PERSISTENT HOMOCLINIC TANGENCIES FOR CONSERVATIVE MAPS NEAR THE IDENTITY

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Abstract: *For families of conservative maps near the identity we prove the existence of open sets of parameters with persistence of homoclinic tangencies between stable and unstable leaves of "thick" horse-shoes. Such families are obtained, for instance, by perturbing integrable Hamiltonian systems in $\mathbb{R}^2$ with a rapidly periodic oscillatory term and then performing a slowing change in the time variable.*

1. **Introduction**

Even for simple dynamical systems as surface diffeomorphisms there is a vast diversity of possible orbit structure. This makes virtually impossible any attempt to extend to all generic diffeomorphisms the well understood classification of structurally stable maps via topological conjugacy. One of the phenomena that best illustrates this diversity is the persistence, in some open set $\mathcal{U}$ of diffeomorphisms, of "homoclinic" tangencies between stable and unstable leaves of the same hyperbolic basic set Λ. If we focus on one of these "homoclinic" tangencies then a convenient perturbation of the system inside $\mathcal{U}$ will destroy it but other tangencies appear instead. This is the meaning of *persistent tangencies*. An example of this phenomenon was first given on a 4 dimensional model by R. Abraham and S. Smale [AS-70] to disprove the density of Ω-stable diffeomorphisms. In the context of surface diffeomorphisms the construction of such examples appears with S.Newhouse [N-70] to disprove the density of Axiom A on S^2. Let us briefly discuss the mechanism behind these examples on surfaces. Look at one "homoclinic" tangency H between stable and unstable leaves of a basic set Λ. Near it we have two Cantor like foliations, those of stable, and unstable, leaves of Λ. Extend them into C^1 foliations over a neighborhood of H. Then there is a C^1 curve ℓ through H consisting of tangencies between these extended foliations. Consider the Cantor sets K^s and K^u formed by the points where the stable, resp. unstable, foliation intersects the curve ℓ. By definition of ℓ, Λ has a "homoclinic" tangency at each point in $K^s \cap K^u$. With this construction the initial problem of finding persistent homoclinic tangencies is transformed into a new problem, that of finding persistent intersections between two Cantor sets. The key in [N-70] to ensure that K^s and K^u intersect persistently is the concept of *thickness* $\tau(K)$ of a Cantor set K, which is used through the following
gap lemma *If the product of the thicknesses of two interconnected Cantor sets K^s and K^u is larger then 1 then they must intersect.*

The thickness of a Cantor set measures the relative size of its gaps, large thickness corresponding to small gaps. It is only invariant under isometries which, in general, makes it much more difficult to compute, or even to estimate, then the Hausdorff dimension, which is invariant under Lipschitz homeomorphisms. Nevertheless, due to their dynamical definitions, it can be proved that the thicknesses of the Cantor sets K^s and K^u depend continuously on the diffeomorphism. Therefor if the initial map is chosen such that K^s and K^u are interconnected and $\tau(K^s)\,\tau(K^u) > 1$, then in a neighborhood of this map these Cantor sets will persistently intersect.

Now, if $\mathcal{U}$ is an open set of diffeomorphisms with persistence of "homoclinic" tangencies between stable and unstable leaves of Λ and we pick a fixed point in Λ then its invariant manifolds will have true homoclinic tangencies for a subset of systems dense in $\mathcal{U}$. Unfolding these tangencies new hyperbolic structures are created around Λ which can only be detected in very long iterates and at microscopic scales. In this creation process non-hyperbolic long periodic orbits are formed close to the orbit of tangencies, which therefore go near Λ for many iterates. These non-hyperbolic periodic orbits are either sinks in the dissipative case or elliptic isles for conservative systems. A simple argument then shows that for almost all systems of $\mathcal{U}$ in a topological sense, i.e. in countable intersection of open dense subsets of $\mathcal{U}$, every arbitrarily small neighborhood of a point in Λ is visited by non-hyperbolic periodic orbits. In other words Λ is accumulated by sinks, in the dissipative case, or by elliptic islands, in the conservative one. These implications of the persistent tangencies phenomenon go back to [N-74]. See [D-94] for a conservative example. This gives us a small idea of how complex the dynamics of such systems can be. But there is more to it. If we look deep into ever smaller dynamical details of ever longer iterates of the initial map in $\mathcal{U}$ we will see the same type of picture repeated indefinitely. In the dissipative case, for instance, we will see the basic set Λ accumulated by periodic sinks with long periods but also by small periodic basic sets Λ_1 which exhibit persistent homoclinic tangencies too. Furthermore we will see transversal heteroclinic orbits flowing from Λ to Λ_1 and vice versa. This means that the hyperbolic structures of the large and the small basic sets are interconnected. Then for Λ_1 we will see a similar picture: Λ_1 accumulated by even longer periodic sinks and smaller periodic basic sets Λ_2 with persistent homoclinic tangencies ... And we will see the same type of picture repeated ad infinitum. Of course at any given scale we need to know where to look for the smaller basic sets with persistence of homoclinic tangencies and to allow ourselves to perform ever smaller perturbation adjustments to the initial system as we go deep into the microscopic dynamical structure. This folkloric description can be stated and proved rigorously from the theorem below, see [N-79].

Theorem [S. Newhouse] *Let f be a surface diffeomorphism with an orbit $\mathcal{O}$ of homoclinic tangencies between stable and unstable manifolds of some dissipative hyperbolic fixed point P. Then there is a sequence of open sets $\mathcal{U}_n$ with persistence of homoclinic tangencies between stable and unstable leaves of some periodic basic set Λ_n such that:*

(1) *$\mathcal{U}_n$ approaches f as $n \to +\infty$,*
(2) *Λ_n converges to the closure $\overline{\mathcal{O}}$ as $n \to +\infty$,*
(3) *There are transversal heteroclinic orbits flowing from P to Λ_n and vice versa.*

Furthermore, for each n there is a residual subset of $\mathcal{U}_n$ for which systems the basic set Λ_n is accumulated by periodic sinks.

A similar result for area preserving diffeomorphisms, as it has been conjectured by J. Palis to be true, is still missing. We should mention however that L. Mora and N. Romero have have exploited other mechanisms, then the persistence of a basic set's homoclinic tangencies, to obtain similar consequences. See [MR1-97] and [MR2-97]. The present work grew out of the proof of Newhouse's conservative version theorem,

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2. **Results**

Let X_δ be a smooth family of Hamiltonian vector fields in $\mathbb{R}^2$, with respect to the canonical symplectic form $\omega = dx \wedge dy$, and $f_{\delta,\mu} : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth family of symplectic diffeomorphisms such that:

H1) X_δ has a saddle connection γ_δ at some hyperbolic fixed point family P_δ,
H2) $f_{\delta,0}$ is the time δ map of the Hamiltonian flow of X_δ.

Through this article smooth will always mean C^∞.

Families like these arise naturally in the following way:

Let $X_0(x,y)$ and $Y_0(x,y,t)$ be a Hamiltonian vector fields in $\mathbb{R}^2$ such that X_0 has a saddle connection γ_0 and $Y_0(x,y,t)$ is periodic with period T in the time variable t. Consider the perturbation of X_0 with a rapidly periodic oscillatory term,

$$\frac{d}{dt}(x,y) = X_0(x,y) + \mu\, Y_0\left(x, y, \frac{t}{\delta}\right) ,$$

where δ and μ are small parameters. If we perform the slowing change in the time variable $\theta = \frac{t}{\delta}$ then the corresponding autonomous system is given by,

$$\begin{cases} \frac{d}{dt}(x,y) &= \delta\ (X_0(x,y) + \mu\, Y_\delta(x,y,\theta)) \\ \frac{d\theta}{dt} &= 1 \end{cases} .$$

It induces a flow $\phi^t_{\delta,\mu} : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{R}^2 \times \mathbb{S}^1$ with a return map $f_{\delta,\mu} = \phi^T_{\delta,\mu}$, to the cross section $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\} \subseteq \mathbb{R}^2 \times \mathbb{S}^1$, that satisfies hypothesis H1) and H2).

Two typical examples to which the main theorem below applies are the following:

Example 2.1.

$$\ddot{x} = 2\,x + x^2 + \mu\,\cos\left(\frac{2\,\pi\,t}{\delta}\right)$$

or equivalently

$$\begin{cases} \dot{x} = \delta\, y \\ \dot{y} = \delta\ \left(2\,x + x^2 + \mu\,\cos(2\,\pi\,\theta)\right) \\ \dot{\theta} = 1 \end{cases}$$

which is a perturbation of the Hamiltonian vector field $X_0(x,y) = \left(y\,, 2\;x + x^2\right)$ *depicted in figure 1.*

Example 2.2.

$$\ddot{x} = 2\,x - x^3 + \mu\,\cos\left(\frac{2\,\pi\,t}{\delta}\right) ,$$

which is equivalent to

$$\begin{cases} \dot{x} = \delta\, y \\ \dot{y} = \delta\ \left(2\,x - x^3 + \mu\,\cos(2\,\pi\,\theta)\right) \\ \dot{\theta} = 1 \end{cases}$$

a perturbation of the Hamiltonian vector field $X_0(x,y) = \left(y\,,\, x - x^3\right)$ *of figure 2.*

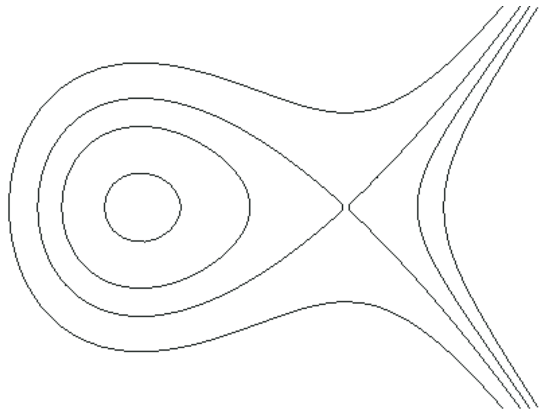

FIGURE 1. The Hamiltonian vector field $X_0(x,y) = \left(y \, , 2\, x + x^2 \right)$

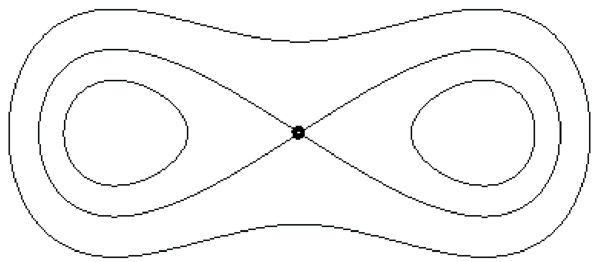

FIGURE 2. The Hamiltonian vector field $X_0(x,y) = \left(y \, , \, x - x^3 \right)$

Given a family $f_{\delta,\mu}$ satisfying the assumptions H1) and H2) above consider the smooth family of Hamiltonians H_δ associated with the vector field family $\delta\, X_\delta$:

$$\delta\, X_\delta = J\, \nabla H_\delta \quad \text{where} \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Definition 2.1. *Let $q_\delta(t) = q(\delta,t)$ be a smooth family of solutions,*

$$\frac{d}{dt}\, q_\delta = \delta\, X_\delta\, (q_\delta(t)) \qquad \forall\, t \in \mathbb{R}\;,$$

whose orbits $\{q_\delta(t) \,:\, -\infty < t < \infty\}$ describe the homoclinic connections γ_δ . Denote by $P_{\delta,\mu}$ the unique (hyperbolic) fixed point of $f_{\delta,\mu}$ near P_δ . For each $t \in \mathbb{R}$ define $q^s_{\delta,\mu}(t)$, respectively $q^u_{\delta,\mu}(t)$, to be the last point in $W^s(P_{\delta,\mu})$, resp. first point in $W^u(P_{\delta,\mu})$, that intersects the line through $q_\delta(t)$ with direction $\nabla H_\delta\,(q_\delta(t))$ in a small neighborhood of $q_\delta(t)$. The Melnikov function of $f_{\delta,\mu}$, w.r.t. $q_\delta(t)$, is defined to be

$$M_\delta(t) = \nabla H_\delta\,(q_\delta(t)) \cdot \frac{\partial}{\partial \mu} \left(q^u_{\delta,\mu}(t) - q^s_{\delta,\mu}(t) \right)_{\mu=0} \; .$$

Although the Melnikov function seems to depend on the euclidean structure of $\mathbb{R}^2$ it really does not. The Melnikov function is a symplectic invariant of the family $f_{\delta,\mu}$, c.f. proposition 5.2. It is a periodic function, with period one, in variable t and it is dominated, in absolute value, by any power δ^N if $\delta > 0$ is small enough. See proposition 5.1. Finally, it is well known that each simple zero t_δ of the Melnikov function, $M_\delta(t_\delta) = 0$ with $\frac{d}{dt} M_\delta(t_\delta) \neq 0$, corresponds to some transversal homoclinic point $H_{\delta,\mu} = q_\delta(t_\delta) + O(\mu) \in W^s(P_{\delta,\mu}) \cap W^u(P_{\delta,\mu})$ for all small enough $\mu \neq 0$.

In the two examples above one can compute, by the Poincaré-Melnikov method, explicit expressions for the Melnikov function which have the following form:

$$M_\delta(t) = \psi(\delta)\,\sin(2\,\pi\,t)$$

where for some constants C and D the smooth function $\psi(\delta)$ has the following asymptotic expression

$$\psi(\delta) \sim \frac{D}{\delta}\,\exp\left(-\frac{C}{\delta}\right)$$

Define for $i = 0, 1, 2,$

$$\begin{aligned} M_i(\delta) &= \max\left\{ \left|\frac{d^i}{dt^i} M_\delta(t)\right| \,:\, t \in \mathbb{R} \right\}, \\ m_1(\delta) &= \min\left\{ \left|\frac{d}{dt} M_\delta(t)\right| \,:\, M_\delta(t) = 0 \right\} \quad \text{and} \\ m_2(\delta) &= \min\left\{ \left|\frac{d^2}{dt^2} M_\delta(t)\right| \,:\, \frac{d}{dt} M_\delta(t) = 0 \right\}. \end{aligned} \tag{1}$$

Theorem 1. *Assume there is $C > 0$ such that for all small enough $\delta > 0$ the Melnikov function satisfies $C\,m_1(\delta) > M_2(\delta)$ and $C\,m_2(\delta) > M_0(\delta)$.*
Then there is an open set U in the (δ, μ) plane, whose closure contains some interval $[0, \delta_0)\times\{0\}$, with $\delta_0 > 0$, and there is a (discontinuous) family of basic sets $\{\Lambda_{\delta,\mu}\}_{(\delta,\mu)\in U}$ of $f_{\delta,\mu}$ containing the fixed point $P_{\delta,\mu}$ and such that:

- i *For each $(\delta, \mu) \in U$ there is a quadratic homoclinic tangency between a pair of stable and unstable leaves of $\Lambda_{\delta,\mu}$ which unfolds generically with μ.*
- ii *There is a residual subset $R \subseteq U$ of parameters $(\delta, \mu) \in R$ such that the closure of $f_{\delta,\mu}$' generic elliptic periodic points contains $\Lambda_{\delta,\mu}$.*
- iii *There is a dense subset $D \subseteq U$ of parameters $(\delta, \mu) \in D$ at which $W^s(P_{\delta,\mu})$ and $W^u(P_{\delta,\mu})$ generically unfold a quadratic homoclinic tangency in the parameter μ.*

Remark 2.1. *The hypothesis is trivially fulfilled for the families $f_{\delta,\mu}$ associated with examples 2.1 and 2.2.*

Remark 2.2. *Consider the open set*

$$W = \{\delta > 0 \,:\, C\,m_1(\delta) > M_2(\delta) \quad \textit{and} \quad C\,m_2(\delta) > M_0(\delta)\,\}$$

and assume $0 \in \overline{W}$. Then there is an open set U in the (δ, μ) plane, whose closure contains $([0, \delta_0) \cap W) \times \{0\}$ for some $\delta_0 > 0$, and there is a family of basic sets $\{\Lambda_{\delta,\mu}\}_{(\delta,\mu)\in U}$ of $f_{\delta,\mu}$, containing the fixed point $P_{\delta,\mu}$, for which the same conclusions of theorem 1 hold. This stronger statement follows also from the proof of theorem 1.

