2)$, $F \mapsto L_{F^n} = (L_F)^n$, $n \in \mathbb{Z}$, are continuous.

Remark 6. *Let α and L be as in remark 5. The region*

$$e^{-\alpha(x\,y)} < \left|\frac{x}{y}\right| < e^{\alpha(x\,y)}$$

is a fundamental domain *for the restriction of L to the invariant set $W_r^* = \{(x,y) \in \mathbb{C}^2 \,:\, 0 < |x\,y| < r\,\}$. Assuming that $|\alpha(\omega)| < A$ for all $|\omega| < r$ it follows that every L orbit in W_r goes through the polydisc $D\left(0, \sqrt{r\,e^A}\right)^2$.*

Remark 7. *Given $r > 0$ and $\mathcal{U}$ as in theorem 1, there is some r_1 such that for each $F \in \mathcal{U}$ we can extend ζ_F holomorphically to the domain W_{r_1}. Let $r_1 < r^2\,e^{-A}/4$ where $|\alpha_F(\omega)| < A$ for all $|\omega| < r$. Then by remark 6, all L_F orbits in W_{r_1} must go through the domain $D\,(0, r/2)^2$, where $\zeta_F(x,y)$ is defined.*

Thus given $(x,y) \in W_{r_1}$ *, we can take some* $n \in \mathbb{Z}$ *such that* $(L_F)^{-n}(x,y) \in D\,(0,r/2)^2$ *, and define*

$$\tilde{\zeta}_F(x,y) := \left(F^n \circ \zeta_F \circ (L_F)^{-n}\right)\,(x,y)\;.$$

By virtue of (5), $\tilde{\zeta}_F(x,y)$ *is well defined and holomorphic in* W_{r_1} *. The transformation* $\tilde{\zeta}:\mathcal{U}\to\mathcal{H}(W_{r_1},\mathbb{C}^2)$ *,* $F \mapsto \tilde{\zeta}_F$ *is easily seen to be continuous.*

Consider an analytic family F_δ of area preserving maps. Suppose all maps F_δ have holomorphic extensions to some open set $U \subseteq \mathbb{C}^2$ containing the origin and the family F_δ is a perturbation of the identity,

$$F_\delta = \mathrm{Id}_{\mathbb{R}^2} + \epsilon\, F_1 + \mathcal{O}(\epsilon^2)\;, \tag{12}$$

where $\epsilon = \epsilon(\delta)$ satisfies

$$\lim_{\delta\to 0} \epsilon(\delta)/\delta = c > 0\;. \tag{13}$$

The variation direction is that of the Hamiltonian vector field F_1, which also extends holomorphically to the same open set $U \subseteq \mathbb{C}^2$. Assume that the origin is a diagonalized hyperbolic fixed point,

$$F_\delta(x,y) = \left(\lambda_\delta\, x + \cdots\;,\; \lambda_\delta^{-1}\, y + \cdots\right)\;, \tag{14}$$

where $\lambda_\delta = 1 + a\,\epsilon + \mathcal{O}(\epsilon^2)$, with $a \neq 0$, and

$$F_1(x,y) = (\,a\,x + \cdots\;,\; -a\,y + \cdots\,)\;, \tag{15}$$

the dots meaning terms in $x^i\,y^j$ of order $i+j \geq 2$. Each map F_δ is, therefore, in the class $\mathcal{S}(U)$. Finally assume that the flow ϕ^t of the vector field F_1 extends to holomorphic maps $\phi^t : U \to \mathbb{C}^2$ for any real time t.

For each $\delta \neq 0$ we define the formal series

$$\begin{aligned} \zeta(\delta,x,y) = \zeta_\delta(x,y) \;&:=\; \zeta_{F_\delta}(x,y) \\ \alpha(\delta,\omega) = \alpha_\delta(\omega) \;&:=\; \alpha_{F_\delta}(\omega)\;. \end{aligned} \tag{16}$$

For $\delta = 0$ let us set

$$\begin{aligned} \zeta(0,x,y) = \zeta_0(x,y) \;&:=\; \zeta_{F_1}(x,y) \\ \alpha(0,\omega) = \alpha_0(\omega) \;&:=\; 0\;. \end{aligned} \tag{17}$$

where $F_1 = (\partial F_\delta/\partial\delta)_{\delta=0}$.

In [6] E. Fontich and C. Simó proved that for some small δ_0 the series $\zeta_\delta(x,y)$ and $\alpha_\delta(\omega)$ converge, uniformly in $\delta \in [-\delta_0,\delta_0]-\{0\}$, over some fixed open domain around the origin. They went through the argument in [19] and checked that the coefficients $a_{n\,m}(\delta)$, $b_{n\,m}(\delta)$ and $c_n(\delta)$ can be uniformly bounded in δ. Next proposition is direct corollary of theorem 1 which generalises proposition 3.1 in[6].

Proposition 4. *Given a family* F_δ *as above, there are constants* $r > 0$ *and* $\delta_0 > 0$ *such that:*

1. *for all* $|\delta| < \delta_0$ *the formal series* $\zeta_\delta(x,y)$ *and* $\alpha_\delta(\omega)$ *converge uniformly to holomorphic functions defined on* $D(0,r/2)^2$ *and* $D(0,r)$*, respectively.*
2. *the maps* $\zeta : [-\delta_0,\delta_0] \to \mathcal{H}(D(0,r/2)^2,\mathbb{C}^2)$ *,* $\delta \mapsto \zeta_\delta$ *, and* $\alpha : [-\delta_0,\delta_0] \to \mathcal{H}(D(0,r),\mathbb{C})$ *,* $\delta \mapsto \alpha_\delta$ *are continuous.*

(3) $\zeta(\delta,x,y)=\zeta_\delta(x,y)$ *and* $\alpha(\delta,\omega)=\alpha_\delta(\omega)$ *are real analytic functions, respectively in the real domains* $\{\,(\delta,x,y)\,:\,0<|\delta|<\delta_0\,,\ \ |x|\,,\,|y|<r/2\,\}$ *and* $\{\,(\delta,\omega)\,:\,0<|\delta|<\delta_0\,,\ \ |\omega|<r\,\}$ *.*

(4) *uniformly in* $\omega\in D(0,r)$ *,*

$$\alpha_{F_1}(\omega)=\lim_{\delta\to 0}\frac{\alpha_\delta(\omega)}{\epsilon(\delta)}\,.$$

To prove this proposition we need a simple lemma. Given $t>0$ denote by $[t/\epsilon]$ the integer part of $t/\epsilon(\delta)$. Let, as before, $\phi^t:U\to\mathbb{C}^2$ be the (real time) flow of the Hamiltonian vector field $F_1(x,y)$. For each compact set $K\subseteq U$ we consider the following semi norm

$$\|f\|_K=\max\{\,\|f(x,y)\|\ :\ (x,y)\in K\,\}\ .$$

Then,

Lemma 1. *the following limit holds in* $\mathcal{H}\,(U,\mathbb{C}^2)$ *, for any* $t>0$ *:*

$$\lim_{\delta\to 0}\,(F_\delta)^{[t/\epsilon(\delta)]}=\phi^t$$

Proof. Since

$$\begin{aligned} F_\delta &= \mathrm{Id}_{\mathbb{C}^2}+\epsilon\,F_1+\epsilon^2\,F_2 \quad \text{and}\\ \phi^\epsilon &= \mathrm{Id}_{\mathbb{C}^2}+\epsilon\,F_1+\mathcal{O}(\epsilon^2)\quad , \end{aligned}$$

there is some constant $C>0$, depending on K, such that $\|DF_\delta\|_K\le 1+C\,\epsilon$ and $\|F_\delta-\phi^\epsilon\|_K\le C\,\epsilon^2$. Thus

$$\begin{aligned} \|F_\delta^n-\phi^{n\,\epsilon}\|_K &\le \left\|F_\delta\,F_\delta^{n-1}-F_\delta\,\phi^{(n-1)\,\epsilon}\right\|_K+\left\|F_\delta\,\phi^{(n-1)\,\epsilon}-\phi^\epsilon\,\phi^{(n-1)\,\epsilon}\right\|_K\\ &\le (1+C\,\epsilon)\,\left\|F_\delta^{n-1}-\phi^{(n-1)\,\epsilon}\right\|_K+C\,\epsilon^2\quad , \end{aligned}$$

and so by induction

$$\|F_\delta^n-\phi^{n\,\epsilon}\|_K\ \le\ C\,\epsilon^2\,\frac{(1+C\,\epsilon)^n-1}{C\,\epsilon}=\epsilon\,((1+C\,\epsilon)^n-1)\,.$$

Finally taking $n=[t/\epsilon]$ we get

$$\begin{aligned} \left\|F_\delta^n-\phi^t\right\|_K &\le \|F_\delta^n-\phi^{n\,\epsilon}\|_K+\left\|\phi^{n\,\epsilon}-\phi^t\right\|_K\\ &\le \epsilon\,e^{C\,t}+\left\|\mathrm{Id}_{\mathbb{C}^2}-\phi^{t-n\,\epsilon}\right\|_K \end{aligned}$$

which converges to zero, as $\delta\to 0$, since $\epsilon=\epsilon(\delta)\to 0$ and $|t-n\,\epsilon|\le\epsilon$. □

Proof. of proposition 4

The time one map ϕ^1 of F_1 is in the class $\mathcal{S}(U)$. For this map ϕ^1 take $r>0$ and $\mathcal{U}$ according to theorem 1. By lemma 1, there is $\delta_0>0$ such that for all $\delta\in[-\delta_0,\delta_0]$, $(F_\delta)^{[1/\epsilon]}\in\mathcal{U}$. Given $|\delta'|<\delta_0$ let us prove the continuity of ζ and α at $\delta=\delta'$. Suppose first that $\delta'\neq 0$ and fix $n=[1/\epsilon(\delta')]$. Of course for all δ near δ' , $(F_\delta)^n$ is in $\mathcal{U}$. By item 2. of theorem 1, ζ and α are continuous on $\mathcal{U}$ and by remark 2, $\zeta_\delta=\zeta_{(F_\delta)^n}$ and $\alpha_\delta=n^{-1}\,\alpha_{(F_\delta)^n}$. Thus the continuity of $\delta\mapsto\zeta_\delta$ and $\delta\mapsto\alpha_\delta$ at $\delta=\delta'$ follows. Notice we also obtain the analyticity of $\zeta(\delta,x,y)$ and $\alpha(\delta,\omega)$, at $\delta=\delta'$, from item 3. of theorem 1.

On the other hand, again by remark 2,

$$\zeta_\delta(x,y) := \zeta_{F_\delta}(x,y) = \zeta_{(F_\delta)^{[1/\epsilon]}}(x,y) \to \zeta_{\phi^1} =: \zeta_0\,,$$

$$\alpha_\delta(x,y) := \alpha_{F_\delta}(x,y) = \frac{\alpha_{(F_\delta)^{[1/\epsilon]}}(x,y)}{[1/\epsilon]} \to 0\,,$$

which proves the continuity at $\delta = 0$.

By lemma 1 and the continuity of $\alpha : \mathcal{U} \to \mathcal{H}(D(0,r),\mathbb{C})$, we have

$$[1/\epsilon]\ \alpha_\delta := \alpha_{(F_\delta)^{[1/\epsilon]}} \to \alpha_{\phi^1} =: \alpha_{F_1}\ ,$$

as $\delta \to 0$, and therefore

$$\lim_{\delta\to 0} \frac{\alpha_\delta}{\epsilon(\delta)} = \lim_{\delta\to 0} \frac{1}{[1/\epsilon]\ \epsilon(\delta)}\ [1/\epsilon]\ \alpha_\delta = \alpha_{F_1}\ .$$

□

4. Splitting of separatrices

In [7] the authors call conservative Hénon family to the following family of maps $\tilde{F}_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$

$$\tilde{F}_\epsilon(x,y) = \left(x + \epsilon\, \left(y + \epsilon\,(x - x^2)\right)\ ,\ y + \epsilon\,(x - x^2)\right)\ . \tag{18}$$

The relation with our model (1) is given by the following rescaling affine mappings $\tau_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\tau_\epsilon(x,y) = \left(\frac{1}{2} + \frac{y+1}{\epsilon^2}, \frac{y-x}{\epsilon^3} \right)\ .$$

One can easily verify that for each $\epsilon \in \mathbb{R}$, setting $a = -1 - \epsilon^4/4$,

$$\tau_\epsilon\left(y, a - x - y^2\right) = (\tilde{F}_\epsilon \circ \tau_\epsilon)\,(x,y)\ .$$

The mappings $\tilde{F}_\epsilon$ are of course reversible for the correspondent involution $I_\epsilon = \tau_\epsilon \circ I \circ \tau_\epsilon^{-1}$, which one computes to be $I_\epsilon(x,y) = (x - \epsilon\, y, -y)$. We can write (18) in the form $\tilde{F}_\epsilon = \mathrm{Id}_{\mathbb{R}^2} + \epsilon\,\tilde{F}_1 + \epsilon^2\,\tilde{F}_2$, where $\tilde{F}_1(x,y) = (y, x - x^2)$ and $\tilde{F}_2(x,y) = (x - x^2, 0)$, which shows that $\tilde{F}_\epsilon$ is a perturbation of the identity in the direction of the Hamiltonian vector field $\tilde{F}_1$. Notice that the origin corresponds to the saddle O_s for all maps $\tilde{F}_\epsilon$,

$$\tilde{F}_\epsilon(x,y) = \left((1+\epsilon^2)\,x + \epsilon\,y + \cdots\ ,\ \epsilon\,x + y + \cdots \right)\ .$$

Denoting by $e^{\pm\delta}$ the eigenvalues of this saddle we have

$$2 + \epsilon^2 = \mathrm{trace} D(\tilde{F}_\epsilon)_{(0,0)} = e^{\delta} + e^{-\delta}\,, \tag{19}$$

with the following asymptotic relation at $\delta = 0$,

$$\epsilon = \delta - \frac{\delta^3}{24} + O(\delta^5)\,. \tag{20}$$

Computing the eigenvectors of $\tilde{F}_\epsilon$'s linear part we construct the following "linearising matrix",

$$M_\epsilon = \frac{1}{2\sqrt[4]{4+\epsilon^2}} \begin{pmatrix} \epsilon + \sqrt{4+\epsilon^2} & -\epsilon + \sqrt{4+\epsilon^2} \\ 2 & -2 \end{pmatrix}, \tag{21}$$

normalised to have determinant equal to -1, which diagonalizes $D(\tilde{F}_\epsilon)_{(0,0)}$ and transforms, by conjugacy, the family of involutions I_δ back to the canonical involution $I(x,y) = (y,x)$. Defining $F_\delta = M_\epsilon^{-1} \circ \tilde{F}_\epsilon \circ M_\epsilon$, where $\epsilon = \epsilon(\delta)$ is implicitly defined by (19), we have $F_\delta = \mathrm{Id}_{\mathbb{R}^2} + \epsilon\, F_1 + \mathcal{O}(\epsilon^2)$, with

$$F_1(x,y) = \left(x - \frac{\sqrt{2}}{4}\,(x+y)^2\,,\; -y + \frac{\sqrt{2}}{4}\,(x+y)^2 \right), \tag{22}$$

and

$$F_\delta(x,y) = \left(e^\delta\, x + \cdots\,,\; e^{-\delta}\, y + \cdots\right).$$

The map $I(x,y) = (y,x)$ being now the time reversing involution for all F_δ. The family F_δ is again a perturbation of the identity, now in the direction of the Hamiltonian vector field F_1.

*

The vector fields $\tilde{F}_1$ and F_1 have homoclinic loops associated with the saddles sitting at the origin, described by the critical level equations of the correspondent Hamiltonians,

$$\tilde{H}_1 = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} = 0\,, \quad \text{and}$$

$$H_1 = \frac{(x-y)^2}{4} - \frac{(x+y)^2}{4} + \frac{(x+y)^3}{6\sqrt{2}} = 0\,.$$

These curves intersect the symmetry lines, of the correspondent involutions I_0 and I, at the symmetric homoclinic points $\tilde{\Omega} = (3/2, 0)$ and $\Omega = (3\sqrt{2}/4, 3\sqrt{2}/4)$, respectively. Let us now compute the corresponding homoclinic lengths, $\lambda(\tilde{\Omega})$ and $\lambda(\Omega)$. See section 7 for the definition of *homoclinic length*. It is straightforward checking that the function $\tilde{\gamma}:\mathbb{R}\to\mathbb{R}^2$, $\tilde{\gamma}(t) = (x(t), y(t))$, where

$$x(t) = \frac{3}{2}\left(1 - \frac{(1-t)^2}{(1+t)^2}\right) \quad \text{and} \quad y(t) = x(t)\,\frac{(1-t)^2}{(1+t)^2}, \tag{23}$$

linearises the unstable curve of the saddle at the origin of $\tilde{F}_1$. By symmetry, $I_0 \circ \tilde{\gamma}$ also linearises the stable curve. Thus, since $\tilde{\gamma}(1) = I_0(\tilde{\gamma}(1)) = (3/2, 0)$, the homoclinic length of $\tilde{F}_1$'s loop is $\lambda(\tilde{\Omega}) = 6\sqrt{2}$, which is the the square root of the area of the parallelogram spanned by the vectors $\tilde{\gamma}'(0)$ and $(I_0 \circ \tilde{\gamma})'(0)$. See definition 39 and the remarks that follow it. The loop of F_1 has exactly the same homoclinic length, $\lambda(\Omega) = 6\sqrt{2}$, since $\tilde{F}_1$ and F_1 are conjugated by the an area preserving linear map with determinant -1.