Remark 2.3. *The symplectic character of $f_{\delta,\mu}$ implies that the Melnikov function $M_\delta(t)$ must have zeros. By the assumption all zeros of $M_\delta(t)$ are simple and, therefore, each of them corresponds to a transversal homoclinic orbit of $f_{\delta,\mu}$ which is created for small $\mu \neq 0$ as the homoclinic connection γ_δ breaks down. It follows also that all critical points of $M_\delta(t)$ are non degenerated.*

Remark 2.4. *From the assumption of theorem 1 it follows easily that all the functions of δ: M_0, M_1, M_2, m_1 and m_2 are asymptoticly equivalent, in the sense that the quotient of any pair is bounded from 0 and from ∞. In particular we see that all them are dominated by powers δ^N, $N \in \mathbb{N}$, as δ tends to zero. See proposition 5.1. Finally, the distance between consecutive zeros of $M_\delta(t)$ is at least $2/C$ and so in each period $M_\delta(t)$ has at most $C/2$ zeros.*

Consider now smooth one parameter families of symplectic diffeomorphisms $f_\delta : \mathbb{R}^2 \to \mathbb{R}^2$ such that: $f_0 = \mathrm{Id}$, and $\left(\frac{d}{d\delta} f_\delta\right)_{\delta=0} = X_0$ is a Hamiltonian vector field with a saddle connection associated to some hyperbolic singularity $X_0(P_0) = 0$, $P_0 \in \mathbb{R}^2$. Such families can be obtained from the previous two parameter families $f_{\delta\,\mu}$ by considering the second parameter $\mu = \mu(\delta)$ to be a smooth function of δ such that $\mu(0) = 0$. For these families a similar theorem could be proved, namely that:

If there is a Melnikov function $M_\delta(t) = \psi(\delta)\,\mu(t)$ "describing the unfolding" of the saddle connection at $\delta = 0$ such that $\mu(t)$ is a smooth periodic Morse function, and the function $\psi(\delta)$ does not vanish identically in any neighborhood of $\delta = 0$, then there is an open set $I \subseteq (0, +\infty)$ accumulating on $\delta = 0$ and there is a family of basic sets $\{\Lambda_\delta\}_{\delta \in I}$ of f_δ containing the (continuation) of the fixed point P_0 for which the same conclusions i), ii), iii) of theorem 1 hold.

A word of caution: in this new context a Melnikov function can not be uniquely determined by the family f_δ as it was in the previous context of two parameter families. A rigorous meaning for the sentence " $M_\delta(t)$ describes the unfolding of the saddle connection at $\delta = 0$ " will be omitted since we will not prove this theorem here. Notice however that, in any concrete analytic example, in order to verify the assumption in the statement above we have to face the hard problem of estimating the exponentially small size (in δ) of the splitting angle of the separatrices of P_0, see for instance [G1-97] and [G2-97]. On the other hand checking the assumption of theorem 1 is usually an easy task using the Poincaré-Melnikov method. Our motivation to prove theorem 1, instead of this seemingly more natural one parameter statement, was to simplify things in [D-98] where we use this theorem to prove the conservative version of the theorem of S. Newhouse mentioned in the introduction.

Proof of theorem 1. Apply proposition 3.2 to the combination of corollary 8.1 with corollary 9.1. Remark, as in the beginning of section 8, that $\Lambda_{\delta,\mu}$ is part of a basic set of $f_{\delta,\mu}$, viewed in the $\Phi_{\delta,\mu}$ coordinates. □

We use the geometric assumption on the Melnikov function to construct a family of basic sets $\Lambda_{\delta,\mu}$, horse-shoes conjugated to the full Bernoulli shift in two symbols, for some return map

$$T_{\delta,\mu}(x,y) = \begin{cases} f_{\delta,\mu}(x,y) & \text{if } (x,y) \in S_{\delta,\mu}(0) \\ f_{\delta,\mu}^{2\,n}(x,y) & \text{if } (x,y) \in S_{\delta,\mu}(1) \end{cases},$$

where $n = n(\delta,\mu)$ tends to infinity as $(\delta,\mu) \to (0,0)$. See figure 3. These horse-shoes will be "thick" in some sense which, once a tangency is found, implies persistent homoclinic tangencies. We use here a generalization of the usual *gap lemma* in [Mo-96] that introduces *lateral thicknesses*.

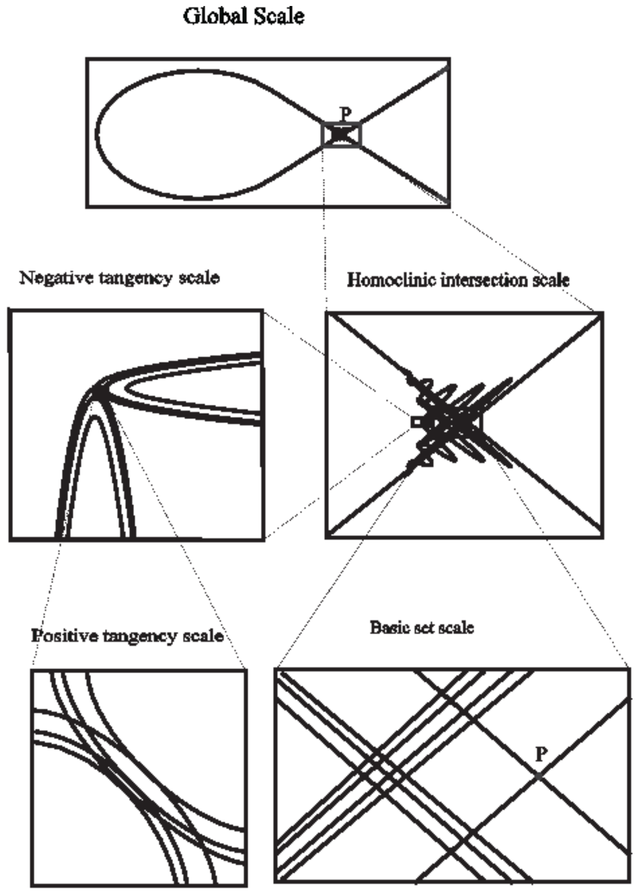

FIGURE 3. Scales of the construction

One of the technical difficulties in the proof of theorem 1 is getting uniform lower bounds of the lateral thicknesses of the Cantor sets associated with $\Lambda_{\delta,\mu}$. For this we need also uniform estimates on the *distortion* of these Cantor sets. The material in section 3 is well known and may be found, with some modifications, in [PT-93] and [Mo-96]. There we recall dynamically defined Cantor sets, their lateral thicknesses, distortion, and relation between them. We also remember the construction of the stable and unstable Cantor sets associated with a horse-shoe type basic set and define the left-right thickness of the basic set. Finally we state and prove an abstract proposition, prop 3.2, relating lateral thicknesses with persistent homoclinic tangencies.

Bounds on distortion involve very technical and fastidious estimates on the differentiability of the invariant foliations (stable and unstable). It would be very nice to have a formula giving an upper bound for the distortion of stable and unstable foliations, of a given basic set, based in the following quantities:

i first and second order derivatives of the map and its inverse on a Markov partition,
ii the scale, or diameter, of the basic set, and
iii the sizes of its larger gaps, i.e. the distances between rectangles of the Markov partition.

To my knowledge there is no such distortion bounding formula available. In section 4 we describe a general setting where distortion of a conservative basic set is bounded in function of the quantities i, ii and iii above. This section was written completely independent of the rest so that the main result in it, theorem 2, could be easily exported. The conservativeness assumption simplifies things but it shouldn't be crucial.

In section 5 we adapt, to our context of families satisfying the assumptions H1) and H2), some well-known facts about Melnikov functions.

In section 6 we translate the geometric assumption on the Melnikov function into conditions on the map along the homoclinic connection. The return map $T_{\delta,\mu}$ is defined without yet specifying its domain. We also obtain some estimates on the derivatives of the map which are needed to obtain the bounds on distortion.

In section 7 we introduce rescaling coordinates which will bring the tiny basic set $\Lambda_{\delta,\mu}$ to unit square. Some more estimates on derivatives are obtained there.

In section 8 the basic set $\Lambda_{\delta,\mu}$, together with its Markov partition, is constructed. All estimates on quantities i,ii and iii, needed to obtain small distortion according to theorem 2, are checked. It is then shown that the basic set $\Lambda_{\delta,\mu}$ has left-right thickness tending to infinity as $(\delta,\mu)\to(0,0)$.

Finally, in section 9 we prove the existence of homoclinic tangencies unfolding generically, which are needed to complete the proof of theorem 1.

3. **Left-right thickness**

We call *dynamically defined Cantor set* to any pair (K,ψ) such that $K\subseteq\mathbb{R}$ is a Cantor set and $\psi:K\to K$ is a locally Lipschitz expanding map, topologically conjugated to some subshift of finite type of a Bernoulli shift $\sigma:\{0,1,\cdots,p\}^{\mathbb{N}}\to\{0,1,\cdots,p\}^{\mathbb{N}}$. For the sake of simplicity, and because this is enough for our purpose, we will restrict ourselves to the case where ψ is conjugated to the full Bernoulli shift $\sigma:\{0,1\}^{\mathbb{N}}\to\{0,1\}^{\mathbb{N}}$. Moreover we will assume that there is a partition of K, $K=K_0\cup K_1$, into disjoint Cantor subsets such that the restriction of ψ to each K_i, $\psi:K_i\to K$ is a strictly monotonous Lipschitz expanding homeomorphism. $\mathcal{P}=\{K_0,K_1\}$ is called a *Markov partition* of (K,ψ). Given a symbolic sequence $(a_0,\cdots,a_{n-1})\in\{0,1\}^n$ writing,

$$K(a_0,\cdots,a_{n-1})=\bigcap_{i=0}^{n-1}\psi^{-i}(K_{a_i})\ ,$$

the map $\psi^n:K(a_0,\cdots,a_{n-1})\to K$ is a Lipschitz expanding homeomorphism.

A bounded component of the complement $\mathbb{R}-K$ is called a *gap* of K. For a dynamically defined Cantor set (K,ψ) the gaps are ordered in the following way. Denote by $\widehat{A}$ the convex hull of a subset $A\subseteq\mathbb{R}$. Then the interval $\widehat{K}-\widehat{K_0}\cup\widehat{K_1}$ is called a gap of order zero. A connected component of

$$\widehat{K}-\bigcup_{(a_0,\cdots,a_{n-1})\in\{0,1\}^n}\widehat{K(a_0,\cdots,a_{n-1})}$$

that is not a gap of order $\leq n-1$ is called a gap of order n. It is straightforward to check that every gap of K is a gap of some finite order, and also that, given a gap $U=(x,y)$ of order n, for every $0\leq k\leq n$ the open interval bounded by $\psi^k(x)$ and $\psi^k(y)$ is a gap of order $n-k$.

Following [Mo-96] we define

Definition 3.1. *Given a gap U of K with order n, we denote by L_U, resp. R_U, the unique interval of the form $\widehat{K(a_0,\cdots,a_{n-1})}$, with $(a_0,\cdots,a_{n-1})\in\{0,1\}^n$, that is* *left, resp. right, adjacent to* U *. The greatest lower bounds*

$$\tau_L(K) = \inf \left\{ \frac{|L_U|}{|U|} \ : \ U \text{ is a gap of } K \ \right\}$$

$$\tau_R(K) = \inf \left\{ \frac{|R_U|}{|U|} \ : \ U \text{ is a gap of } K \ \right\}$$

are respectively called the left and right thickness *of* K *. Similarly, the ratios*

$$\tau_L(\mathcal{P}) = \frac{|L_{U_0}|}{|U_0|} \quad \text{and} \quad \tau_R(\mathcal{P}) = \frac{|R_{U_0}|}{|U_0|} ,$$

where U_0 *is the unique gap of order zero, are called the left and right thickness of the Markov partition* $\mathcal{P}$ *.*

We should remark that this definition differs slightly from the correspondent in [Mo-96]. The reason is that we have ordered gaps not by their lengths but by the order in which they appear as pull-backs of gaps of zero order by the map ψ. However, for the binary Cantor sets, to which we have restricted our attention, the two definitions coincide.

When the restriction of ψ to each K_i is affine we have

$$\tau_L(K) = \tau_L(\mathcal{P}) \text{ and } \quad \tau_R(K) = \tau_R(\mathcal{P}) ,$$

but in general these thicknesses may be very different due to the nonlinearity of ψ.

We now give precise definition of *distortion*. Given a Lipschitz expanding map (injective in particular) $g : J \to \mathbb{R}$, defined on some subset $J \subseteq \mathbb{R}$, we call distortion of g on J to the lowest upper bound,

$$\mathrm{Dist}\,(g, J) = \sup_{x,y,z \in J} \log \left\{ \frac{|g(y) - g(x)|}{|g(z) - g(x)|} \, \frac{|z - x|}{|y - x|} \right\} \in [0, +\infty]$$

where the sup is taken over all $x,\, y,\, z \in J$ such that $z \neq x$ and $y \neq x$, which implies, because g is injective, $g(z) \neq g(x)$ and $g(y) \neq g(x)$. Reversing the roles of y and z we see that the distortion is always $\geq \log 1 = 0$. If $\mathrm{Dist}\,(g, J) = c$ then for all $x,\, y,\, z \in J$ with $z \neq x$ and $y \neq x$ we have

$$e^{-c} \, \frac{|y - x|}{|z - x|} \leq \frac{|g(y) - g(x)|}{|g(z) - g(x)|} \leq e^{c} \, \frac{|y - x|}{|z - x|} \tag{2}$$

The *distortion* of a dynamically defined Cantor set (K, ψ) is defined to be the lowest upper bound

$$\mathrm{Dist}_\psi(K) = \sup \ \ \mathrm{Dist}\,(\psi^n, K(a_0, \cdots, a_{n-1}))$$

taken over all sequences $(a_0, \cdots, a_{n-1}) \in \{0,1\}^n$. Distortion is the key to estimate thickness.