*

Finally we will analyse the rescaled Hénon family F_δ at the symmetric homoclinic point Ω_δ. Let $(x,y)=\zeta_\delta(\overline{x},\overline{y})$ be the Birkhoff co-ordinates, as in proposition 4 applied to the family F_δ. These co-ordinate transformations ζ_δ take each mapping F_δ to the Birkhoff normal form

$$L_\delta(\overline{x},\overline{y}) = \left(e^{\alpha_\delta(\overline{x}\,\overline{y})}\,\overline{x}\,,\ e^{-\alpha_\delta(\overline{x}\,\overline{y})}\,\overline{y}\right)\ . \tag{24}$$

Then by remark 7, there is some $r>0$ such that the co-ordinates $(x,y)=\zeta_\delta(\overline{x},\overline{y})$ extend analytically to the domain W_r where L_δ acts as a global diffeomorphism, see remark 5. The mappings $\zeta_\delta(\overline{x},\overline{y})$, and all their derivatives, are equicontinuous on the domain W_r, depending continuously in $\delta\in[-\delta_0,\delta_0]$ (analytically in $\delta\in[-\delta_0,\delta_0]-\{0\}$).

Since all maps F_δ are reversible w.r.t. the canonical involution I we have, by remark 4, that $\zeta_\delta\circ I = I\circ\zeta_\delta$ for all small δ. Thus, if $(w(\delta),0)\in W_r$ is such that $\zeta_\delta(w(\delta),0)=\Omega_\delta$ then

$$\zeta_\delta(0,w(\delta)) = \zeta_\delta\circ I(w(\delta),0) = I\circ\zeta_\delta(w(\delta),0) = I(\Omega_\delta) = \Omega_\delta\,.$$

Observing that $\gamma_u(\overline{x})=\zeta_\delta(\overline{x},0)$ and $\gamma_s(\overline{y})=\zeta_\delta(0,\overline{y})$ linearise the invariant manifolds of the saddle at the origin of F_δ we see that $w(\delta)$ is precisely the homoclinic length of Ω_δ. Thus $w(0)=\lambda(\Omega_0)=6\sqrt{2}$, and $w(\delta)$ converges to this number as $\delta\to 0$.

We are now going to define, using Birkhoff co-ordinates $(\overline{x},\overline{y})$, a new system of co-ordinates (t,E), in a neighbourhood of the homoclinic points Ω_δ, in which the mapping F_δ is described by the following shift translation

$$\sigma^\delta\ :\ \ (t,E)\mapsto(t+\delta,E)\ , \tag{25}$$

where the unstable curve is described by the axis $\{E=0\}$, and the stable one is the graph of a periodic function $E=g_\delta(t)$.

We begin by taking a small, but fixed size, neighbourhood U of the point Ω_0 which is covered by all images $\zeta_\delta(W_r)$ and where we have well defined inverse branches $(\overline{x},\overline{y})=(\zeta_\delta)^{-1}(x,y)$ depending continuously, with all their derivatives, on δ. Define then $(t,E)=\eta_\delta(x,y)$, in such a neighbourhood, by setting

$$\begin{aligned} E=E_\delta(x,y) &= \overline{x}\,\overline{y}\\ t=t_\delta(x,y) &= (\,\delta/\alpha_\delta(\overline{x}\,\overline{y})\,)\,\log\,(\,\overline{x}/w(\delta)\,)\ ,\end{aligned}$$

with $(\overline{x},\overline{y})=(\zeta_\delta)^{-1}(x,y)$. Clearly these new co-ordinates are defined for all small δ in a fixed neighbourhood U of Ω_0 and depend, with their derivatives, continuously on δ. One can trivially verify that these co-ordinates conjugate F_δ with the shift σ^δ in (25). In other words for all small δ, given $(x,y)\in U$ such that $F_\delta(x,y)\in U$,

$$\eta_\delta\circ F_\delta(x,y) = \eta_\delta(x,y)+(\delta,0)\,.$$

From the definition one has $\eta_\delta(\Omega_\delta)=(0,0)$. By remark 1 one computes easily

$$\det D\eta_\delta\,(\Omega_\delta)\ =\ 1\ . \tag{26}$$

It is also clear that the line $\{E=0\}$ corresponds to the $\overline{x}-$ axis, which in turn represents the the unstable manifold. Since the stable manifold must be invariant under the shift σ^δ, in co-ordinates (t,E), it has to be the graph of a periodic function $E=g_\delta(t)$ with period δ. Of course $g_\delta(0)=0$ because $(t,E)=(0,0)$ are the co-ordinates of the homoclinic point Ω_δ. At $\delta = 0$ both invariant manifolds merge in the homoclinic loop of F_1 and we have $g_0(t) \equiv 0$.

Consider the function $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\gamma(t) = M_0 \cdot \tilde{\gamma}(e^t) ,$$

where M_0 is the matrix (21) with $\epsilon = 0$ and $\tilde{\gamma}$ was defined in (23). Then $\gamma(t)$ is the homoclinic solution of the vector field F_1 which satisfies the initial condition $\gamma(0) = \Omega_0$. This homoclinic solution $\gamma(t)$ extends to complex time with poles at the numbers $t = i\,\pi + 2\,n\,\pi\,i$, $n \in \mathbb{Z}$. Therefore $\gamma(t)$ is holomorphic in the horizontal complex strip $|\mathrm{Im} t| \leq r = 3\,\pi/4 < \pi$. It then follows from [6] that the Fourier coefficients of $g_\delta(t)$,

$$c_n(\delta) = \frac{1}{\delta} \int_0^\delta g_\delta(t)\, e^{i\,2\,\pi\,n\,t/\delta}\, dt \quad (n \in \mathbb{Z}) ,$$

have the following uniform upper bounds

$$|c_n(\delta)| \leq C \exp\left\{ -\frac{2\,\pi\,r\ |n|}{\delta} \right\} = C \exp\left\{ -\frac{3\,\pi^2\ |n|}{2\,\delta} \right\} \tag{27}$$

where the constant $C > 0$ is independent of δ and $n \in \mathbb{Z}$.

In [7] the authors have proved that there is some constant $\theta_0 > 0$ (which numerically is known to be large, $\theta_0 \approx 2.474 \cdot 10^6$) such that the Lazutkin invariant of F_δ at Ω_δ has the following asymptotic behaviour

$$\theta(\Omega_\delta) = \frac{64\,\pi\, e^{-2\,\pi^2/\epsilon}}{9\,\epsilon^7} \left(\theta_0 + \mathcal{O}(\epsilon) \right) . \tag{28}$$

It can easily be checked, using (20), that

$$\frac{e^{2\,\pi^2/\epsilon}}{\epsilon^7} = \frac{e^{2\,\pi^2/\delta}}{\delta^7} \left(1 + \mathcal{O}(\delta) \right) ,$$

and therefore we can replace ϵ by δ in (28).

In the co-ordinates $(t, E) = \eta_\delta(x, y)$ the Lazutkin invariant is, up to some factor of order $1 + \mathcal{O}(\delta)$, minus the derivative $(g_\delta)'(0)$. By (26), the co-ordinate transformations $(t, E) = \eta_\delta(x, y)$ preserve the Lazutkin invariant at Ω_δ. Thus for all small $\delta > 0$,

$$(g_\delta)'(0) = \frac{64\,\pi\, e^{-2\,\pi^2/\delta}}{9\,\delta^7} \left(-1 + \mathcal{O}(\delta) \right) \tag{29}$$

But from the Fourier series of $g_\delta(t)$, and the upper bounds (27),

$$\begin{aligned} (g_\delta)'(0) &= \frac{2\,\pi\,i}{\delta} \left(c_1(\delta) - c_{-1}(\delta) \right) + \frac{2\,\pi\,i}{\delta} \sum_{|n| \geq 2} n\, c_n(\delta) \\ &= \frac{2\,\pi\,i}{\delta} \left(c_1(\delta) - c_{-1}(\delta) \right) + \mathcal{O}\left(e^{-3\,\pi^2/\delta} \right) \end{aligned}$$

and since the rest $\mathcal{O}\left(e^{-3\,\pi^2/\delta} \right)$ is exponentially small compared with the right hand side in (29) we obtain that $c_1(\delta) = \overline{c_{-1}(\delta)} = i\, b_1(\delta) \in i\,\mathbb{R}$ and for all small enough $\delta > 0$,

$$b_1(\delta) = \frac{16\, e^{-2\,\pi^2/\delta}}{9\,\delta^6}\, (-1 + \mathcal{O}(\delta))\,. \tag{30}$$

Thus

$$g_\delta(t) = b_1(\delta)\, \left(\sin\left(\frac{2\,\pi\, t}{\delta}\right) \,+\, \mathcal{O}\left(e^{-\pi^2/\delta}\right) \right)\,, \tag{31}$$

where the remainder is the sum of a Fourier series with all coefficients exponentially small.

Clearly, the family of periodic functions $g_\delta(t)$ has *bounded C^2- geometry* in sense of the following definition. This concept synthesises everything that will be used in the next section.

Given a family of periodic smooth functions $g_\delta(t)$ depending on a small parameter $\delta > 0$ and satisfying the period condition $g_\delta(t+\delta) = g_\delta(t)$, define for $i = 0, 1, 2$,

$$\begin{aligned} M_i(\delta) \;&=\; \max\left\{\, \delta^i \left|\frac{d^i g_\delta}{dt^i}(t)\right| \;:\; t \in \mathbb{R} \,\right\}\,, \\ m_1(\delta) \;&=\; \min\left\{\, \delta \left|\frac{d g_\delta}{dt}(t)\right| \;:\; g_\delta(t) = 0 \,\right\} \quad \text{and} \\ m_2(\delta) \;&=\; \min\left\{\, \delta^2 \left|\frac{d^2 g_\delta}{dt^2}(t)\right| \;:\; \frac{d g_\delta}{dt}(t) = 0 \,\right\}\,. \end{aligned} \tag{32}$$

Definition 1. *Let us say that $g_\delta(t)$ has bounded C^2- geometry if and only if there is some constant $C > 0$ such that for all small enough $\delta > 0$, $C\, m_1(\delta) > M_2(\delta)$ and $C\, m_2(\delta) > M_0(\delta)$.*

Remark 8. *It follows from definition 1 that all the functions of δ: M_0, M_1, M_2, m_1 and m_2 are asymptotically equivalent, in the sense that the quotient of any pair is bounded from 0 and from ∞. In particular we see that all them are of exponentially small order.*

The family of periodic functions $g_\delta(t)$ will play the role of the Melnikov function $M_\delta(t)$ in [3]. The quantities (32) here correspond exactly to the quantities defined in (1) there. Notice that, in the quoted work, the "flow time" t has been scaled so that all Melnikov functions $M_\delta(t)$ have period one. This accounts for the factors δ^i appearing in (32) but missing in (1) of [3].

5. Thick horse-shoes

The basic set construction is based on an abstract bounding distortion result in [3]. There we define $\mathcal{F}$ to be the class of all (positive binary horse-shoe) maps $f : S_0 \cup S_1 \to \mathbb{R}^2$ where: (1) S_0 and S_1 are compact subsets, diffeomorphic to rectangles, with nonempty interior; (2) f is a map of class C^2, in a neighborhood of $S_0 \cup S_1$, mapping this compact set diffeomorphically onto its image $f(S_0) \cup f(S_1)$; (3) the maximal invariant set $\Lambda(f) = \bigcap_{n\in\mathbb{Z}} f^{-n}(S_0 \cup S_1)$ is a hyperbolic basic set conjugated to the Bernoulli shift $\sigma : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$; (4) $\mathcal{P} = \{S_0, S_1\}$ is a Markov partition for $f : \Lambda(f) \to \Lambda(f)$, in particular f has two fixed points, $P_0 \in S_0$ and $P_1 \in S_1$, whose stable and unstable manifolds contain the boundaries of S_0 and S_1; and, finally, (5) both fixed points P_0 and P_1 have positive eigenvalues.

Given two positive small numbers $\epsilon > 0$ and $\gamma > 0$, we define there $\mathcal{F}(\epsilon,\gamma)$ to be the class of all maps $f : S_0 \cup S_1 \to \mathbb{R}^2$, $f \in \mathcal{F}$, such that:

(1) f preserves area;
(2) $\mathrm{diam}\,(S_0 \cup S_1) = \mathrm{diam}\,(f(S_0) \cup f(S_1)) = 1\,$;
(3) writing $f(x,y) = (f_1(x,y), f_2(x,y))$ and $f^{-1}(x,y) = (\tilde{f}_1(x,y), \tilde{f}_2(x,y))$ we have
   (a) $\left|\frac{\partial f_2}{\partial y}\right| < 1 < \left|\frac{\partial f_1}{\partial x}\right| \leq 2/\epsilon$
   (b) $\displaystyle \left|\frac{\partial f_1}{\partial y}\right| \,,\; \left|\frac{\partial f_2}{\partial x}\right| \leq \epsilon\,(\left|\frac{\partial f_1}{\partial x}\right| - 1)$
(4) the second derivatives satisfy
   (a) $\displaystyle \left|\frac{\partial^2 \tilde{f}_1}{\partial x \partial y}\right| \,,\; \left|\frac{\partial^2 \tilde{f}_1}{\partial y^2}\right| \,,\; \left|\frac{\partial^2 \tilde{f}_2}{\partial x^2}\right| \,,\; \left|\frac{\partial^2 \tilde{f}_2}{\partial x \partial y}\right| \leq \gamma \left(\left|\frac{\partial \tilde{f}_2}{\partial y}\right| - 1\right)$
   (b) $\displaystyle \left|\frac{\partial^2 f_1}{\partial x \partial y}\right| \,,\; \left|\frac{\partial^2 f_1}{\partial y^2}\right| \,,\; \left|\frac{\partial^2 f_2}{\partial x^2}\right| \,,\; \left|\frac{\partial^2 f_2}{\partial x \partial y}\right| \leq \gamma \left(\left|\frac{\partial f_1}{\partial x}\right| - 1\right)$
   (c) $\displaystyle \left|\frac{\partial^2 \tilde{f}_2}{\partial y^2}\right| \,,\; \left|\frac{\partial^2 \tilde{f}_1}{\partial x^2}\right| \leq \gamma \left|\frac{\partial \tilde{f}_2}{\partial y}\right| \left(\left|\frac{\partial \tilde{f}_2}{\partial y}\right| - 1\right)$
   (d) $\displaystyle \left|\frac{\partial^2 f_1}{\partial x^2}\right| \,,\; \left|\frac{\partial^2 f_2}{\partial y^2}\right| \leq \gamma \left|\frac{\partial f_1}{\partial x}\right| \left(\left|\frac{\partial f_1}{\partial x}\right| - 1\right)\,.$
(5) the variation of $\log\left|\frac{\partial f_1}{\partial x}(x,y)\right|$ in each rectangle S_i is less or equal than $\gamma\,(1 - \alpha_i^{-1})$, where $\alpha_i = \max_{(x,y)\in S_i} \left|\frac{\partial f_1}{\partial x}(x,y)\right|$.
(6) the gap sizes satisfy:

$$\mathrm{dist}\,(S_0, S_1) \geq \frac{\epsilon}{\gamma} \quad \text{and} \quad \mathrm{dist}\,(f(S_0), f(S_1)) \geq \frac{\epsilon}{\gamma}\,.$$

The normalising condition (2) avoids having all subsequent items referring to the scale of the basic set. Then, with this notation, we prove in [3], c.f. *theorem 2* there, that:

Theorem 2. *For all small enough $\epsilon > 0$ and $\gamma > 0$, given $f \in \mathcal{F}(\epsilon,\gamma)$, the basic set $\Lambda(f)$ has dynamically defined Cantor sets (Λ^u, ψ^u) and (Λ^s, ψ^s) with small distortion, bounded by $D(\epsilon,\gamma) = 20\,\gamma + 2\,\epsilon$. In particular*

$$e^{-2\,D(\epsilon,\gamma)}\,\tilde{\tau}_{LR}\,(\Lambda(f)) \leq \tau_{LR}\,(\Lambda(f)) \leq e^{2\,D(\epsilon,\gamma)}\,\tilde{\tau}_{LR}\,(\Lambda(f))\;.$$

In the rest of this section we outline the basic set construction, in the class $\mathcal{F}(\epsilon,\gamma)$, and shall estimate the corresponding top scale $\tilde{\tau}_{LR}$-thickness, in order to apply theorem above. All proofs will refer to [3] with the following notation adaptations. References to propositions, lemmas and formulas there will be written in *italic text mode*. In all formulas of [3] one should either drop the variable μ, when it appears as an argument, or take it to be $\mu = 1$, when it is a factor in some expression. Here the eigenvalue is $\lambda = e^{\delta}$ so that $\log\lambda$ should be replaced by δ

in all expressions there. Of course $\lambda_{\delta,\mu}(t)$ is to be understood as $\lambda_\delta(t) = e^{\alpha_\delta(t)}$. The filter $\mathcal{N}$ in our setting will be just the filter of all neighbourhoods of $\delta = 0$ in the parameter half line $[0, +\infty)$. Many computations here will be reduced to half due to the reversible character of the Henn map, which was not assumed of maps $f_{\delta,\mu}$ there. A crucial quantity in [3] is $\theta = \theta_\delta$ defined in *formula 4*, and which appears in "almost all" derivative bounds given thereafter. To play this role we define here

$$\theta = \theta_\delta = -(g_\delta)'(0) > 0 \;, \tag{33}$$

for which one has the upper and lower bounds given by (27) and (29) .