Lemma 3.1. *Let* (K, ψ) *be a dynamically defined Cantor set with a Markov Partition* $\mathcal{P}$ *and distortion* $Dist_\psi(K) = c$ *. Then*

$$e^{-c} \, \tau_L(\mathcal{P}) \leq \tau_L(K) \leq e^{c} \, \tau_L(\mathcal{P})$$

$$e^{-c} \, \tau_R(\mathcal{P}) \leq \tau_R(K) \leq e^{c} \, \tau_R(\mathcal{P})$$

Proof. Follows by a well known argument. See [PT-93]. □

We will need the following improvement of the usual gap lemma [Mo-96]:

Lemma 3.2 (left-right gap lemma)**.** *Let* (K^s,ψ^s) , (K^u,ψ^u) *be dynamically defined Cantor sets such that the intervals supporting* K^s *and* K^u *do intersect,* K^s *is not contained inside a gap of* K^u *and neither* K^u *is contained inside a gap of* K^s*. If* $\tau_L(K^s)\,\tau_R(K^u)>1$ *and* $\tau_R(K^s)\,\tau_L(K^u)>1$ *, then both Cantor sets intersect,* $K^s\cap K^u\neq\emptyset$*.*

Proof. The same proof as in [Mo-96] works. Just argue that one could obtain pairs of linked gaps with ever higher order, instead of ever smaller lengths. The result is the same because as we consider gaps with strictly increasing order their lengths converge to zero. □

Let us see now how this lemma will be applied to get open sets of persistent homoclinic tangencies in the following class of "horse-shoes".

Definition 3.2. *Define* $\mathcal{F}$ *to be the set of all maps* $f:S_0\cup S_1\to\mathbb{R}^2$ *such that:*

1. S_0 *and* S_1 *are compact subsets, diffeomorphic to rectangles, with nonempty interior.*
2. f *is a map of class* C^2 *, in a neighborhood of* $S_0\cup S_1$ *, mapping this compact set diffeomorphically onto its image* $f(S_0)\cup f(S_1)$ *.*
3. *the maximal invariant set* $\Lambda(f)=\bigcap_{n\in\mathbb{Z}} f^{-n}(S_0\cup S_1)$ *is a hyperbolic basic set conjugated to the Bernoulli shift* $\sigma:\{0,1\}^{\mathbb{Z}}\to\{0,1\}^{\mathbb{Z}}$ *.*
4. $\mathcal{P}=\{S_0,S_1\}$ *is a Markov partition for* $f:\Lambda(f)\to\Lambda(f)$ *, in particular* f *has two fixed points,* $P_0\in S_0$ *and* $P_1\in S_1$ *, whose stable and unstable manifolds contain the boundaries of* S_0 *and* S_1 *.*
5. *Both fixed points* P_0 *and* P_1 *have positive eigenvalues.*

The action of f and f^{-1} resp. on the stable, and unstable, foliation of Λ ,

$$\mathcal{F}^s=\{\text{ connected comp. of } W^s(\Lambda)\cap(S_0\cup S_1)\ \},$$

$$\mathcal{F}^u=\{\text{ connected comp. of } W^s(\Lambda)\cap(f(S_0)\cup f(S_1))\ \}\ ,$$

is described in the following way. Define

$$I_*^s=W^s_{\rm loc}(P_0)\cap S_0 \quad\text{and}\quad I_*^u=W^u_{\rm loc}(P_0)\cap f(S_0)\ .$$

I_*^s and I_*^u are stable and unstable leaves of Λ respectively transversal to the foliation $\mathcal{F}^u$ and $\mathcal{F}^s$. Then the Cantor sets

$$K^s=\Lambda\cap I_*^u \quad\text{and}\quad K^u=\Lambda\cap I_*^s\ ,$$

can be identified with the set of stable leaves of $\mathcal{F}^s$, resp. unstable leaves of $\mathcal{F}^u$. Define the projections $\pi_s:\Lambda\to K^s$ and $\pi_u:\Lambda\to K^u$ in the obvious way: $\pi_s(P)$ is the unique point in $W^s_{\rm loc}(P)\cap I_*^u$, and similarly $\pi_u(P)$ is the unique point in $W^u_{\rm loc}(P)\cap I_*^s$. The maps $\psi^s:K^s\to K^s$ and $\psi^u:K^u\to K^u$

$$\psi^s=\pi_s\circ f \quad\text{and}\quad \psi^u=\pi_u\circ f^{-1}\ ,$$

describe the action of f , resp. f^{-1} , on stable, resp. unstable leaves of Λ . The pairs (K^s,ψ^s) and (K^u,ψ^u) are dynamically defined Cantor sets, topologically conjugated to the Bernoulli shift $\sigma:\{0,1\}^{\mathbb{N}}\to\{0,1\}^{\mathbb{N}}$, with Markov partitions $\mathcal{P}^s=\{I_*^u\cap S_0, I_*^s\cap S_1\}$ and $\mathcal{P}^u=\{I_*^s\cap f(S_0), I_*^s\cap f(S_1)\}$.

Given a map $f\in\mathcal{F}$ we define the *left-right thickness* of $\Lambda(f)$ to be

$$\tau_{LR}(\Lambda)=\min\{\tau_L(K^s)\,\tau_R(K^u),\ \tau_L(K^u)\,\tau_R(K^s)\}\ .$$

In order to estimate this thickness we define the left-right thickness of $\mathcal{P}$ as

$$\tau_{LR}(\mathcal{P}) = \min\{\tau_L(\mathcal{P}^s)\,\tau_R(\mathcal{P}^u),\ \tau_L(\mathcal{P}^u)\,\tau_R(\mathcal{P}^s)\}\ .$$

If the distortion on both dynamically defined Cantor sets (K^s, ψ^s), (K^u, ψ^u) is small, say less or equal than c, then it follows from proposition 3.1 that

$$e^{-2c}\,\tau_{LR}(\mathcal{P}) \le \tau_{LR}(\Lambda) \le e^{2c}\,\tau_{LR}(\mathcal{P})\ .$$

Bounding distortion is the key to the next

Proposition 3.1. *The map $\tau_{LR} : \mathcal{F} \to \mathbb{R}$, $f \mapsto \tau_{LR}(\Lambda(f))$, is continuous.*

Proof. As it is remarked in [Mo-96] the lateral thicknesses may be discontinuous for non binary dynamically defined Cantor sets. However if one adapts instead the more dynamical definition 3.1 then lateral thicknesses are always continuous. The same argument as for the usual tickness applies, e.g. see [PT-93]. $\square$

For one parameter families of maps, the usual gap lemma establishes a connection between thickness of the invariant, stable and unstable, foliations and the existence of parameter intervals with persistence of homoclinic tangencies. The left-right gap lemma establishes a new, but entirely similar, connection which we now state.

Definition 3.3. *Given a hyperbolic fixed point P of some surface diffeomorphism, with both its eigenvalues positive, we orient the stable and unstable branches of $W^s(P) - P$ and $W^u(P) - P$ so that orbits increase along them. A homoclinic tangency of P is called* positive *if both the orientations, on the stable and unstable branches, agree near the point of tangency.*

Proposition 3.2. *Let $\varphi_\mu \in \mathrm{Diff}^2(M^2)$ be a smooth family of diffeomorphisms, on some surface M^2, and $\Lambda_\mu = \varphi_\mu(\Lambda_\mu)$ be an invariant basic set family. Assume there are coordinates (possibly depending on the parameter) in a neighborhood of Λ_μ in which the diffeomorphism φ_μ is expressed by a map $f_\mu \in \mathcal{F}$, and that the left-right thickness of the basic set, in these coordinates, satisfies $\tau_{LR}(\Lambda_\mu) > 1$. If a fixed point $P_\mu \in \Lambda_\mu$, with both its eigenvalues positive, unfolds generically, at the parameter μ_0, an orbit of quadratic positive homoclinic tangencies , then there is an open interval I with $\mu_0 \in \overline{I}$ such that:*

- i *For each $\mu \in I$ there is a quadratic homoclinic tangency between a pair of stable and unstable leaves of Λ_μ which unfolds generically with μ.*
- ii *If all φ_μ preserve some area form then there is a residual subset $R \subseteq I$ of parameters $\mu \in R$ such that the closure of φ_μ's generic elliptic periodic points contains Λ_μ.*
- iii *There is a dense subset $D \subseteq I$ of parameters $\mu \in D$ where $W^s(P_\mu)$ and $W^u(P_\mu)$ generically unfold quadratic homoclinic tangencies.*

Proof. We will just outline the proof to stress the point where the positive tangency hypothesis plays its role. Let us identify φ_μ with f_μ and look at $\mathcal{F}$ as a subset of $\mathrm{Diff}^2(M^2)$. For every parameter μ sufficiently close to μ_0, we extend the stable and unstable foliations of Λ_μ filling in all $S_0(\mu) \cup S_1(\mu)$ with a foliation $\mathcal{F}^s_\mu$ and all $f_\mu(S_0(\mu)) \cup f_\mu(S_1(\mu))$ with a foliation $\mathcal{F}^u_\mu$, respectively invariant by f_μ^{-1} and f_μ. Let $H_0 \in W^s(P_{\mu_0}) \cap W^u(P_{\mu_0})$ be a positive homoclinic tangency of f_{μ_0}, which unfolds generically with μ. Take $n, m \ge 1$ such that $f^n_{\mu_0}(H_0) \in S_0(\mu_0) \cup S_1(\mu_0)$ and $f^{-m}_{\mu_0}(H_0) \in f_{\mu_0}(S_0(\mu_0)) \cup f_{\mu_0}(S_1(\mu_0))$. Then for all μ in a small neighborhood U of μ_0 there is, near H_0, a line of tangencies ℓ_μ between $f^{-n}_\mu(\mathcal{F}^s_\mu)$ and $f^m_\mu(\mathcal{F}^u_\mu)$. ℓ_μ is a curve of class C^1 which depends C^1 continuously on μ. The line ℓ_{μ_0} goes through H_0. We denote by $\pi_s : I^u_* \to \ell_\mu$ and $\pi_u : I^s_* \to \ell_\mu$ the projections along $f_\mu^{-n}(\mathcal{F}^s_\mu)$ and $f_\mu^{m}(\mathcal{F}^u_\mu)$ respectively. These are C^1 diffeomorphisms depending continuously on μ. Defining $C^u_\mu = \pi_s(K^u_\mu)$ and $C^s_\mu = \pi_s(K^s_\mu))$ we have:

- Points in $C^s_\mu \cap C^u_\mu$ are homoclinic tangencies of Λ_μ.
- Because *the tangency is positive*, both orientations induced in ℓ_μ by the projections π_u and π_s agree. Therefor the products $\tau_L(C^s_\mu)\,\tau_R(C^u_\mu)$ and $\tau_R(C^s_\mu)\,\tau_L(C^u_\mu)$ are close to $\tau_L(K^s_\mu)\,\tau_R(K^u_\mu)$ and $\tau_R(K^s_\mu)\,\tau_L(K^u_\mu)$, at least if we restrict the Cantor sets C^s_μ, C^u_μ to a very small neighborhood of H_0, where the distortion due to the nonlinearity of the projection diffeomorphisms π_u, π_s is also small. Thus, by continuity of τ_{LR}, if U is a sufficiently small neighborhood of μ_0 we have $\tau_L(C^s_\mu)\,\tau_R(C^u_\mu) > 1$ and $\tau_R(C^s_\mu)\,\tau_L(C^u_\mu) > 1$.

At μ_0 we have $C^u_{\mu_0} \cap C^s_{\mu_0} \neq \emptyset$. Now define I to be the open subset of U formed by all μ such that the Cantor sets C^s_μ and C^u_μ have supporting intervals whose interiors intersect but such that no one of them is contained inside a gap of the other. By the left-right gap lemma it follows that $\forall \mu \in I$, $C^s_\mu \cap C^u_\mu \neq \emptyset$ and so I is an open interval with persistence of homoclinic tangencies of Λ_μ. By definition it is clear that $\mu_0 \in \overline{I}$. Because the tangency H_0 at parameter μ_0 is quadratic and unfolds generically, if I is sufficiently small then all these tangencies will be quadratic and unfold generically too.

Finally, items ii and iii are standard consequences of item i. See [D-94] for the conclusion in ii, and [PT-93] for item iii. □

4. **Uniformly Bounded Distortion**

In this section we describe a class of conservative horse-shoe maps where the dynamics of the stable and unstable foliations have small uniformly bounded distortion.

Definition 4.1. *Given positive small numbers $\epsilon > 0$ and $\gamma > 0$ define $\mathcal{F}(\epsilon,\gamma)$ to be the class of all maps $f : S_0 \cup S_1 \to \mathbb{R}^2$, $f \in \mathcal{F}$, such that:*

(1) *$diam\,(S_0 \cup S_1) = diam\,(f(S_0) \cup f(S_1)) = 1$.*

(2) *The derivative of f, $Df_{(x,y)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where a, b, c and d are C^1 functions, satisfies all over $S_0 \cup S_1$*
   - (a) $\det Df = a\,d - b\,c = 1$
   - (b) $|d| < 1 < |a| \leq 2/\epsilon$
   - (c) $|b|\,,\ |c| \leq \epsilon\,(|a| - 1)$

(3) *The C^1 functions on $f(S_0) \cup f(S_1)$, $\tilde a = a \circ f^{-1}$, $\tilde b = b \circ f^{-1}$, $\tilde c = c \circ f^{-1}$ and $\tilde d = d \circ f^{-1}$, i.e. $Df^{-1}_{(x,y)} = \begin{pmatrix} \tilde d & -\tilde b \\ -\tilde c & \tilde a \end{pmatrix}$, satisfy*
   - (a) $\left|\dfrac{\partial \tilde b}{\partial x}\right| = \left|\dfrac{\partial \tilde d}{\partial y}\right|\,,\ \left|\dfrac{\partial \tilde b}{\partial y}\right|\,,\ \left|\dfrac{\partial \tilde c}{\partial x}\right|\,,\ \left|\dfrac{\partial \tilde a}{\partial x}\right| = \left|\dfrac{\partial \tilde c}{\partial y}\right| \leq \gamma\ (|\tilde a| - 1)$
   - (b) $\left|\dfrac{\partial a}{\partial y}\right| = \left|\dfrac{\partial b}{\partial x}\right|\,,\ \left|\dfrac{\partial b}{\partial y}\right|\,,\ \left|\dfrac{\partial c}{\partial x}\right|\,,\ \left|\dfrac{\partial c}{\partial y}\right| = \left|\dfrac{\partial d}{\partial x}\right| \leq \gamma\ (|a| - 1)$
   - (c) $\left|\dfrac{\partial \tilde a}{\partial y}\right|\,,\ \left|\dfrac{\partial \tilde d}{\partial x}\right| \leq \gamma\ |\tilde a|\,(|\tilde a| - 1)$

(d) $\left|\dfrac{\partial a}{\partial x}\right|$, $\left|\dfrac{\partial d}{\partial y}\right| \le \gamma\ |a|\,(|a|-1)$.

(4) *The variation of* $\log|a(x,y)|$ *in each rectangle* S_i *is less or equal than* $\gamma\,(1-\alpha_i^{-1})$ *, where* $\alpha_i = \max_{(x,y)\in S_i} |a(x,y)|$.