The half return time. We define here the *half return time* as

$$n(\delta) = \text{ the integer part of } \frac{-\log\left(\theta_\delta\, \delta^{3/2}\right)}{2\,\delta} \;.$$

Clearly $\lim_{\delta\to 0} n(\delta) = +\infty$. From item 4. in proposition 4 we see that

Lemma 2. *There are constants $C > 0$ and $\delta_0 > 0$ such that the following inequalities hold for all $0 < \delta < \delta_0$ and all $|t| < 2$,*

(1) $C^{-1}\delta \le \alpha_\delta(t) \le C\,\delta$,
(2) $|(\alpha_\delta)'(t)| \le C\,\delta$,
(3) $|(\alpha_\delta)''(t)| \le C\,\delta$.

Using these facts we prove as in *lemma 6.4* , that

Lemma 3. *Writing $n = n(\delta)$ for all small enough $\delta > 0$,*

(1) $e^{-2\,n\,\delta} = \delta^{3/2}\,\theta_\delta\,(1 + \mathcal{O}(\delta))$
(2) $n\,\theta_\delta = o(\sqrt{\theta_\delta})$,
(3) $e^{2\,n\,(\alpha_\delta(t)-\delta)} = 1 + \mathcal{O}(\sqrt{\theta_\delta})$, *for* $0 \le t \le e^{-2\,n\,\alpha_\delta(t)}$.

The transition map. Consider the normal form L_δ, with the notation in (24), associated with the map F_δ. Conformally rescaling F_δ, we can make $w(\delta) = 1$, or, in other words, we can make the homoclinic length of Ω_δ to be constant and equal to 1. Therefore $\zeta_\delta(1,0) = \Omega_\delta = \zeta_\delta(0,1)$, and for some small $r > 0$ both restriction maps $\zeta_\delta^- = \zeta_\delta|_{B_r(1,0)}$ and $\zeta_\delta^+ = \zeta_\delta|_{B_r(0,1)}$ are one to one onto a neighbourhood of Ω_0. For each $\delta > 0$ we define the *transition map* $G_\delta = (\zeta_\delta^+)^{-1} \circ \zeta_\delta^-$ in a small but fixed neighbourhood of $(1,0)$. Clearly these maps satisfy $G_\delta(1,0) = (0,1)$, the compatibility relation, $L_\delta \circ G_\delta = G_\delta \circ L_\delta$, and also the reversibility equation

$$G_\delta \circ I = I \circ (G_\delta)^{-1} \;. \tag{34}$$

Denote the components of $G_\delta(x,y)$ by,

$$G_\delta(x,y) = (\; g_1(\delta,x,y)\;,\; g_2(\delta,x,y)\;) \;\;.$$

Then by (34) one has

$$G_\delta^{-1}(x,y) = (\; g_2(\delta,y,x)\;,\; g_1(\delta,y,x)\;) \;\;,$$

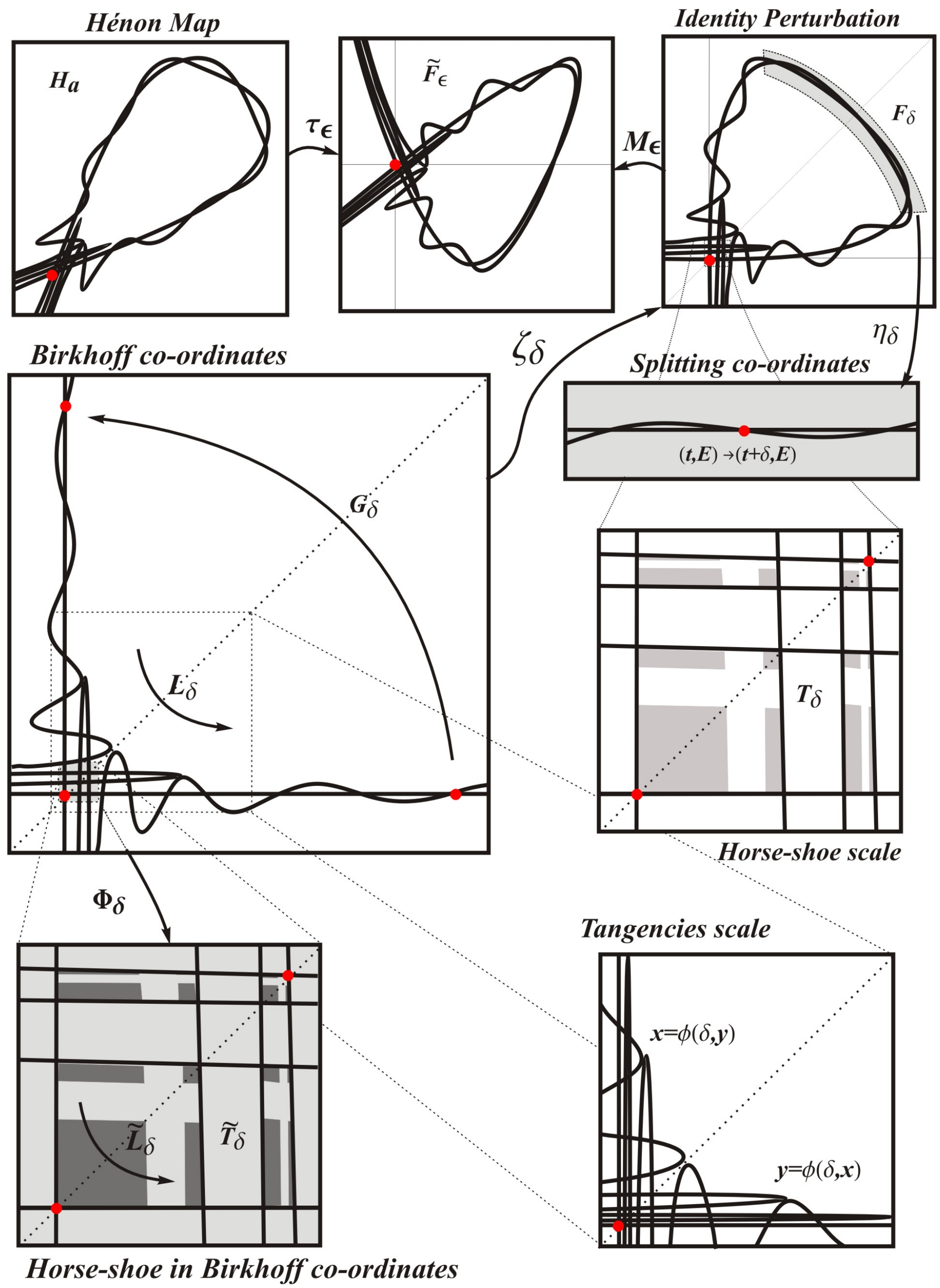

FIGURE 2. Construction co-ordinates

and so all bounds on G_δ's derivatives give, automatically, bounds on G_δ^{-1}'s derivatives. One can easily prove that, at $\delta = 0$, $g_1(0, x, 0) = 0$ and $g_2(0, x, 0) = x^{-1}$. Thus using the symplectic character of G_0 along the homoclinic loop, see remark 1, and also *lemma 6.2*,

Lemma 4. *for all* $x \in [1/2, 3/2]$, $(g_2)_x(0, x, 0) = -x^{-2} = -1 + O(x - 1)$ *and* $(g_1)_y(0, x, 0) = x^{-2} = 1 + O(x - 1)$.

Lemma 5. *for some constant $C > 0$ and all small enough $\delta > 0$ the function $g_1(\delta, x, 0)$ its first and second derivative w.r.t. x are respectively bounded by $C\,\delta\,\theta_\delta$, $C\,\theta_\delta$ and $C\,\theta_\delta/\delta$.*

Moreover, relating the transition map G_δ with the periodic function $g_\delta(t)$, through the co-ordinate transformation η_δ, one proves easily that

Lemma 6. $(g_1)_x(\delta, 1, 0) = -\theta_\delta$.

Rescaling the basic set. The map T_δ will be defined by two different branches where it coincides with the maps L_δ and $(L_\delta)^n \circ G_\delta \circ (L_\delta)^n$. These two branches are respectively defined on the very small rectangles

$$S_0 = \left\{ (x,y) \,:\, |x| < e^{-(n+1/2)\,\delta} \quad \text{and} \quad |y| < e^{-(n-1/2)\,\delta} \right\}, \quad \text{and}$$

$$S_1 = \left\{ (x,y) \,:\, \left|x - e^{-n\,\delta}\right| < 2\,\delta^{3/2}\,e^{-n\,\delta} \quad \text{and} \quad |y| < e^{-(n-1/2)\,\delta} \right\}.$$

The domain $S_0 \cup S_1$ has diameter of order $e^{-n\,\delta} \sim \sqrt{\delta^{3/2}\,\theta_\delta}$. In order to scale this domain up to the unit square we introduce the scaling maps, $\Phi_\delta : \mathbb{R}^2 \to \mathbb{R}^2$

$$\Phi_\delta(x,y) = \left(e^{n\,\alpha_\delta(xy)}\,x\,,\, e^{n\,\alpha_\delta(xy)}\,y\right),$$

where $n = n(\delta)$ is the half return time. The product of Φ_δ's components, $e^{2\,n\,\alpha_\delta(xy)}x\,y$, is a function of the product $x\,y$. Therefore the inverse map $\Phi_\delta^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$, is given by

$$\Phi_\delta^{-1}(x,y) = \left(e^{-n\,\alpha_\delta(t_\delta(xy))}x\,,\, e^{-n\,\alpha_\delta(t_\delta(xy))}y\right)$$

where $t_\delta(s)$ is defined implicitly by $t_\delta(0) = 0$ and

$$e^{2\,n\,\alpha_\delta(t_\delta(s))}\,t_\delta(s) = s \qquad \text{for all} \qquad |s| < 2\,.$$

We scale T_δ setting $\tilde{T}_\delta = \Phi_\delta \circ T_\delta \circ \Phi_\delta^{-1}$. This map has the following two branches $\tilde{L}_\delta = \Phi_\delta \circ L_\delta \circ \Phi_\delta^{-1}$ and $\tilde{G}_\delta = \Phi_\delta \circ L_\delta^n \circ G_\delta \circ L_\delta^n \circ \Phi_\delta^{-1}$. To compute the first define

$$\tilde{\alpha}(s) = \tilde{\alpha}_\delta(s) := \alpha_\delta \circ t_\delta\,(s)\,.$$

Then it is easily checked that

$$\tilde{L}_\delta(x,y) = \left(e^{\tilde{\alpha}_\delta(x\,y)}\,x\,,\, e^{-\tilde{\alpha}_\delta(x\,y)}\,y \right).$$

A simple computation gives for all δ, $\tilde{G}_\delta(1,0) = (0,1)$. It is obvious that the scaling map Φ_δ commutes with the involution I. Therefore, using (34), we obtain the reversibility of $\tilde{G}_\delta$: $\tilde{G}_\delta \circ I = I \circ \tilde{G}_\delta^{-1}$. To explicit $\tilde{G}_\delta$ we introduce an auxiliary function

$$p(x,y) = p_\delta(x,y) \;:=\; g_1\left(x,\, e^{-2\,n\,\tilde{\alpha}(xy)}y\right) \cdot g_2\left(x,\, e^{-2\,n\,\tilde{\alpha}(xy)}y\right)$$

where $\tilde{\alpha} = \tilde{\alpha}_\delta$, $g_1(\cdot,\cdot) = g_1(\delta,\cdot,\cdot)$, $g_2(\cdot,\cdot) = g_2(\delta,\cdot,\cdot)$, and $n = n(\delta)$. Then, see *formula 11*, we get

$$\tilde{G}_\delta(x,y) = \left(e^{2\,n\,\alpha\circ p\,(x,y)}\,g_1\left(x, e^{-2\,n\,\tilde{\alpha}(xy)}y\right),\, g_2\left(x, e^{-2\,n\,\tilde{\alpha}(xy)}y\right) \right).$$

We will consider the domains of the rescaled maps G_δ and G_δ^{-1} to be, respectively, the following rectangles

$$\tilde{S}_1(\delta) = \left\{(x,y) \,:\, |x-1| \le 2\,\delta^{3/2} \quad \text{and} \quad 0 \le y \le 1 + \delta/2 \right\},$$

$$\tilde{S}_1'(\delta) = \left\{(x,y)\,:\, 0 \le x \le 1+\delta/2 \quad \text{and} \quad |y-1| \le 2\,\delta^{3/2}\right\} \,.$$

To estimate $\tilde{G}_\delta$'s derivatives notice first that

Lemma 7. *There are some constants $C>0$ and $\delta_0>0$ such that for all $0<\delta<\delta_0$ and $s\in[0,2]$, $|t_\delta(s)|$, $|t_\delta'(s)|$ and $|t_\delta''(s)| \le C\,\delta^{3/2}\,\theta_\delta$.*

Proof. This proof depends on lemma 3 here. See *lemma 7.1.* □

Then, from these bounds and lemmas 2, 4 and 5, we can prove that

Lemma 8. *There are constants $C>0$ and $\delta_0>0$ such that for all $(x,y)\in\tilde{S}_1(\delta)$*

1 $|p(x,y)| \le C\,\delta^{3/2}\,\theta_\delta$,

2 $\left|\dfrac{\partial p}{\partial x}(x,y)\right| \le C\,\theta_\delta$,

3 $\left|\dfrac{\partial p}{\partial y}(x,y)\right| \le C\,\delta^{3/2}\,\theta_\delta$,

4 $\left|\dfrac{\partial^2 p}{\partial x^2}(x,y)\right| \le C\,\delta^{-1}\,\theta_\delta$.

Proof. See *lemma 7.2.* □

Bounds on the derivatives of $\tilde{T}_\delta$. Consider the first branch $\tilde{L}_\delta$ of $\tilde{T}_\delta$ to be defined on the rectangle $\tilde{S}_0(\delta)=[0,1-\delta/2]\times[1+\delta/2]$, while $\tilde{L}_\delta^{-1}$ is defined on $\tilde{S}_0'(\delta)=[0,1+\delta/2]\times[1-\delta/2]$. Then

Lemma 9. *There is $\delta_0>0$ such that for $0<\delta<\delta_0$ the Jacobian matrix of $\tilde{L}_\delta$ satisfies*

$$D\tilde{L}_\delta = \begin{pmatrix} e^{\delta} & 0 \\ 0 & e^{-\delta} \end{pmatrix} + \mathcal{O}(\delta^{3/2}\,\theta_\delta)$$

on $\tilde{S}_0(\delta)$ and all the second derivatives of $\tilde{L}_\delta$ are of exponentially small order $\mathcal{O}(\delta^{3/2}\,\theta_\delta)$. Similar bounds hold for $\tilde{L}_\delta^{-1}$ on $\tilde{S}_0'$.

Proof. See *lemmas 8.1 and 8.2.* □

On the other branch we have

Lemma 10. *There is $\delta_0>0$ such that for $0<\delta<\delta_0$ the Jacobian matrix of $\tilde{G}_\delta$ satisfies*

$$D\tilde{G}_\delta = \begin{pmatrix} -\delta^{-3/2} & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\delta^{-1}) & o(1) \\ o(1) & o(1) \end{pmatrix}$$

on the domain $\tilde{S}_1$, and the second derivatives of $\tilde{G}_\delta$ are all uniformly bounded, except for $\frac{\partial^2 g_1}{\partial x^2} = \mathcal{O}(\delta^{-5/2})$. Again, similar bounds hold for $\tilde{G}_\delta^{-1}$ on S_1'.

Proof. This result follows from lemma 8. See *lemmas 8.3 and 8.4* . In the proof of *lemma 8.3* the following estimate

$$\frac{\partial g_1}{\partial x}(\delta,\mu,1,0) = -\mu\,\theta_\delta + \mathcal{O}(\mu^2)\ ,$$

should be replaced by the explicit value given in lemma 6 here. □

Lemma 11. *There are constants $C > 2$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the maps $\tilde{L}_\delta$ and $\tilde{L}_\delta^{-1}$ over $\tilde{S}_0$, the map $\tilde{G}_\delta$ over $\tilde{S}_1$, and the map $\tilde{G}_\delta^{-1}$ over $\tilde{S}_1'$, fulfil conditions (3), (4) and (5) of class $\mathcal{F}\left(3\,\delta^{3/2}/2,\, 3\,C\,\delta^{1/2}\right)$ definition.*

Proof. This lemma follows from the previous ones 9 and 10. See the proof of *Lemma 8.5.* □

Thick horse shoes.

Lemma 12. *Besides $(0,0) \in \tilde{S}_0$, the map $\tilde{T}_\delta$ has a second symmetric fixed point with co-ordinates $(x_1, x_1) \in \tilde{S}_1 \cap \tilde{S}_1'$, where $x_1 = 1 + \mathcal{O}(\delta^{5/2})$. Moreover, there is a family of smooth functions $\gamma_\delta : [0, 1+\delta/2] \to \mathbb{R}$ such that $\gamma_\delta(\,x_1\,) = x_1$ and for all $t \in [0, 1+\delta/2]$,*

(1) $-\dfrac{3}{2}\,\delta^{3/2} \le (\gamma_\delta)'(t) \le -\dfrac{2}{3}\,\delta^{3/2}$,

(2) $|\gamma_\delta(t) - 1| \le \dfrac{7}{4}\,\delta^{3/2}$.