(5) *Finally, the gap sizes satisfy:*

$$dist\,(S_0,S_1) \ge \frac{\epsilon}{\gamma} \quad and \quad dist\,(f(S_0),f(S_1)) \ge \frac{\epsilon}{\gamma}\ .$$

Remark 4.1. *Coments on some items of the definition of class* $\mathcal{F}(\epsilon,\gamma)$ *,*

(1) *This normalizing condition is to avoid having all subsequent items refering to the scale of the basic set.*
(2) *This item says that* f *is symplectic and hyperbolic with stable direction close to vertical and unstable direction nearly horizontal. Notice, however, that expansion and contraction in (2b) may be arbitrarily weak.*
(3) *Here bounds are given on the second derivatives of* f *and* f^{-1} *in terms of the norms* $|a(x,y)| = \left\|Df_{(x,y)}\right\|$ *and* $|\tilde a(x,y)| = \left\|Df^{-1}_{(x,y)}\right\|$ *. Notice that the second derivatives in (3c) and (3d) can be very large compared with linear terms.*

Remark 4.2. *The class* $\mathcal{F}(\epsilon,\gamma)$ *is symmetric with respect to inversion. Take the linear involution* $I:\mathbb{R}^2\to\mathbb{R}^2$ $I(x,y)=(y,x)$ *as a coordinate transformation. Then if we write, as above,* $Df_{(x,y)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *the inverse map, in the new coordinates,* $g = I^{-1}\circ f^{-1}\circ I$ *has derivative* $Dg_{(x,y)} = \begin{pmatrix} \tilde a & -\tilde c \\ -\tilde b & \tilde d \end{pmatrix}$ *. Looking at the conditions defining* $\mathcal{F}(\epsilon,\gamma)$ *we see that*

$$f\in\mathcal{F}(\epsilon,\gamma) \quad\Leftrightarrow\quad g = I^{-1}\circ f^{-1}\circ I \in \mathcal{F}(\epsilon,\gamma)\ .$$

Thus any proof of a statement about differentiability of the stable foliations of maps in $\mathcal{F}(\epsilon,\gamma)$ *can be transformed into a proof of a similar statement about unstable foliations.*

Remark 4.3. *For the sake of simplicity, and because this is enough for our purposes, we have restricted ourselves to the case of basic sets conjugated to the full Bernoulli shift* $\sigma:\{0,1\}^{\mathbb{Z}}\to\{0,1\}^{\mathbb{Z}}$ *. But adapting assumption 5., in the definition of class* $\mathcal{F}(\epsilon,\gamma)$ *, the main theorem in this section holds for any class of basic sets modeled in any subshift of finite type.*

Theorem 2. *For all small enough* $\epsilon>0$ *and* $\gamma>0$ *, given* $f\in\mathcal{F}(\epsilon,\gamma)$ *, the basic set* $\Lambda(f)$ *gives dynamically defined Cantor sets* (K^u,ψ^u) *and* (K^s,ψ^s) *with small distortion, bounded by* $D(\epsilon,\gamma) = 20\,\gamma + 2\,\epsilon$ *. In particular*

$$e^{-2\,D(\epsilon,\gamma)}\,\tau_{LR}\,(\mathcal{P}) \le \tau_{LR}\,(\Lambda(f)) \le e^{2\,D(\epsilon,\gamma)}\,\tau_{LR}\,(\mathcal{P})\ .$$

Lemma 4.1. *For all small enough* $\epsilon>0$ *and* $\gamma>0$ *, given* $f\in\mathcal{F}(\epsilon,\gamma)$ *there are functions of class* C^1

$$\sigma^s: W^s(\Lambda)\cap(S_0\cup S_1)\to\mathbb{R} \quad and \quad \sigma^u: W^u(\Lambda)\cap(f(S_0)\cup f(S_1))\to\mathbb{R}$$

such that vector fields $X_u(x,y) = (1,\sigma^u(x,y))$ *and* $X_s(x,y) = (\sigma^s(x,y),1)$ *respectively generate the line fields of stable and unstable directions of* $\Lambda(f)$ *and the following estimations hold:*

(1) $|\sigma^s|$, $|\sigma^u| \leq \epsilon$,
(2) *Lip* (σ^s) , *Lip* $(\sigma^u) \leq 8\,\gamma$,

where the Lipschitz seminorm Lip(σ) *is taken w.r.t. the norm* $\|(x,y)\| = |x| + |y|$ *in* $\mathbb{R}^2$.

In the lemma just stated we have chosen the norm $\|(x,y)\| = |x| + |y|$ of $\mathbb{R}^2$ in order to have the following easy properties:

(1) Given functions $\sigma : \mathbb{R}^2 \to \mathbb{R}$ and $T : S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$, respectively Lipschitz and of class C^1 , we have
$$\mathrm{Lip}(\sigma \circ T) \leq \mathrm{Lip}(\sigma)\ \|DT\|\ ,$$
where $\|DT\|_S$ is the maximum absolute value, over S , of the first partial derivatives of the components of T .
(2) If $\sigma : \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 then for all $(x,y) \in \mathbb{R}^2$,
$$\left|\frac{\partial \sigma}{\partial x}(x,y)\right|\ ,\ \left|\frac{\partial \sigma}{\partial y}(x,y)\right| \leq \mathrm{Lip}(\sigma)\ .$$

It is well known, see [HP-70], that the stable and unstable invariant line fields of any hyperbolic set of a surface diffeomorphism of class C^2 are always of class C^1 . The point here is to give explicit bounds on the derivatives of these line fields. Before doing it, in fact we will only estimate the unstable line field σ^u , let us introduce some notation and state some easy facts. Using the notation of assumption 6. on the class $\mathcal{F}(\epsilon,\gamma)$ define $\rho = \rho_f : f(S_0) \cup f(S_1) \times \mathbb{R} \to \mathbb{R}$
$$\rho(x,y,s) = \frac{\tilde{c}(x,y) + \tilde{d}(x,y)\,s}{\tilde{a}(x,y) + \tilde{b}(x,y)\,s}\ .$$
The relation between $Df_{(x,y)}$ and $\rho(x,y,s)$ is given in the formula,
$$Df_{(x,y)}\,(\,1,\ s\,) = (a + b\,s)\ (\,1,\ \rho(\,f(x,y), s)\,)\ .$$
Notice that since $\tilde{a}\,\tilde{d} - \tilde{b}\,\tilde{c} = 1$ we have,
$$\frac{\partial \rho}{\partial s}(x,y,s) = \frac{1}{(\tilde{a} + \tilde{b}\,s)^2}\ .$$
In the lemma below we give a list of inequalities which are simple consequences of the mean value theorem and the definition of class $\mathcal{F}(\epsilon,\gamma)$. We will use the notation $\|h\|_S = \max\{|h(x,y)|\ :\ (x,y) \in S\}$ for a given continuous function $h(x,y)$ on S .

Lemma 4.2. *For all small enough* $\epsilon > 0$ *and* $\gamma > 0$ *, given* $f \in \mathcal{F}(\epsilon,\gamma)$ *and* $|s| \leq \epsilon$ *, if* $\rho = \rho_f$ *then the following inequalities hold:*

(1) $\|\rho(s)\|_{f(S_i)} \leq \epsilon$.
(2) $\left\|\frac{\partial \rho}{\partial s}(s)\right\|_{f(S_i)} \leq \frac{1}{\alpha_i^2}\, e^{2(\epsilon+\gamma)(1-\alpha_i^{-1})} \leq \frac{1}{\alpha_i}$.
(3) $\left\|\frac{\partial \rho}{\partial x}(s)\right\|_{f(S_i)}\ ,\ \left\|\frac{\partial \rho}{\partial y}(s)\right\|_{f(S_i)} \leq 6\,\gamma\,(1 - \alpha_i^{-1})$.

Proof of lemma 4.1. We only estimate σ^u, the stable case being entirely analogous. Given $f \in \mathcal{F}(\epsilon,\gamma)$ consider the forward f^{-1}-invariant compact set,

$$\Sigma^u(f) = W^u(\Lambda) \cap (f(S_0) \cup f(S_1)) = \bigcap_{k\geq 1} f^k(S_0 \cap S_1)$$

and define the operator $F_f : C^0(\Sigma^u(f)) \to C^0(\Sigma^u(f))$,

$$F_f(\sigma)\,(x,y) = \rho\left(\, x,\, y,\, \sigma\, f^{-1}(x,y)\,\right) \ .$$

Here $C^0(\Sigma^u(f))$ denotes the Banach space of all real valued functions over the compact set $\Sigma^u(f)$, with the usual maximum norm denoted by $\|\cdot\|$. Remarking that $F_f(\sigma) = \sigma$ is equivalent to

$$Df\ (1,\sigma) = (a + b\,\sigma)\circ f\ \cdot\ (\ 1\,,\ \sigma\circ f\ )$$

we see that a fixed point σ of this operator corresponds to a f-invariant line field generated by $X(x,y) = (1,\sigma(x,y))$. To prove that σ^u satisfies the inequalities 1. and 2. we define the closed subset $\mathcal{X} \subseteq C^0\,(\Sigma^u(f))$,

$$\mathcal{X} = \{\ \sigma \in C^0(\Sigma^u(f))\ :\ \|\sigma\| \leq \epsilon\ \text{ and }\ \mathrm{Lip}(\sigma) \leq 8\,\gamma\ \ \}\ ,$$

and show that $F_f(\mathcal{X}) \subseteq \mathcal{X}$ and F_f is a Lipschitz contraction on $\mathcal{X}$. Then F_f has a unique fixed point in $\mathcal{X}$ which must coincide with σ^u, therefore proving that $\sigma^u \in \mathcal{X}$. Let us now prove that $F_f(\mathcal{X}) \subseteq \mathcal{X}$. Item 1. of lemma 4.2 immediately shows that $\|\sigma\| \leq \epsilon$ implies $\|F_f(\sigma)\| \leq \epsilon$. Assume $\sigma \in \mathcal{X}$. We have to see that $\sigma^* = F_f(\sigma)$ is also Lipschitz with constant $\mathrm{Lip}(\sigma^*) \leq 8\,\gamma$. Given $(x,y) \in f(S_0)$ and $(x',y') \in f(S_1)$, we have

$$\begin{aligned}
\left|\sigma^*(x,y) - \sigma^*(x',y')\right| &\leq\ 2\,|\sigma^*| \leq \frac{2\,\epsilon}{\|(x-x',y-y')\|}\,\left\|(x-x',y-y')\right\| \\
&\leq\ \frac{2\,\epsilon}{\epsilon/\gamma}\,\left\|(x-x',y-y')\right\| = 2\,\gamma\,\left\|(x-x',y-y')\right\|\ .
\end{aligned}$$

This inequality takes care of the case when the two points belong to different rectangles $f(S_i)$ $i = 0,1$. If both (x,y) and (x',y') belong to the same $f(S_i)$ then, by inequalities 2. and 3. of the previous lemma, we have

$$\begin{aligned}
\frac{|\sigma^*(x,y) - \sigma^*(x',y')|}{\|(x-x',y-y')\|} &\leq \left\|D_{(x,y)}\rho\right\|_{f(S_i)} + \left\|\frac{\partial\rho}{\partial s}\right\|_{f(S_i)} \mathrm{Lip}(\sigma)\ \left\|Df^{-1}\right\|_{f(S_i)} \\
&\leq\ 6\,\gamma\,(1-\alpha_i^{-1}) + \frac{e^{2\,(\gamma+\epsilon)\,(1-\alpha_i^{-1})}}{\alpha_i^2}\,8\,\gamma\,\alpha_i \\
&=\ 8\,\gamma\,(\frac{3}{4}\,(1-\alpha_i^{-1}) + \alpha_i^{-1}\,e^{2\,(\gamma+\epsilon)\,(1-\alpha_i^{-1})}) \leq\ 8\,\gamma
\end{aligned}$$

Finally from item 2. of lemma 4.2 we get that F_f is a Lipschitz contraction with constant $k = \max\{\alpha_0^{-1},\alpha_1^{-1}\}$. □

In the following three lemmas we will be dealing with a map $f \in \mathcal{F}(\epsilon,\gamma)$ where $\epsilon > 0$ and $\gamma > 0$ are small enough constants fixed according to lemmas 1. and 2.

Lemma 4.3. *Each leaf $F_0^u \in \mathcal{F}^u$, resp. $F_0^s \in \mathcal{F}^s$, is the graph of a class C^2 function $g_0 : I_0 \to \mathbb{R}$, $F_0^u = \{(x,g_0(x))\ :\ x \in I_0\}$, resp. $F_0^s = \{(g_0(x),x)\ :\ x \in I_0\}$, such that $|g_0'(x)| \leq \epsilon$.*

Proof. Take one point $(x_0, y_0) \in \Lambda$ and the unstable leaf $F_0 \in \mathcal{F}^u$ through it. Since the vector field $X^u(x,y) = (1, \sigma^u(x,y))$ is tangent to F_0 at every point $(x,y) \in F_0$, the graph of the maximal solution $g_0 : I_0 \to \mathbb{R}$ of the Cauchy problem

$$\begin{cases} {g_0}'(x) = \sigma^u(x, g_0(x)) \\ g_0(x_0) = y_0 \end{cases}$$

must coincide with F_0. By the previous lemma

$$\left|{g_0}'(x)\right| \le |\sigma^u| \le \epsilon \ .$$

For stable leaves a similar proof holds. □

Lemma 4.4. *Given $F^s, \tilde{F}^s \in \mathcal{F}^s$ and $F_0^u, F_1^u \in \mathcal{F}^u$ with intersection points $P_0 = (x_0, y_0) \in F^s \cap F_0^u$, $P_1 = (x_1, y_1) \in F^s \cap F_1^u$, $\tilde{P}_0 = (\tilde{x}_0, \tilde{y}_0) \in \tilde{F}^s \cap F_0^u$, and $\tilde{P}_1 = (\tilde{x}_1, \tilde{y}_1) \in \tilde{F}^s \cap F_1^u$ the following inequalities hold*

(1) $\exp(-\epsilon - 8\gamma) \le \dfrac{|x_0 - \tilde{x}_0|}{|x_1 - \tilde{x}_1|} \le \exp(\epsilon + 8\gamma)$

(2) $\exp(-\epsilon - 8\gamma) \le \dfrac{|y_0 - y_1|}{|\tilde{y}_0 - \tilde{y}_1|} \le \exp(\epsilon + 8\gamma)$

Proof. Let $g : I \to \mathbb{R}$, $\tilde{g} : \tilde{I} \to \mathbb{R}$, $g_0 : I_0 \to \mathbb{R}$ and $g_1 : I_1 \to \mathbb{R}$ be C^2 functions such that

$$F^s = \{(\, g(y),\, y\,) \,:\, y \in I\,\}\,, \qquad \tilde{F}^s = \{(\, \tilde{g}(y),\, y\,) \,:\, y \in \tilde{I}\,\}\,,$$

$$F_0^u = \{(x,\, g_0(x)\,) \,:\, y \in I_0\,\}\,, \qquad F_1^u = \{(x,\, g_1(x)\,) \,:\, y \in I_1\,\}\,.$$

Using the mean value theorem and the fact that $|\tilde{g}'|$, $|{g_1}'| \le \epsilon$ we have

$$\begin{aligned} |x_1 - \tilde{x}_1| &\le |x_1 - \tilde{g}(y_1)| + |\tilde{g}(y_1) - \tilde{x}_1| \\ &\le |x_1 - \tilde{g}(y_1)| + \epsilon\, |y_1 - \tilde{y}_1| \\ &\le |x_1 - \tilde{g}(y_1)| + \epsilon^2\, |x_1 - \tilde{x}_1| \end{aligned}$$