The graphs $\{\,(t,\gamma_\delta(t))\,:\, t\in[0,1+\delta/2]\,\}$ and $\{\,(\gamma_\delta(t),t)\,:\, t\in[0,1+\delta/2]\,\}$ are the local invariant manifolds of the fixed point (x_1,x_1), respectively the unstable and the stable one. In particular, these pieces of invariant manifolds are contained in $\tilde{S}_1'$ and $\tilde{S}_1$, respectively.

Proof. See *lemmas 8.6* and *8.7.* Notice that the remainder $\mathcal{O}(\delta^{5/2})$ of the expression for x_1 can be obtained here, instead of $\mathcal{O}(\mu\,\log^{3/2}\lambda)$ there, replacing $\frac{\partial g_1}{\partial y}(\delta,\mu,1,0) = 1 + \mathcal{O}(\mu)$ by $\frac{\partial g_1}{\partial y}(\delta,1,0) = 1 + \mathcal{O}(\delta)$, which in turn follows from $\frac{\partial g_1}{\partial y}(0,1,0) = 1$. See lemma 4 here. □

Lemma 13. *For all small enough $\delta > 0$ the local invariant manifolds of the fixed points $(0,0)$ and (x_1,x_1) are the boundary curves of a Markov Partition $\mathcal{P}_\delta = \{\,S_0(\delta),\, S_1(\delta)\,\}$ such that $S_0 \subseteq \tilde{S}_0$, $\tilde{T}_\delta(S_0) \subseteq \tilde{S}_0'$, $S_1 \subseteq \tilde{S}_1$ and $\tilde{T}_\delta(S_1) \subseteq \tilde{S}_1'$.*

Proof. See the proof of *lemma 8.8* . □

Since both fixed points have positive eigenvalues we get that

Lemma 14. *For all small enough $\delta > 0$, the restriction $\tilde{T}_\delta|_{S_0\cup S_1} : S_0 \cup S_1 \to \mathbb{R}^2$ is a positive binary horse-shoe map, i.e., $\tilde{T}_\delta \in \mathcal{F}$.*

Lemma 15. *For all small enough $\delta > 0$, $dist(S_0, S_1) = dist(\tilde{T}(S_0), \tilde{T}(S_1)) = \mathcal{O}(\delta)$.*

Proof. This follows essentially from lemma 12. See the proof of *lemma 8.9* . □

Thus, from lemmas 14 and 15, it follows that

Corollary 5. *There are constants $C > 0$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, $\tilde{T}_\delta \in \mathcal{F}\left(3\,\delta^{3/2}/2,\ 3\,C\,\delta^{1/2} \right)$.*

In particular, by theorem 2, the $\tilde{T}_\delta$-invariant horse-shoe $\Lambda_\delta = \bigcap_{j=-\infty}^{\infty} (\tilde{T}_\delta)^{-j} \left(\tilde{S}_0 \cup \tilde{S}_1 \right)$ has stable and unstable distortion of order $\mathcal{O}(\delta)$. It is also easy to compute that the width of the rectangles $S_0(\delta)$, $S_1(\delta)$, and the gap between them are of orders 1, $\delta^{3/2}$ and δ, respectively. Therefore, the top scale left-right thickness of $\left(\Lambda_\delta, \tilde{T}_\delta\right)$ is of order $\mathcal{O}(\delta^{-1/2})$.

Lemma 16. *There is some $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$,*

(1) $\tilde{\tau}_L\left(\Lambda_\delta^s\right)\ ,\quad \tilde{\tau}_L\left(\Lambda_\delta^u\right)\ \geq \dfrac{1}{\delta}$

(2) $\tilde{\tau}_R\left(\Lambda_\delta^s\right)\ ,\quad \tilde{\tau}_R\left(\Lambda_\delta^u\right)\ \geq \dfrac{1}{4}\sqrt{\delta}$

In particular, $\tilde{\tau}_{LR}\left(\Lambda_\delta\right) \geq \dfrac{1}{4\sqrt{\delta}}$.

Proof. See the proof of *lemma 8.10.* □

Finally, combining theorem 2 with the lemma above we conclude that

$$\lim_{\delta\to 0} \tau_{LR}\left(\Lambda_\delta, \tilde{T}_\delta\right) = +\infty\ .$$

6. Tangencies in the Hénon Map

In this section we will prove Lemma B, stated in section 2. The graph $E = g_\delta(t)$ describes a piece of the stable manifold in the co-ordinates $(t, E) = \eta_\delta(x, y)$. Translating this into Birkhoff co-ordinates $(\overline{x}, \overline{y}) = \zeta_\delta^{-1}(x, y)$, the same arc of stable manifold is characterised as the graph $\overline{y} = \phi(\delta, \overline{x})$ of a function $\phi(\delta, x)$ implicitly defined by

$$\begin{aligned} \phi(\delta, x) &= g_\delta\left(\frac{\delta}{\alpha(\delta,\ x\,\phi)}\ \log x \right)/x \\ &= g_\delta\left(\,\log x - \tau\,\right)/x \end{aligned} \tag{35}$$

where ϕ stands for $\phi(\delta, x)$, and from $\alpha(\delta, 0) = \log\lambda_\delta = \delta$ we can derive the following expression for $\tau = \tau(\delta, x)$,

$$\tau(\delta, x) = x\ \log x\,\phi(\delta, x)\,\alpha(\delta,\ x\,\phi)^{-1} \int_0^1 \frac{d\,\alpha}{d\,\omega}(\delta,\ s\,x\,\phi)\,ds \tag{36}$$

We are going to consider the following domain for $\phi(\delta, x)$,

$$\Xi = \left\{ (\delta, x)\ :\ |\delta| < \delta_0 \quad \text{and} \quad \sqrt{|b_1(\delta)|}/3 < x < 3/\sqrt{|b_1(\delta)|}\ \right\}\ .$$

A simple application of implicit function theorem, computing derivatives of the implicitly defined functions $\phi(\delta, x)$, and $\tau(\delta, x)$ shows that

Lemma 17. *The unique functions $\phi(\delta,x)$ and $\tau(\delta,x)$ defined implicitly on Ξ by the equations (35) and (36) have the following asymptotics over Ξ :*

$\phi(\delta,x)=\mathcal{O}(\sqrt{|b_1(\delta)|})$ $\qquad \tau(\delta,x)=\mathcal{O}(|b_1(\delta)|)$
$\phi_x(\delta,x)=\mathcal{O}(\delta^{-1})$ $\qquad \tau_x(\delta,x)=\mathcal{O}(\delta^{-1}\sqrt{|b_1(\delta)|})$
$\phi_{xx}(\delta,x)=\mathcal{O}(\delta^{-2}\sqrt{|b_1(\delta)|}^{-1})$ $\qquad \tau_{xx}(\delta,x)=\mathcal{O}(\delta^{-2})$

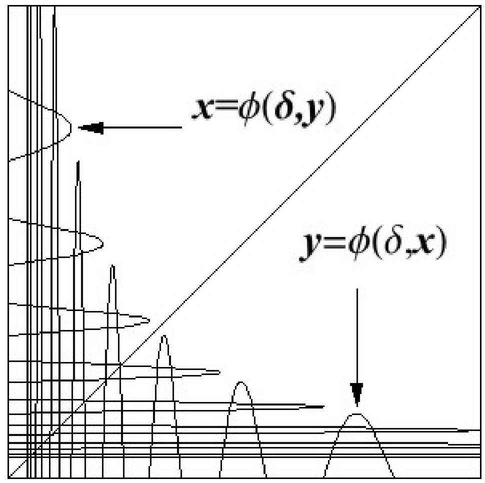

FIGURE 3. Symmetric homoclinic tangencies

Remark that for $(\delta,x)\in\Xi$ both points $(x,\phi(\delta,x))$ and $(\phi(\delta,x),x)$ belong to the open set W_r where Birkhoff co-ordinates can be extended. The graphs $\{\,(x,\phi(\delta,x))\,:\,(\delta,x)\in\Xi\,\}$, and $\{\,(\phi(\delta,x),x)\,:\,(\delta,x)\in\Xi\,\}$, respectively represent, in these co-ordinates, arcs of stable and unstable manifolds of the origin for the map F_ϵ $(\,\epsilon=\epsilon(\delta)\,)$. The following proposition is a statement about existence of quadratic homoclinic tangencies between these symmetric invariant manifolds.