Therefore

$$|x_1 - \tilde{x}_1| \le (1 - \epsilon^2)^{-1}\, |x_1 - \tilde{g}(y_1)| \ .$$

On the other hand, since for $y \in I \cap \tilde{I}$ we have $|y - y_0| \le 1$, and by item 2 of lemma 4.1

$$\begin{aligned} |g(y) - \tilde{g}(y)| &\le |g(y_0) - \tilde{g}(y_0)| + \int_{y_0}^{y} |\sigma^s(g(s), s) - \sigma^s(\tilde{g}(s), s)|\, ds \\ &\le |x_0 - \tilde{g}(y_0)| + 8\gamma \int_{y_0}^{y} |g(s) - \tilde{g}(s)|\, ds\,, \end{aligned}$$

using Gronwall inequality we obtain

$$|x_1 - \tilde{g}(y_1)| = |g(y_1) - \tilde{g}(y_1)| \le |x_0 - \tilde{g}(y_0)| \exp(8\gamma)\ ,$$

and so

$$|x_1 - \tilde{x}_1| \le (1 - \epsilon^2)^{-1} \exp(8\gamma)\, |x_0 - \tilde{g}(y_0)| \ .$$

Finally using the mean value theorem again and the fact that $|\tilde{g}'|$, $|{g_0}'| \le \epsilon$ we obtain

$$\begin{aligned} |x_0 - \tilde{g}(y_0)| &\le |x_0 - \tilde{x}_0| + |\tilde{x}_0 - \tilde{g}(y_0)| \\ &\le |x_0 - \tilde{x}_0| + \epsilon\, |\tilde{y}_0 - y_0| \\ &\le |x_0 - \tilde{x}_0| + \epsilon^2\, |\tilde{x}_0 - x_0| = (1 + \epsilon^2)\, |x_0 - \tilde{x}_0| \end{aligned}$$

This proves that

$$|x_1 - \tilde{x}_1| \leq \frac{1+\epsilon^2}{1-\epsilon^2} \exp(8\,\gamma)\ |x_0 - \tilde{x}_0| \leq \exp(8\,\gamma + \epsilon)\ |x_0 - \tilde{x}_0|\ .$$

Reversing the roles of x_0, $\tilde{x}_0$ and x_1, $\tilde{x}_1$ we obtain the opposite inequality, thus proving item 1. A completely analogous calculation proves item 2. ☐

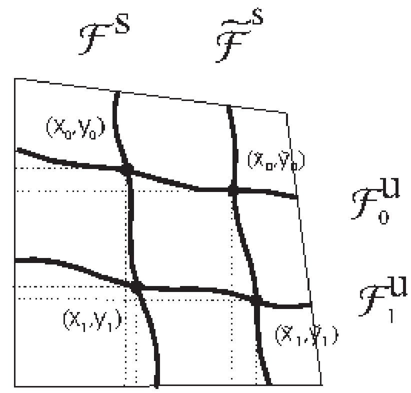

FIGURE 4. F^s, $\tilde{F}^s$, F^u_0, F^u_1

Lemma 4.5. *Given two unstable leaves* F^u_0, $F^u_1 \in \mathcal{F}^u$ *consider the functions of class* C^2 $g_0 : I_0 \to \mathbb{R}$ *and* $g_1 : I_1 \to \mathbb{R}$ *such that* $F_0 = \mathit{graph}(g_0)$ *and* $F_1 = \mathit{graph}(g_1)$ *and define for* $i = 0, 1$ $\phi_i : I_0 \cap S_i \to I_1$ *by,*

$$f\,(\,x,\,g_0(x)\,) = (\,\phi_i(x),\,g_1\,\phi_i(x)\,) \qquad \forall x \in I_0 \cap S_i\ .$$

Then ϕ_i *is of class* C^2 *and the following inequalities hold*

(1) $|\phi_i'(x)| \geq \alpha_i\, e^{-(\gamma+\epsilon)(1-\alpha_i^{-1})}$
(2) $|\log|\phi_i'(x)| - \log|\phi_i'(y)|\,| \leq 2\,\gamma\,(1-\alpha_i^{-1})\ |\phi_i(x) - \phi_i(y)|$

A similar result holds for stable leaves.

Proof. Differentiating the identity that defines ϕ_i we get

$$Df_{(x,g_0(x))}\,(\,1, \sigma^u(x,\,g_0(x))\,) = \phi_i'(x)\ (\,1,\,\sigma^u(x, g_1\,\phi_i(x))\,)$$

Thus

$$\phi_i'(x) = (a + b\,\sigma^u)(x, g_0(x))\ .$$

A simple estimation, using the mean value theorem, shows that

$$\begin{aligned} \left|\phi_i'(x)\right| \quad &\geq \quad |a| - \epsilon^2\,(|a|-1)\,) \geq \alpha_i \left|\frac{a}{\alpha_i}\right|\ (1 - \epsilon^2\,(1-\alpha_i^{-1})\,) \\ &\geq \quad \alpha_i\, e^{-(\gamma+\epsilon)(1-\alpha_i^{-1})}\ . \end{aligned}$$

Using the notation introduced in lemma 4.1 we can write the second derivative as

$$\phi_i''(x) = (\partial_u a + \partial_u b\ \sigma^u + b\ \partial_u \sigma^u)\,(x, g_0(x))$$

Thus,

$$\begin{aligned} \left|\phi_i''(x)\right| &\leq |\partial_u a| + \epsilon\, |\partial_u b| + |b|\ |\partial_u \sigma^u| \\ &\leq (1+\epsilon)\,\gamma\,\alpha_i\,(\alpha_i - 1) + \epsilon\,(1+\epsilon)\,\gamma\,(\alpha_i - 1) + \epsilon\,(\alpha_i - 1)\,8\,\gamma \\ &\leq (1+10\,\epsilon)\,\gamma\,\alpha_i\,(\alpha_i - 1)\ , \end{aligned}$$

and

$$\left|\frac{\phi_i''(x)}{\phi_i'(x)^2}\right| \leq e^{(\gamma+\epsilon)(1-\alpha_i^{-1})}(1+10\,\epsilon)\,\gamma\,(1-\alpha_i^{-1}) \leq 2\ \gamma\,(1-\alpha_i^{-1})\ .$$

Finally, given $x,\, y \in I \cap S_i$ write $z_t = t\,x + (1-t)\,y$ for $t \in [0,1]$. Then using the mean value theorem, and the fact that $\phi_i'(z)$ has constant sign, we have

$$\begin{aligned} \left|\log\, \left|\phi_i'(x)\right| - \log\, \left|\phi_i'(y)\right|\right| &\leq \int_0^1 \left|\frac{\phi_i''(z_t)}{\phi_i'(z_t)}\right|\, dt\ \ |x-y| \\ &\leq 2\,\gamma\,(1-\alpha_i^{-1}) \int_0^1 \left|\phi_i'(z_t)\right|\, dt\ \ |x-y| \\ &= 2\,\gamma\,(1-\alpha_i^{-1}) \left|\int_0^1 \phi_i'(z_t)\, dt\ (x-y)\right| \\ &= 2\,\gamma\,(1-\alpha_i^{-1})\ |\phi_i(x) - \phi_i(y)|\ . \end{aligned}$$

□

Proof of theorem 2 . We will estimate the distortion of (K^s, ψ^s). The unstable case is worked in the same way. Recall that the Cantor set K^s lies inside the unstable leaf I_*^u, which is the graph of a C^2 function g_*. Consider in I_*^u the length induced by the parameterization $x \mapsto (x, g_*(x))$. Distortion will be estimated w.r.t. this metric. For notation convenience we will write ψ instead of ψ^s. Now, fix some symbolic sequence $(\nu_0, \cdots, \nu_{n-1}) \in \{0,1\}^n$ and the corresponding Cantor set

$$K^s(\nu_0, \cdots, \nu_{n-1}) = \bigcap_{i=0}^{n-1} \psi^{-i}\,(I_*^u \cap S_{\nu_i})\ .$$

We are going to estimate the distortion of the map $\psi^n : K^s(\nu_0, \cdots, \nu_{n-1}) \to K^s$. Take three points $(x_0, y_0),\, (\tilde{x}_0, \tilde{y}_0),\, (\hat{x}_0, \hat{y}_0) \in K^s(\nu_0, \cdots, \nu_{n-1})$. Then for every $i = 0, 1, \cdots, n-1$ the three iterates

$$(x_i, y_i) = f^i(x_0, y_0), \quad (\tilde{x}_i, \tilde{y}_i) = f^i(\tilde{x}_0, \tilde{y}_0), \quad (\hat{x}_i, \hat{y}_i) = f^i(\hat{x}_0, \hat{y}_0),$$

belong to the same rectangle S_{ν_i}. Defining

$$\begin{aligned} (x_n^*, y_n^*) &= \pi_s(x_n, y_n) = \pi_s\, f^n(x_0, y_0)\,, \\ (\tilde{x}_n^*, \tilde{y}_n^*) &= \pi_s(\tilde{x}_n, \tilde{y}_n) = \pi_s\, f^n(\tilde{x}_0, \tilde{y}_0)\,, \\ (\hat{x}_n^*, \hat{y}_n^*) &= \pi_s(\hat{x}_n, \hat{y}_n) = \pi_s\, f^n(\hat{x}_0, \hat{y}_0)\,, \end{aligned}$$

our goal is to find the upper bound $D(\epsilon, \gamma) = 20\,\gamma + 2\,\epsilon$ for the logarithm,

$$\log \frac{|\tilde{x}_n^* - x_n^*|}{|\hat{x}_n^* - x_n^*|}\ \frac{|\hat{x}_0 - x_0|}{|\tilde{x}_0 - x_0|}\ .$$

This expression is the sum of the following two logarithms:

$$\log \frac{|\tilde{x}_n^* - x_n^*|}{|\hat{x}_n^* - x_n^*|}\ \frac{|\hat{x}_n - x_n|}{|\tilde{x}_n - x_n|} = \log \frac{|\tilde{x}_n^* - x_n^*|}{|\tilde{x}_n - x_n|} + \log \frac{|\hat{x}_n - x_n|}{|\hat{x}_n^* - x_n^*|}\ ,$$

that, by lemma 4.4, is dominated by $16\,\gamma + 2\,\epsilon$ and

$$\log \frac{|\tilde{x}_n - x_n|}{|\hat{x}_n - x_n|}\,\frac{|\hat{x}_0 - x_0|}{|\tilde{x}_0 - x_0|}\ ,$$

to be estimated next. Consider the sequence of unstable leaves recursively defined by $F_0 = I^u_*$ and $F_{i+1} = f(F_i \cap S_{\nu_i})$. For each $i = 0, 1, \cdots, n-1$ let $g_i : I_i \to \mathbb{R}$ be the class C^2 function whose graph coincides with F_i and define $\phi_i : I_i \cap S_{\nu_i} \to I_{i+1}$ by

$$f\,(x, g_i(x)) = (\,\phi_i(x)\,,\, g_{i+1}\,\phi_i(x)\,) \qquad (x, g_i(x)) \in F_i \cap S_{\nu_i}$$

as in lemma 4.5. Then the map $\Phi = \phi_{n-1} \circ \cdots \phi_1 \circ \phi_0$ satisfies

$$f^n\,(x, \Phi(x)) = (\,\Phi(x)\,,\, g_n\,\Phi(x)\,) \quad \text{ for } \quad (x, g_0(x)) \in K^s(\nu_0, \cdots, \nu_{n-1})\ ,$$

which shows that

$$x_n = \Phi(x_0)\ , \quad \tilde{x}_n = \Phi(\tilde{x}_0)\ , \quad \text{ and } \quad \hat{x}_n = \Phi(\hat{x}_0)\ .$$

By the mean value theorem there are points $\tilde{\xi}_0$, between $\tilde{x}_0$ and x_0, and $\hat{\xi}_0$, between $\hat{x}_0$ and x_0, such that

$$\Phi'(\tilde{\xi}_0) = \frac{\tilde{x}_n - x_n}{\tilde{x}_0 - x_0} \quad \text{ and } \quad \Phi'(\hat{\xi}_0) = \frac{\hat{x}_n - x_n}{\hat{x}_0 - x_0}\ .$$

Thus

$$\begin{aligned}
\log \frac{|\tilde{x}_n - x_n|}{|\hat{x}_n - x_n|}\,\frac{|\hat{x}_0 - x_0|}{|\tilde{x}_0 - x_0|} \quad &= \quad \log \frac{\left|\Phi'(\tilde{\xi}_0)\right|}{\left|\Phi'(\hat{\xi}_0)\right|} \le \left|\,\log\left|\Phi'(\tilde{\xi}_0)\right| - \log\left|\Phi'(\hat{\xi}_0)\right|\,\right| \\
&\le \quad \sum_{i=0}^{n-1} \left|\,\log\left|\phi_i'(\tilde{\xi}_i)\right| - \log\left|\phi_i'(\hat{\xi}_i)\right|\,\right| \\
&\le \quad \sum_{i=0}^{n-1} 2\,\gamma\,(1 - \alpha_{\nu_i}^{-1})\,\left|\,\phi_i(\tilde{\xi}_i) - \phi_i(\hat{\xi}_i)\,\right| \\
&\le \quad \sum_{i=0}^{n-1} 2\,\gamma\,(1 - \alpha_{\nu_i}^{-1})\,\left|\tilde{\xi}_{i+1} - \hat{\xi}_{i+1}\right|
\end{aligned}$$

where $\tilde{\xi}_i = \phi_{i-1} \circ \cdots \circ \phi_0(\tilde{\xi}_0)$ and $\hat{\xi}_i = \phi_{i-1} \circ \cdots \circ \phi_0(\hat{\xi}_0)$. Now for each $i = 0, 1, \cdots, n-1$ define

$$\lambda_i = \alpha_{\nu_i}^{-1}\, e^{(\gamma+\epsilon)(1-\alpha_{\nu_i}^{-1})} \qquad (0 < \lambda_i < 1)$$

From lemma 4.5 one has,

$$\left|\tilde{\xi}_{i+1} - \hat{\xi}_{i+1}\right| \le \lambda_{i+1} \cdots \lambda_{n-1}\, \underbrace{\left|\tilde{\xi}_n - \hat{\xi}_n\right|}_{\le 1} \le \lambda_{i+1} \cdots \lambda_{n-1}\ .$$

A simple computation gives

$$\left(1 - \alpha_{\nu_i}^{-1}\right) \le \frac{1 - \lambda_i}{1 - 2\,(\gamma + \epsilon)}$$

and so

$$
\begin{aligned}
\log \frac{|\tilde{x}_n - x_n|}{|\hat{x}_n - x_n|} \frac{|\hat{x}_0 - x_0|}{|\tilde{x}_0 - x_0|} &\leq \sum_{i=0}^{n-1} 2\,\gamma\,(1-\alpha_{\nu_i}^{-1}) \left|\tilde{\xi}_{i+1} - \hat{\xi}_{i+1}\right| \\
&\leq \sum_{i=0}^{n-1} \frac{2\,\gamma}{1-2\,(\gamma+\epsilon)}\,(1-\lambda_i)\,\lambda_{i+1}\,\cdots\,\lambda_{n-1} \\
&= \frac{2\,\gamma}{1-2\,(\gamma+\epsilon)}\,(1-\lambda_0\,\lambda_1\,\cdots\,\lambda_{n-1}) \leq 3\,\gamma
\end{aligned}
$$

This gives the desired bound $D(\epsilon,\gamma) = 20\,\gamma + 2\,\epsilon$ for the logarithm,

$$
\log \frac{|\tilde{x}_n^* - x_n^*|}{|\hat{x}_n^* - x_n^*|} \frac{|\hat{x}_0 - x_0|}{|\tilde{x}_0 - x_0|} \leq 16\,\gamma + 2\,\epsilon + 3\,\gamma \leq 20\,\gamma + 2\,\epsilon
$$

□

5. The Melnikov Function

Let X_δ and $f_{\delta,\mu} : \mathbb{R}^2 \to \mathbb{R}^2$ be smooth families of Hamiltonian vector fields and symplectic maps satisfying hypothesis H1) and H2) of section 2.

The first lemma shows that the Melnikov function is unique up to time shifts.

Lemma 5.1. *Given two smooth families $q_\delta(t)$ and $\tilde{q}_\delta(t)$ of homoclinic solutions parametrizing the connection γ_δ, if we denote the respective Melnikov functions by $M_\delta(t)$ and $\tilde{M}_\delta(t)$ then for some smooth function $\tau(\delta)$ we will have for all $t \in \mathbb{R}$, $\tilde{M}_\delta(t) = M_\delta(t+\tau(\delta))$.*

Proof. Since $q_\delta(t)$ and $\tilde{q}_\delta(t)$ parametrize the same orbit, for some $\tau(\delta)$ and all $t \in \mathbb{R}$, $\tilde{q}_\delta(t) = q_\delta(t+\tau(\delta))$. It follows that $\tilde{M}_\delta(t) = M_\delta(t+\tau(\delta))$. □

Lemma 5.2. *Given open sets $I \subseteq \mathbb{R}$, $U \subseteq \mathbb{R}^2$, and smooth families of maps $\gamma^s_{\delta,\mu}$, $\gamma^u_{\delta,\mu}$: $I \to U$ and $H_{\delta,\mu} : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ such that:*

1. *$\delta\, X_\delta = J\,\nabla H_{\delta,0}$ on U,*
2. *$f_{\delta,\mu}$ is the time one map of $J\,\nabla H_{\delta,\mu}$ on U,*
3. *$\gamma^s_{\delta,0}(t) = \gamma^u_{\delta,0}(t) = q_\delta(t)$,*
4. *$\gamma^s_{\delta,\mu}$ and $\gamma^u_{\delta,\mu}$ are, respectively, parametrizations of $W^s(P_{\delta,\mu})$ and $W^u(P_{\delta,\mu})$.*

Then, for all $t \in I$, the Melnikov function is given by

$$
M_\delta(t) = \frac{\partial}{\partial \mu} \left(H_{\delta,\mu}(\gamma^u_{\delta,\mu}(t)) - H_{\delta,\mu}(\gamma^s_{\delta,\mu}(t)) \right)_{\mu=0} .
$$

Proof. From item 1. it follows that $H_{\delta,0} = H_\delta + \text{Const}$ where H_δ is the family of Hamiltonians that appears in definition 2.1. Write

$$
\Delta_\delta(t) = \frac{\partial}{\partial \mu} \left(H_{\delta,\mu}(\gamma^u_{\delta,\mu}(t)) - H_{\delta,\mu}(\gamma^s_{\delta,\mu}(t)) \right)_{\mu=0} .
$$

Then

$$\begin{aligned}
\Delta_\delta(t) &= \frac{\partial H}{\partial \mu}\left(\delta,0,\gamma^u_{\delta,0}(t)\right) - \frac{\partial H}{\partial \mu}\left(\delta,0,\gamma^s_{\delta,0}(t)\right) \\
&\quad + \nabla H_{\delta,0}\left(\gamma^u_{\delta,0}(t)\right)\cdot \frac{\partial}{\partial\mu}\left(\gamma^u_{\delta,\mu}(t)\right)_{\mu=0} \\
&\quad - \nabla H_{\delta,0}\left(\gamma^s_{\delta,0}(t)\right)\cdot \frac{\partial}{\partial\mu}\left(\gamma^s_{\delta,\mu}(t)\right)_{\mu=0} \\
&= \frac{\partial H}{\partial \mu}\left(\delta,0,q_\delta(t)\right) - \frac{\partial H}{\partial \mu}\left(\delta,0,q_\delta(t)\right) \\
&\quad + \nabla H_{\delta,0}\left(q_\delta(t)\right)\cdot \frac{\partial}{\partial\mu}\left(\gamma^u_{\delta,\mu}(t) - \gamma^s_{\delta,\mu}(t)\right)_{\mu=0} \\
&= \nabla H_{\delta,0}\left(q_\delta(t)\right)\cdot \frac{\partial}{\partial\mu}\left(\gamma^u_{\delta,\mu}(t) - \gamma^s_{\delta,\mu}(t)\right)_{\mu=0}
\end{aligned}$$

Since $\gamma^u_{\delta,\mu}(t)$ and $q^u_{\delta,\mu}(t)$, resp. $\gamma^s_{\delta,\mu}(t)$ and $q^s_{\delta,\mu}(t)$, parametrize the same invariant manifolds we can find smooth families of reparametrizations $\sigma^u_{\delta,\mu}(t)$ and $\sigma^s_{\delta,\mu}(t)$ such that $\gamma^u_{\delta,\mu}(t) = q^u_{\delta,\mu}\left(\sigma^u_{\delta,\mu}(t)\right)$ and $\gamma^s_{\delta,\mu}(t) = q^s_{\delta,\mu}\left(\sigma^s_{\delta,\mu}(t)\right)$. Of course $\sigma^u_{\delta,0}(t) = \sigma^s_{\delta,0}(t) = t$ since $\gamma^u_{\delta,0}(t) = \gamma^s_{\delta,0}(t) = q^u_{\delta,0}(t) = q^s_{\delta,0}(t) = q_\delta(t)$. Therefor

$$\begin{aligned}
\frac{\partial}{\partial\mu}\left(\gamma^u_{\delta,\mu}(t) - \gamma^s_{\delta,\mu}(t)\right)_{\mu=0} &= \frac{\partial}{\partial\mu}\left(q^u_{\delta,\mu}\left(\sigma^u_{\delta,\mu}(t)\right) - q^s_{\delta,\mu}\left(\sigma^s_{\delta,\mu}(t)\right)\right)_{\mu=0} \\
&= \frac{\partial q^u}{\partial\mu}\left(\delta,0,\sigma^u_{\delta,0}(t)\right) - \frac{\partial q^s}{\partial\mu}\left(\delta,0,\sigma^s_{\delta,0}(t)\right) \\
&\quad + \frac{\partial q^u}{\partial t}\left(\delta,0,\sigma^u_{\delta,0}(t)\right)\ \frac{\partial \sigma^u}{\partial\mu}(\delta,0,t) \\
&\quad - \frac{\partial q^s}{\partial t}\left(\delta,0,\sigma^s_{\delta,0}(t)\right)\ \frac{\partial \sigma^s}{\partial\mu}(\delta,0,t) \\
&= \frac{\partial}{\partial\mu}\left(q^u_{\delta,\mu}(t) - q^s_{\delta,\mu}(t)\right)_{\mu=0} \\
&\quad + q_\delta'(t)\left(\frac{\partial\sigma^u}{\partial\mu} - \frac{\partial\sigma^s}{\partial\mu}\right)(\delta,0,t)
\end{aligned}$$

and so

$$\begin{aligned}
\Delta_\delta(t) &= \nabla H_{\delta,0}\left(q_\delta(t)\right)\cdot\frac{\partial}{\partial\mu}\left(q^u_{\delta,\mu}(t) - q^s_{\delta,\mu}(t)\right)_{\mu=0} \\
&\quad + \left(\frac{\partial\sigma^u}{\partial\mu} - \frac{\partial\sigma^s}{\partial\mu}\right)\underbrace{\nabla H_\delta\left(q_\delta(t)\right)\cdot q_\delta'(t)}_{=0} = M_\delta(t)\ .
\end{aligned}$$

□

Proposition 5.1. *The Melnikov function of* $f_{\delta,\mu}$, $M_\delta(t)$, *is periodic in* t *with period one* $M_\delta(t+1) = M_\delta(t)$, *vanishing with all its* δ*-derivatives at* $\delta = 0$*,i.e. given* $N \in \mathbb{N}$, *if* $\delta > 0$ *is small enough then for all* $t \in \mathbb{R}$ $|M_\delta(t)| \le \delta^N$.

Proof. Choose $H_{\delta,\mu}$ according to items 1. and 2. of proposition 5.2 and take $\gamma^s_{\delta,\mu}$, $\gamma^u_{\delta,\mu}$: $\mathbb{R}\to\mathbb{R}^2$ linearizing the invariant manifolds of $P_{\delta,\mu}$, in the sense that

$$f_{\delta,\mu}(\gamma^s_{\delta,\mu}(t)) = \gamma^s_{\delta,\mu}(t+1) \quad , \quad f_{\delta,\mu}(\gamma^u_{\delta,\mu}(t)) = \gamma^u_{\delta,\mu}(t+1)\ ,$$

and such that $\gamma^s_{\delta,0}(t) = \gamma^u_{\delta,0}(t) = q_\delta(t)$. Then $H_{\delta,\mu}(\gamma^u_{\delta,\mu}(t)) - H_{\delta,\mu}(\gamma^s_{\delta,\mu}(t))$ is periodic in t with period one, which according to lemma 5.2 ensures the same periodicity for the Melnikov function.

The second statement follows from the results in [FS1-90]. $\square$

The Melnikov function is a symplectic invariant.

Proposition 5.2. *Let $\psi_{\delta,\mu}:\mathbb{R}^2 \to \mathbb{R}^2$ be a smooth family of symplectic change of variables and consider the family $f_{\delta,\mu}$ expressed in the new coordinates, $\tilde{f}_{\delta,\mu} = \psi_{\delta,\mu} \circ f_{\delta,\mu} \circ \psi^{-1}_{\delta,\mu}$. Given the family of homoclinic solutions $q_\delta(t)$ and the correspondent family in the new coordinates $\tilde{q}_\delta(t) = \psi_{\delta,0} \circ q_\delta(t)$. Then the Melnikov functions of $\tilde{f}_{\delta,\mu}$ w.r.t. $\tilde{q}_\delta(t)$ is equal to the Melnikov function of $f_{\delta,\mu}$ w.r.t. $q_\delta(t)$.*

Proof. Choose families $H_{\delta,\mu}(x,y)$, $\gamma^u_{\delta,\mu}(t)$ and $\gamma^s_{\delta,\mu}(t)$ under the hypothesis of proposition 5.2 and take $\tilde{U}$ such that $\tilde{U} \subseteq \psi^{-1}_{\delta,\mu}(U)$ for all (δ,μ) .

Then, since $f_{\delta,\mu}$ is the time one map of $J\nabla H_{\delta,\mu}$ on U and $\psi_{\delta,\mu}$ is symplectic, $\tilde{f}_{\delta,\mu} = \psi_{\delta,\mu} \circ f_{\delta,\mu} \circ \psi^{-1}_{\delta,\mu}$ is the time one map of $J\nabla \tilde{H}_{\delta,\mu}$ on $\tilde{U}$, where $\tilde{H}_{\delta,\mu} = H_{\delta,\mu} \circ \psi^{-1}_{\delta,\mu}$. On the other hand, since $\gamma^u_{\delta,\mu}(t)$ and $\gamma^s_{\delta,\mu}(t)$ parametrize the invariant manifolds $W^u(P_{\delta,\mu})$ and $W^s(P_{\delta,\mu})$, respectively, the families of curves $\tilde{\gamma}^u_{\delta,\mu} = \psi_{\delta,\mu} \circ \gamma^u_{\delta,\mu}$ and $\tilde{\gamma}^s_{\delta,\mu} = \psi_{\delta,\mu} \circ \gamma^s_{\delta,\mu}$ are parametrizations of the correspondent invariant manifolds of the fixed point $\tilde{P}_{\delta,\mu} = \psi_{\delta,\mu}(P_{\delta,\mu})$ of $\tilde{f}_{\delta,\mu}$. Of course $\tilde{\gamma}^u_{\delta,0}(t) = \psi_{\delta,0}(\gamma^u_{\delta,0}(t)) = \psi_{\delta,0}(q_\delta(t)) = \tilde{q}_\delta(t)$ and similarly $\tilde{\gamma}^s_{\delta,0}(t) = \tilde{q}_\delta(t)$. Thus $\tilde{H}_{\delta,\mu}$, $\tilde{\gamma}^u_{\delta,\mu}$ and $\tilde{\gamma}^s_{\delta,\mu}$ fulfill conditions 1.,2.,3., and 4. of proposition 5.2. The invariance then follows from this proposition, since

$$\tilde{H}_{\delta,\mu}\left(\tilde{\gamma}^u_{\delta,\mu}(t)\right) - \tilde{H}_{\delta,\mu}\left(\tilde{\gamma}^s_{\delta,\mu}(t)\right) = H_{\delta,\mu}\left(\gamma^u_{\delta,\mu}(t)\right) - H_{\delta,\mu}\left(\gamma^s_{\delta,\mu}(t)\right) .$$

$\square$

6. **The Return Map**

We are going to construct a return map, of $f_{\delta,\mu}$'s iterates along the homoclinic connection, to a small neighborhood of the fixed point $P_{\delta,\mu}$.

Let us introduce now some convenient terminology. Define $\mathcal{N}$ to be the class of all subsets of the (δ,μ)-plane which have the form $U \cap]0,+\infty[^2$ where U is a neighborhood of a segment line $\{(\delta,\mu) \,:\, \mu = 0,\ 0 < \delta < \delta_0\}$ for some small $\delta_0 > 0$. Any finite intersection of elements in $\mathcal{N}$ is again in $\mathcal{N}$ and any union of elements in $\mathcal{N}$ belongs to $\mathcal{N}$. In other words $\mathcal{N}$ is a filter of subsets of $]0,+\infty[^2$.

The construction of the return map will work for all parameters (δ,μ) in some small enough $N \in \mathcal{N}$. First choose any zero of $M_\delta(t)$ with negative derivative. By the implicit function theorem we can choose these zeros depending smoothly on δ . There is a smooth function $\delta \mapsto t(\delta)$, $\delta > 0$, such that

$$M_\delta(t(\delta)) = 0 \quad \text{and} \quad \frac{d}{dt} M_\delta(t(\delta)) < 0 \,.$$

Redefining the Melnikov function, for instance doing a time shift in the family of solutions $q_\delta(t)$, we may just assume that $t(\delta) = 0$ for all δ . Define $H_{\delta,0} = q_\delta(0) \in \gamma_\delta$. Then, as we have pointed out in remark 2.3, we can extend this family, continuously, to a family of transversal homoclinic points $H_{\delta,\mu} \in W^s(P_{\delta,\mu}) \cap W^u(P_{\delta,\mu})$ defined for all (δ,μ) in some open set $N \in \mathcal{N}$.

Next we take smooth (δ, μ)-dependent coordinates reducing the families X_δ and $f_{\delta,\mu}$ to their Birkhoff normal form around P.