Proposition 6. *There are sequences δ_n and x_n converging to zero and such that for all $n\in\mathbb{N}$, $(\delta_n,x_n)\in\Xi$ and:*

(1) $\lim_{n\to\infty} e^{2\,n\,\delta}\,|b_1(\delta)|=1\,,\quad \lim_{n\to\infty} e^{n\,\delta}\,x_n=1\,,$
(2) $\phi(\delta_n,x_n)=x_n\,,$
(3) $\phi_x(\delta_n,x_n)=1\,,$
(4) $\phi_{xx}(\delta_n,x_n)<0\,,$
(5) *for some* $\delta_n'>\delta_n$ *and all* $\delta_n<\delta<\delta_n'\,,$ $\phi_\delta(\delta,x_n)>0\,.$

Proof of Lemma B. Take now δ_n and x_n as in proposition 6. By item 2. of this proposition, $\zeta_{\delta_n}(x_n,x_n)$ are homoclinic points, which are non transversal by item 3. By item 4. these homoclinic tangencies are quadratic. If we would know that $\phi_\delta(\delta_n,x_n)>0$ then these tangencies would unfold generically. But since there are transversal homoclinic orbits, the stable and unstable local manifolds of the origin are accumulated on the right side by other arcs of the same manifolds. Then, because the tangency at (x_n,x_n) is negative, δ_n is the limit of a decreasing sequence of parameters where other homoclinic tangencies unfold near (x_n,x_n). Statement in item 5. implies these new tangencies unfold generically. $\square$

Proof of proposition 6. The graph $\overline{y} = \phi(\delta, \overline{x})$ oscillates from exponentially small amplitudes at $x = 1$ to very large oscillations near $x = 0$. We re-scale each concave part in this graph, scaling its width, measured in the $\overline{x}-$ axis, to an interval of size δ and scaling its height, measured in the $\overline{y}-$ axis, to an interval of size one. Define

$$\Delta_n = \left\{ \delta \,:\, 1/2 < e^{2\,n\,\delta}\, |b_1(\delta)| < 2 \right\} \,,$$

and

$$\Xi_n = \{ (\delta, x) \,:\, \delta_n \in \Delta_n \quad \text{and } x \in [1 - 3\,\delta/8\,,\, 1 - \delta/8] \} \,.$$

It is straightforward to check that for each $n \geq 1$,

$$(\delta, x) \in \Xi_n \quad \Rightarrow \quad (\delta, e^{-n\,\delta}\, x) \in \Xi \,.$$

Therefore we may define for $(\delta, x) \in \Xi_n$,

$$\begin{aligned} \phi_n(\delta, x) &= e^{n\,\delta}\, \phi(\delta, e^{-n\,\delta}\, x) \\ &= e^{2\,n\,\delta}\, g_\delta\,(\log\, x - \tau_n(\delta, x)) \,. \end{aligned} \tag{37}$$

where $\tau_n(\delta, x) = \tau(\delta, e^{-n\,\delta}\, x)$. From lemma 17 we get that

(1) $\tau_n(\delta, x) = \mathcal{O}(|b_1(\delta)|)$,
(2) $(\tau_n)_x(\delta, x) = \mathcal{O}(\delta^{-1}\, |b_1(\delta)|)$ and
(3) $(\tau_n)_{xx}(\delta, x) = \mathcal{O}(\delta^{-2}\, |b_1(\delta)|)$.

Combining (31) with (37), and noticing that (37) is strictly positive and bounded away from zero in its domain Ξ_n, one obtains easily for $(\delta, x) \in \Xi_n$,

$$\phi_n(\delta, x) = \gamma_n\,(1 + \sigma_n)\, \sin\left(\frac{2\,\pi\,(1 - x)}{\delta}\right) \,,$$

where $\gamma_n = -e^{2\,n\,\delta}\, b_1(\delta)$ and where $\sigma_n = \sigma_n(\delta, x)$, together with its derivatives $(\sigma_n)_x$ and $(\sigma_n)_{xx}$, are small functions of order $\mathcal{O}(\delta)$. By (30) the factor γ_n is positive. Notice also that

$$(\delta, x) \in \Xi_n \quad \Leftrightarrow \quad \frac{\pi}{4} < \frac{2\,\pi\,(1 - x)}{\delta} < \frac{3\,\pi}{4} \,,$$

and the *sin* function is strictly positive and concave in the interval $[\pi/4\,,\, 3\,\pi/4]$. By definition of Δ_n, as δ goes through Δ_n the factor γ_n runs across the interval $[1/2, 2]$. Therefore there is some parameter $\delta = \delta_n$ for which γ_n is close to 1 and the graph $y = \phi_n(\delta_n, x)$ is tangent to the diagonal $y = x$ at some point (x_n^*, x_n^*) near $(1, 1)$. Then, since ϕ_n is just a rescaling of ϕ, the sequences δ_n and $x_n = e^{-n\,\delta}\, x_n^*$ satisfy items 1., 2., 3. and 4. of proposition 6. But if we choose δ_n to be the last parameter $\delta \in \Delta_n$ such that $y = \phi_n(\delta_n, x)$ has some tangency with $y = x$, then clearly $(\phi_n)_\delta(\delta, x_n) \geq 0$ for all $\delta \geq \delta_n$ which are sufficiently close to δ_n. If $(\phi_n)_\delta(\delta_n, x_n) > 0$ item 5. is obvious. If not, by analyticity of $(\phi_n)_\delta$ its zeros are isolated and one must have $(\phi_n)_\delta(\delta, x_n) > 0$ for all $\delta > \delta_n$ which are sufficiently close to δ_n. Thus item 5. follows anyway.

□

7. Symplectic invariants

In this section we recall two symplectic invariants associated to homoclinic orbits. Given a fixed point P of a symplectic map $\varphi : M^2 \to M^2$, with multipliers $0 < \lambda^{-1} < 1 < \lambda$, assume two separatrices $\gamma^s(P)$ of $W^s(P)$, and $\gamma^u(P)$ of $W^u(P)$ intersect at some homoclinic point Q. Take vectors v_s and v_u, respectively tangent to $\gamma^s(P)$ and $\gamma^u(P)$ at point P. Consider the (unique) "linearising" maps $\gamma_s : \mathbb{R} \to W^s(P) \subseteq M^2$ and $\gamma_u : \mathbb{R} \to W^u(P) \subseteq M^2$ such that:

(1) $\gamma_s(0) = P$, and $\gamma_u(0) = P$,
(2) $(\gamma_s)'(0) = v_s$, and $(\gamma_u)'(0) = v_u$,
(3) $\varphi(\gamma_s(t)) = \gamma_s(\lambda^{-1} t)$ and $\varphi(\gamma_u(t)) = \gamma_u(\lambda t)$

There are positive real numbers $T_s, T_u \in \mathbb{R}^+$ such that $Q = \gamma_s(T_s) = \gamma_u(T_u)$.

The first invariant, *Lazutkin invariant*, is the ratio,

$$\theta(Q) = \frac{\omega_p\left((\gamma_s)'(T_s), (\gamma_u)'(T_u) \right)}{\omega_p\left(v_s, v_u\right)} . \tag{38}$$

The second one might be known in the literature, but we call it the *homoclinic length* of Q

$$\lambda(Q) = \sqrt{T_s \cdot T_u \cdot |\omega_P(v_u, I(v_u))|} . \tag{39}$$

Both these numbers do not depend on the choice of the vectors v_s and v_u, they both remain constant along the orbits, and they are easily seen to be invariant under area preserving changes of co-ordinates.

Lazutkin invariant is used as a symplectic invariant measure of the splitting angle at some transversal homoclinic point. $\theta(Q) = 0$ means that the two separatrices are tangent or coincide.

The homoclinic length is always positive. Suppose we take (area preserving) Birkhoff co-ordinates taking the map φ to its normal form around P, so that the co-ordinates of Q are $(c, 0)$ and $(0, c)$. Then c is precisely the homoclinic length of Q.

In the case of Hamiltonian vector fields with a homoclinic loop the length is defined w.r.t. the induced flow maps ϕ^t, $t \neq 0$. Is not difficult to see that all points in the loop have the same length which is also the same for all flow maps ϕ^t. Assume now that X is a Hamiltonian vector field in surface M^2 which is reversible w.r.t. some involution I, meaning that $DI_x \cdot X(x) = -X(I(x))$. Suppose we are given a symmetric fixed saddle $P = I(P)$, $X(P) = 0$, with multipliers $-\lambda < 0 < \lambda$, together with a homoclinic loop cutting the fixed point set of I in some symmetric homoclinic point Q. In order to compute the length of the homoclinic connection we take a nonzero vector $v_u \in T_P(M^2)$ tangent to the unstable manifold at P and then find a solution $\gamma_u : \mathbb{R}^+ \to M^2$ of the following problem: For all $t > 0$,

(1) $\lambda t \, (d\gamma_u/dt) = X(\gamma_u(t))$,
(2) $\lim_{t\to 0} \gamma_u(t) = P$,
(3) $\lim_{t\to 0} t^{-1} \, (d\gamma_u/dt) = v_u$,

The curve γ_u linearises the unstable manifold of P and, by symmetry, $\gamma_s = I \circ \gamma_u$ linearises the stable manifold of P. Let $T \in \mathbb{R}^+$ be the time corresponding to the symmetric homoclinic point, $\gamma_u(T) = Q = (I \circ \gamma_u)(T)$. Then the homoclinic length of the connection is given by $\sqrt{T^2 \cdot |\omega(v_u, I(v_u))|}$.

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