Proposition 6.1. *In some small enough $N \in \mathcal{N}$ there is a smooth family of symplectic coordinates $\{\psi_{\delta,\mu} : (\delta,\mu) \in N\}$, defined in a neighborhood of P_0 and conjugating both families $f_{\delta,\mu}$ and X_δ to their Birkhoff normal forms,*

$$L_{\delta,\mu}(x,y) = \psi_{\delta,\mu} \circ f_{\delta,\mu} \circ \psi_{\delta,\mu}^{-1}(x,y) = \left(\lambda(x\,y)\,x\,,\ \lambda(x\,y)^{-1}\,y \right)\,,$$

where $\lambda(t) = \lambda_{\delta,\mu}(t) = \lambda(\delta,\mu,t) > 1$ is a smooth function of (δ,μ,t), and

$$D\psi_{\delta,0}\left(\psi_{\delta,0}^{-1}(x,y)\right)\ X_\delta\left(\psi_{\delta,0}^{-1}(x,y)\right) = (\ x\,\log\,\lambda_{\delta,0}(xy)\,,\ -y\,\log\,\lambda_{\delta,0}(xy)\ )\ .$$

Proof. The problem of reducing a C^∞ volume preserving local map $f:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ to a normal form is addressed in [St-58]. The two dimensional case, of area preserving local maps at hyperbolic fixed points, is a special case to which theorem 1. in [St-58] applies. One has to check, going through the proof of this theorem, that all choices necessary to bring together X_δ (when $\mu = 0$) and $f_{\delta,\mu}$ to their normal forms, first on the formal level and then on the C^∞ level, can be made smoothly in (δ,μ). □

One could avoid this proposition working with the Birkhoff normal form of second degree, i.e where $\lambda_{\delta,\mu}(t)$ is a polynomial of second degree in t, and dealing with a remainder of order $\left(x\,O(x^3y^3)\,,\ y\,O(x^3y^3)\right)$. Since the construction of the basic set will be done remarkably close to the axis, the stable and unstable manifolds of P, the remainder errors can be controlled and proved to be negligible.

These coordinates extend smoothly to a multiple valued map defined in a neighborhood of $W^s(P_{\delta,\mu}) \cup W^u(P_{\delta,\mu})$. Scaling, if necessary, we may assume that the homoclinic point $H_{\delta,\mu}$ has coordinates $(1,0)$ and $(0,1)$.

For each $(\delta,\mu) \in N$ let $G_{\delta,\mu}:\left(\mathbb{R}^2,(0,0)\right)\to\left(\mathbb{R}^2,(0,0)\right)$,

$$G_{\delta,\mu}(x,y) = (\ g_1(\delta,\mu,x,y)\ ,\ g_2(\delta,\mu,x,y)\ )$$

be the transition map from a neighborhood of $(1,0)$ onto a neighborhood of $(0,1)$, mapping $(1,0)$ to $(0,1)$. For each (x,y) close to $(1,0)$, the vectors (x,y) and $G_{\delta,\mu}(x,y)$ represent coordinates of the same point. This transition map must, of course, satisfy the following compatibility relation,

$$L_{\delta,\mu} \circ G_{\delta,\mu} = G_{\delta,\mu} \circ L_{\delta,\mu}\ . \tag{3}$$

Denote the components of the inverse map by,

$$G_{\delta,\mu}^{-1}(x,y) = (\ h_1(\delta,\mu,x,y)\ ,\ h_2(\delta,\mu,x,y)\ )\ .$$

Lemma 6.1. *For all (δ,μ), x and y, where the functions below are defined,*

1 a) $g_1(\delta,0,x,0) = 0$, *1 b)* $g_2(\delta,0,x,0) = x^{-1}$,

1 c) $\dfrac{\partial g_2}{\partial x}(\delta,0,x,0) = -x^{-2} = -1 + O(x-1)$,

1 d) $\dfrac{\partial g_1}{\partial y}(\delta,0,x,0) = x^{-2} = 1 + O(x-1)$,

2 a) $h_1(\delta,0,0,y) = y^{-1}$, *2 b)* $h_2(\delta,0,0,y) = 0$,

2 c) $\dfrac{\partial h_2}{\partial x}(\delta,0,0,y) = y^{-2} = 1 + O(y-1)$, *2 d)* $\dfrac{\partial h_1}{\partial y}(\delta,0,0,y) = -y^{-2} = -1 + O(y-1)\,,$

Proof. Items a) and b) follow from the compatibility relation (3) plus the fact that the stable and unstable manifolds of $P_{\delta,0}$ form a saddle-connection. These two items in turn imply c) and d) using the symplectic character of $G_{\delta,\mu}$. $\square$

The following quantity will be related to the angle at the homoclinic point $H_{\delta,\mu}$. Define

$$\theta = \theta_\delta = -\frac{\partial^2 g_1}{\partial\mu\,\partial x}(\delta,0,1,0) = -\frac{\partial^2 h_2}{\partial\mu\,\partial y}(\delta,0,0,1)\,. \tag{4}$$

The equality between these two partial derivatives comes from the symplectic character of G.

Lemma 6.2. *The function $\delta\mapsto\theta_\delta$ is strictly positive and vanishes with all its derivatives at $\delta=0$. Moreover there is a constant $C>0$ and some open set $N\in\mathcal{N}$ such that for all $(\delta,\mu)\in N$ and all x or y in $[\lambda^{-1},\lambda]$, with $\lambda=\lambda_{\delta,\mu}(0)$,*

1 a) $|\,g_1(\delta,\mu,x,0)\,| \le C\,\theta_\delta\,\mu\,\log\lambda\,,$ *1 b)* $|\,h_2(\delta,\mu,0,y)\,| \le C\,\theta_\delta\,\mu\,\log\lambda\,,$

2 a) $\left|\dfrac{\partial g_1}{\partial x}(\delta,\mu,x,0)\right| \le C\,\theta_\delta\,\mu\,,$ *2 b)* $\left|\dfrac{\partial h_2}{\partial y}(\delta,\mu,0,y)\right| \le C\,\theta_\delta\,\mu\,,$

3 a) $\left|\dfrac{\partial^2 g_1}{\partial x^2}(\delta,\mu,x,0)\right| \le C\,\dfrac{\theta_\delta\,\mu}{\log\lambda}\,,$ *3 b)* $\left|\dfrac{\partial^2 h_2}{\partial y^2}(\delta,\mu,0,y)\right| \le C\,\dfrac{\theta_\delta\,\mu}{\log\lambda}\,.$

Proof. Define $\sigma_{\delta,\mu}(t) = \int_0^t \log\lambda_{\delta,\mu}(s)\,ds$ and $H_{\delta,\mu}(x,y) = \sigma_{\delta,\mu}(x\,y)$. The flow of the Hamiltonian vector field $J\,\nabla H_{\delta,\mu} = (x\,\log\lambda_{\delta,\mu}(xy), -y\,\log\lambda_{\delta,\mu}(xy))$ is easily seen to be $\phi^t_{\delta,\mu}(x,y) = \left(\lambda_{\delta,\mu}(xy)^t\,x, \lambda_{\delta,\mu}(xy)^{-t}y\right)$. Thus items 1. and 2. of proposition 5.2 are automatically fulfilled. We will compute the Melnikov function in two different ways using this proposition.

First consider the homoclinic solution $q_\delta(t) = \left(0,\lambda_{\delta,0}(0)^{-t}\right)$, which satisfies $q_{\delta,0}(0) = (0,1)$, and define $\gamma^u_{\delta,\mu}(t) = G_{\delta,\mu}\left(\lambda_{\delta,\mu}(0)^t,0\right)$ and $\gamma^s_{\delta,\mu}(t) = \left(0,\lambda_{\delta,\mu}(0)^{-t}\right)$. Clearly these parametrizations satisfy the conditions 3. and 4. of proposition 5.2. Define also

$$\zeta_{\delta,\mu}(t) = g_1\left(\delta,\mu,\lambda_{\delta,\mu}(0)^t,0\right)\,g_2\left(\delta,\mu,\lambda_{\delta,\mu}(0)^t,0\right)\,.$$

Then $H_{\delta,\mu}(\gamma^s_{\delta,\mu}(t)) = 0$ and

$$M_\delta(t) = \frac{\partial}{\partial\mu}\left(H_{\delta,\mu}(\gamma^u_{\delta,\mu}(t))\right)_{\mu=0} = \frac{\partial}{\partial\mu}\left(\sigma_{\delta,\mu}\circ\zeta_{\delta,\mu}\,(t)\right)_{\mu=0}\,. \tag{Ma}$$

Alternatively consider the homoclinic solution $q_\delta(t) = \left(\lambda_{\delta,0}(0)^t,0\right)$, which satisfies $q_{\delta,0}(0) = (1,0)$, and define $\gamma^u_{\delta,\mu}(t) = \left(\lambda_{\delta,\mu}(0)^t,0\right)$ and $\gamma^s_{\delta,\mu}(t) = G^{-1}_{\delta,\mu}\left(0,\lambda_{\delta,\mu}(0)^{-t}\right)$. Again these parametrizations satisfy the conditions 3. and 4. of proposition 5.2. Thus defining

$$\eta_{\delta,\mu}(t) = h_1\left(\delta,\mu,0,\lambda_{\delta,\mu}(0)^{-t}\right)\,h_2\left(\delta,\mu,0,\lambda_{\delta,\mu}(0)^{-t}\right)\,,$$

in this case $H_{\delta,\mu}(\gamma^u_{\delta,\mu}(t)) = 0$ and

$$M_\delta(t) = \frac{\partial}{\partial\mu}\left(-H_{\delta,\mu}(\gamma^s_{\delta,\mu}(t))\right)_{\mu=0} = -\frac{\partial}{\partial\mu}\left(\sigma_{\delta,\mu}\circ\eta_{\delta,\mu}\,(t)\right)_{\mu=0}\,. \tag{Mb}$$

Next we are going to exploit these relations to translate the assumptions on the Melnikov function into conditions on the components g_1, g_2 of $G_{\delta,\mu}$ and h_1, h_2 of $G_{\delta,\mu}^{-1}$.

Actually we will only estimate 1a, 2a and 3a out of the relation (Ma). An entirely analogous calculation proves 1b,2b and 3b from the relation (Mb). Writing $\lambda = \lambda_{\delta,\mu}(0)$ we have

$$\begin{aligned} M_\delta(t) &= \sigma'_{\delta,0}\left(\zeta_{\delta,0}(t)\right) \frac{\partial \zeta}{\partial \mu}(\delta,0,t) + \frac{\partial \sigma}{\partial \mu}\left(\delta,0,\zeta_{\delta,0}(t)\right) \\ &= \log\lambda \left(\frac{\partial g_1}{\partial \mu}(\delta,0,\lambda^t,0)\, g_2(\delta,0,\lambda^t,0) + g_1(\delta,0,\lambda^t,0)\, \frac{\partial g_2}{\partial \mu}(\delta,0,\lambda^t,0) \right) && (5) \\ &= \lambda^{-t} \log\lambda \,\frac{\partial g_1}{\partial \mu}(\delta,0,\lambda^t,0) && (6) \end{aligned}$$

Notice that by lemma 6.1 $g_1(\delta,0,\lambda^t,0)=0$, which implies $\zeta_{\delta,0}(t)=0$. By definition of $\sigma_{\delta,\mu}$, $\sigma'_{\delta,0}(\zeta_{\delta,0}(t)) = \sigma'_{\delta,0}(0) = \log\lambda$. Notice also that

$$\frac{\partial \sigma}{\partial \mu}(\delta,0,\zeta_{\delta,0}(t)) = \frac{\partial \sigma}{\partial \mu}(\delta,0,0) = 0$$

because $\sigma(\delta,\mu,0)=0$ for all μ.

$$\begin{aligned} M'_\delta(t) &= -\lambda^{-t}\log^2\lambda\ \frac{\partial g_1}{\partial \mu}(\delta,0,\lambda^t,0)\ +\ \log^2\lambda\ \frac{\partial^2 g_1}{\partial \mu \partial x}(\delta,0,\lambda^t,0) \\ &= \log^2\lambda\ \left(\frac{\partial^2 g_1}{\partial \mu \partial x}(\delta,0,\lambda^t,0) - \lambda^{-t}\frac{\partial g_1}{\partial \mu}(\delta,0,\lambda^t,0) \right) && (7) \end{aligned}$$

For $t=0$ one has $\dfrac{\partial g_1}{\partial \mu}(\delta,0,1,0)=0$ since $g_1(\delta,\mu,1,0)=0$ for all μ. Thus by (4)

$$0 > M'_\delta(0) = \log^2\lambda\ \frac{\partial^2 g_1}{\partial \mu \partial x}(\delta,0,1,0) = -\theta_\delta\ \log^2\lambda\,,$$

which proves that $\theta_\delta > 0$. From remark 2.4 it follows that $\delta \mapsto \theta_\delta$ has all its derivatives vanishing at $\delta = 0$. Using the notation introduced in (1) we have for some constant $C>0$, $M_i(\delta) \le C\ |M'_\delta(0)| = C\,\theta_\delta\ \log^2\lambda$ for $i=0,1,2$. Define now,

$$\Lambda_i(\delta) = \max_{\lambda^{-1}\le x\le \lambda} \left| \frac{\partial^{i+1} g_1}{\partial \mu \partial x}(\delta,0,x,0) \right| \qquad i=0,1,2\,.$$

Equality (5) implies that

$$\Lambda_0(\delta) \le \frac{M_0(\delta)}{\log\lambda} \le C\,\theta_\delta\ \log\lambda\,,$$

which implies 1a, taking μ small enough. From (7) we get

$$\Lambda_1(\delta) \le \frac{M_1(\delta)}{\log^2\lambda} + \Lambda_0(\delta) \le C'\,\theta_\delta\,,$$

implying that 2a holds for all $(\delta,\mu)\in N$ in some small $N \in \mathcal{N}$. Finally, differentiating (7) we have

$$M''_\delta(t) = \log^3\lambda\ \left(\lambda^t\ \frac{\partial^3 g_1}{\partial \mu \partial x^2} - \frac{\partial^2 g_1}{\partial \mu \partial x} + \lambda^{-t}\frac{\partial g_1}{\partial \mu} \right)(\delta,0,\lambda^t,0)\,,$$

and therefore

$$\Lambda_2(\delta) \leq \frac{M_2(\delta)}{\log^3 \lambda} + \Lambda_1(\delta) + \Lambda_0(\delta) \leq C'' \frac{\theta_\delta}{\log \lambda} \ ,$$

which implies 3a. ☐

Lemma 6.3. *For some constant $C > 0$ and some small enough open set $N \in \mathcal{N}$ the following inequalities hold for all $(\delta, \mu) \in N$ and all $t \in [0,1]$,*

(1) $\log \lambda_{\delta,\mu}(t) \geq C^{-1} \delta$,

(2) $\left|\lambda'_{\delta,\mu}(t)\right| \leq C\,\delta$,

(3) $\left|\lambda''_{\delta,\mu}(t)\right| \leq C\,\delta$,

where $\lambda = \lambda_{\delta,\mu}(0)$.

Proof. Let $\phi_\delta^t(x,y) = \left(\Lambda_\delta(xy)^t\, x, \Lambda_\delta(xy)^{-t} y\right)$ be the Birkhoff normal form, around P_δ, of X_δ's Hamiltonian flow. We have $f_{\delta,0} = \phi_\delta^\delta$, the time δ map of X_δ. Therefor $\lambda_{\delta,0}(t) = \exp\left(\delta\,\Lambda_\delta(t)\right)$ and for some constant $C > 0$ we must have, $\log \lambda_{\delta,0}(t) \geq 2\,C^{-1}\delta$, $\left|\lambda'_{\delta,0}(t)\right| \leq C\,\delta/2$ and $\left|\lambda''_{\delta,0}(t)\right| \leq C\,\delta/2$. Thus taking $N \in \mathcal{N}$ small enough the inequalities above hold for all $(\delta,\mu) \in N$ with C instead of $C/2$. ☐

We define the *half return time* by

$$n(\delta,\mu) = \text{ the integer part of } \frac{-\log\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)}{2\,\log\lambda} \ , \tag{8}$$

where $\lambda = \lambda_{\delta,\mu}(0)$. Because θ_δ has infinite zero jet at $\delta = 0$, $\mu\,\theta_\delta\,\log^{3/2}\lambda \to 0$ as $(\delta,\mu) \to (0,0)$ and therefore $\lim_{(\delta,\mu)\to(0,0)} n(\delta,\mu) = +\infty$.

Lemma 6.4. *There is $N \in \mathcal{N}$ such that, if we write $\lambda = \lambda_{\delta,\mu}(0)$ and $n = n(\delta,\mu)$, then we have*

(1) $\lambda^{-2\,n} = (1 + O(\delta))\ \mu\,\theta_\delta\,\log^{3/2}\lambda$,

(2) $n\,\mu\,\theta_\delta = o\left(\sqrt{\mu\,\theta_\delta}\right)$,

(3) $\left(\dfrac{\lambda_{\delta,\mu}(t)}{\lambda_{\delta,\mu}(0)}\right)^{2\,n} = 1 + O\left(\sqrt{\mu\,\theta_\delta}\right)$, *for* $0 \leq t \leq \lambda_{\delta,\mu}(t)^{-2\,n}$.

Proof. By definition of $n(\delta,\mu)$ there is some $s \in [0,1]$ such that

$$n + s = \frac{-\log\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)}{2\,\log\lambda}$$

which implies

$$\lambda^{-2\,(n+s)} = \lambda^{-2\,n}\lambda^{-2\,s} = \mu\,\theta_\delta\,\log^{3/2}\lambda \ .$$

Since $\lambda^{2\,s} = 1 + O(\delta)$, this proves item 1.

For the second item it is enough proving that

$$\lim_{(\delta,\mu)\to(0,0)} n(\delta,\mu)\,\sqrt{\mu\,\theta_\delta} = 0 \ .$$

Because θ_δ has infinite zero jet at $\delta = 0$, we see that

$$\frac{\sqrt{\mu\,\theta_\delta}}{2\,\log\lambda} \ll \left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)^{1/4} \;.$$

Thus

$$n\,\sqrt{\mu\,\theta_\delta} \ll -\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)^{1/4}\log\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)\;,$$

which converges to zero as $(\delta,\mu)\to(0,0)$. Note that $\lim_{x\to 0} x^{1/4}\log x = 0$.

If $\delta > 0$ is small enough, because by lemma 6.3 the variation of $\lambda_{\delta,\mu}$ is small, there is some small constant $d > 0$ such that $\lambda_{\delta,\mu}(t)^{-1} \le \lambda_{\delta,\mu}(0)^{-d}$. Therefor

$$\begin{aligned}
\left|\log\left(\frac{\lambda_{\delta,\mu}(t)}{\lambda_{\delta,\mu}(0)}\right)^{2\,n}\right| &\le 2\,n\,\left|\log\lambda_{\delta,\mu}(t) - \log\lambda_{\delta,\mu}(0)\right| \\
&\le 2\,n\,\frac{\left|\lambda'_{\delta,\mu}\right|}{\lambda_{\delta,\mu}}\,t \le 2\,n\,C\,\delta\,\lambda_{\delta,\mu}(t)^{-2\,n} \\
&\le 2\,n\,C^2\,\log\lambda_{\delta,\mu}(0)\,\left(\lambda_{\delta,\mu}(0)^{-2\,n}\right)^d \\
&\le -C^2\,\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)^d\log\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right) \\
&\le C^2\,(\mu\,\theta_\delta)^{2\,d/3} \to 0
\end{aligned}$$

Thus for small $N \in \mathcal{N}$, $0 \le t \le \lambda_{\delta,\mu}(t)^{-2\,n} \le 2\,\lambda_{\delta,\mu}(0)^{-2\,n} \ll \delta$ which means that we may take $d > 0$ above to be much closer to 1 and still have $\lambda_{\delta,\mu}(t)^{-1} \le \lambda_{\delta,\mu}(0)^{-d}$. For instance if we take $d = \frac{3}{4}$ we will end up with

$$\left|\log\left(\frac{\lambda_{\delta,\mu}(t)}{\lambda_{\delta,\mu}(0)}\right)^{2\,n}\right| \le (\mu\,\theta_\delta)^{\frac{2}{3}\,\frac{3}{4}} = \sqrt{\mu\,\theta_\delta}\;.$$

□

Latter, in section 8, we will find disjoint rectangles $S_{\delta,\mu}(0) \subseteq [0,\lambda^{-n+1}]^2$ and

$$S_{\delta,\mu}(1) \subseteq \left\{(x,y)\;:\;0 \le y < \lambda^{-n+1}\quad\text{and}\quad |x-1| \le 2\,\frac{\log^{3/2}\lambda}{\lambda^{2\,n}}\right\}\;,$$

with $n = n(\delta,\mu)$, $\lambda = \lambda_{\delta,\mu}(0)$, and define the *Return Map* as

$$T_{\delta,\mu}(x,y) = \begin{cases} L_{\delta,\mu}(x,y) & \text{if } (x,y)\in S_{\delta,\mu}(0) \\ L^n_{\delta,\mu}\circ G_{\delta,\mu}\circ L^n_{\delta,\mu}(x,y) & \text{if } (x,y)\in S_{\delta,\mu}(1) \end{cases}\;.$$

7. **Rescaling Coordinates**

In this section we introduce coordinates that will be used to scale the domain of the return map up to the unit square. For all $(\delta,\mu)\in N$ in some small enough $N\in\mathcal{N}$ we define the scaling map $\Phi_{\delta,\mu}:\mathbb{R}^2\to\mathbb{R}^2$

$$\Phi_{\delta,\mu}(x,y) = \left(\lambda_{\delta,\mu}(xy)^{n(\delta,\mu)}x\,,\;\lambda_{\delta,\mu}(xy)^{n(\delta,\mu)}y\right)$$

The product of $\Phi_{\delta,\mu}$'s components, $\lambda_{\delta,\mu}(xy)^{2\,n} x\,y$, is a function of the product $x\,y$. Therefor the inverse map is $\Phi_{\delta,\mu}^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$\Phi_{\delta,\mu}^{-1}(x,y) = \left(\lambda_{\delta,\mu}(t_{\delta,\mu}(xy))^{-n(\delta,\mu)} x\,,\, \lambda_{\delta,\mu}(t_{\delta,\mu}(xy))^{-n(\delta,\mu)} y \right)$$

where $t_{\delta,\mu}(s)$ is defined implicitly by $t_{\delta,\mu}(0) = 0$ and

$$\lambda_{\delta,\mu}(t_{\delta,\mu}(s))\, t_{\delta,\mu}(s) = s \qquad \text{for all} \quad s \in [0,2]\ . \tag{9}$$

Lemma 7.1. *For some constant $C > 0$, some open set $N \in \mathcal{N}$ and all $s \in [0,2]$, $|t_{\delta,\mu}(s)|$, $|t'_{\delta,\mu}(s)|$ and $|t''_{\delta,\mu}(s)| \leq C\,\mu\,\theta_\delta\, \log^{3/2} \lambda_{\delta,\mu}(0)$.*

Proof. By lemma 6.4, items 1. and 3. , we have

$$\begin{aligned} 0 \leq t_{\delta,\mu}(s) &\leq s\,\lambda_{\delta,\mu}(t_{\delta,\mu}(s))^{-2\,n} \leq 2\,\left(1 + O(\sqrt{\mu\,\theta_\delta})\right)\, \lambda_{\delta,\mu}(0)^{-2\,n} \\ &\leq C\,\mu\,\theta_\delta\, \log^{3/2} \lambda\ . \end{aligned}$$

Differentiating relation 9 , we obtain

$$2\,n\,\lambda\,(t(s))^{2\,n-1}\,\lambda'\,(t(s))\; t'(s)\, t(s) + \lambda\,(t(s))^{2\,n}\; t'(s) = 1\ ,$$

and therefore,

$$(*) \qquad t'(s)\, \left(2\,n\, \frac{\lambda'\,(t(s))}{\lambda\,(t(s))}\, s + \lambda\,(t(s))^{2\,n} \right) = 1\ ,$$

which implies $0 \leq \dfrac{\lambda\,(t(s))^{2\,n}}{2}\, t'(s) \leq 1$, and so

$$0 < t'(s) \leq 2\,\lambda\,(t(s))^{-2\,n} \leq C\,\mu\,\theta_\delta\, \log^{3/2} \lambda\ .$$

Differentiating now $(*)$ we obtain

$$t''(s)\; \left(O(n) + \lambda\,(t(s))^{2\,n} \right) + t'(s)\, O(n) = 0\ ,$$

which implies

$$\begin{aligned} \left| t''(s) \right| &\leq \frac{O(n)\, t'(s)}{\lambda\,(t(s))^{2\,n} + O(n)} \leq \frac{C\,n\,\mu\,\theta_\delta}{\lambda\,(t(s))^{2\,n}} \\ &\leq C\,\mu\,\theta_\delta\, \log^{3/2} \lambda\ . \end{aligned}$$

$\square$

We now scale the return map to the unit square, setting $\tilde{T}_{\delta,\mu} = \Phi_{\delta,\mu} \circ T_{\delta,\mu} \circ \Phi_{\delta,\mu}^{-1}$.

The *first branch* is $\tilde{L}_{\delta,\mu} = \Phi_{\delta,\mu} \circ L_{\delta,\mu} \circ \Phi_{\delta,\mu}^{-1}$. We will analyze it in the larger square $\tilde{S}_0 = [0,2]^2$. Define

$$\tilde{\lambda}(s) = \tilde{\lambda}_{\delta,\mu}(s) = \tilde{\lambda}(\delta,\mu,s) := \lambda_{\delta,\mu} \circ t_{\delta,\mu}\,(s)\ .$$

Then it is easily verified that

$$\tilde{L}_{\delta,\mu}(x,y) = \left(\tilde{\lambda}_{\delta,\mu}(x\,y)\, x\,,\, \tilde{\lambda}_{\delta,\mu}(x\,y)^{-1}\, y \right)\ . \tag{10}$$

The *second branch* of $\tilde{T}_{\delta,\mu}$ is just,

$$\tilde{G}_{\delta,\mu} = \Phi_{\delta,\mu} \circ L^n_{\delta,\mu} \circ G_{\delta,\mu} \circ L^n_{\delta,\mu} \circ \Phi_{\delta,\mu}^{-1}\ .$$

To compute $\tilde{G}_{\delta,\mu}$ and its inverse $\tilde{G}^{-1}_{\delta,\mu}$ we need the auxiliary functions

$$p(x,y) = p_{\delta,\mu}(x,y) \;:=\; g_1\left(x, \tilde{\lambda}(xy)^{-2\,n} y\right)\; g_2\left(x, \tilde{\lambda}(xy)^{-2\,n} y\right)$$

$$r(x,y) = r_{\delta,\mu}(x,y) \;:=\; h_1\left(\tilde{\lambda}(xy)^{-2\,n} x, y\right)\; h_1\left(\tilde{\lambda}(xy)^{-2\,n} x, y\right)$$

where $\tilde{\lambda} = \tilde{\lambda}_{\delta,\mu}$, $g_1(\cdot,\cdot) = g_1(\delta,\mu,\cdot,\cdot)$, $g_2(\cdot,\cdot) = g_2(\delta,\mu,\cdot,\cdot)$, $h_1(\cdot,\cdot) = h_1(\delta,\mu,\cdot,\cdot)$, $h_2(\cdot,\cdot) = h_2(\delta,\mu,\cdot,\cdot)$ and $n = n(\delta,\mu)$. Then, with these notation, we do the substitutions and compute the following expressions,

$$\tilde{G}_{\delta,\mu}(x,y) = \left((\lambda \circ p\,(x,y))^{2\,n}\; g_1\left(x, \tilde{\lambda}(xy)^{-2\,n} y\right) ,\, g_2\left(x, \tilde{\lambda}(xy)^{-2\,n} y\right) \right) \tag{11}$$

$$\tilde{G}^{-1}_{\delta,\mu}(x,y) = \left(h_1\left(\tilde{\lambda}(xy)^{-2\,n} x, y\right) ,\, (\lambda \circ r\,(x,y))^{2\,n}\; h_2\left(\tilde{\lambda}(xy)^{-2\,n} x, y\right) \right) . \tag{12}$$

Notice that for all (δ,μ),

$$\tilde{G}_{\delta,\mu}(1,0) = (0,1)\,. \tag{13}$$

We will analyze $G_{\delta,\mu}$ and $G^{-1}_{\delta,\mu}$ respectively on the following rectangles

$$\tilde{S}_1 = \left\{ (x,y)\,:\, |x-1| \le 2\,\log^{3/2}\lambda \quad \text{and} \quad 0 \le y \le 2 \right\}\,,$$

$$\tilde{S}'_1 = \left\{ (x,y)\,:\, 0 \le x \le 2 \quad \text{and} \quad |y-1| \le 2\,\log^{3/2}\lambda \right\}\,,$$

to which all subsequent estimates on the derivatives of $G_{\delta,\mu}$ and $G^{-1}_{\delta,\mu}$ will refer.

Lemma 7.2. *For some small enough $N \in \mathcal{N}$ and some constant $C > 0$ we have for all $(x,y) \in \tilde{S}_1$*

1. $|p(x,y)| \le C\,\mu\,\theta_\delta\,\log^{3/2}\lambda$
2. $\left|\dfrac{\partial p}{\partial x}(x,y)\right| \le C\,\mu\,\theta_\delta$
3. $\left|\dfrac{\partial p}{\partial y}(x,y)\right| \le C\,\mu\,\theta_\delta\,\log^{3/2}\lambda$
4. $\left|\dfrac{\partial^2 p}{\partial x^2}(x,y)\right| \le C\,\dfrac{\mu\,\theta_\delta}{\log\lambda}$

where, as usual, $\lambda = \lambda_{\delta,\mu}(0)$. Similar inequalities hold for all $(x,y) \in \tilde{S}'_1$ if we change p to r and each derivative with respect to $\frac{\partial}{\partial x}$ by the correspondent derivative with respect to $\frac{\partial}{\partial y}$.

Proof. Once again because the two cases are entirely similar we will only prove the inequalities corresponding to the function $p(x,u)$. Consider the differential operators Id, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial^2}{\partial x^2}$. If D stands for any of these operators and $i = 1,2$ then of course $\left|Dg_i(\delta,\mu,x,\tilde{\lambda}(xy)^{-2n}y)\right|$ is bounded by some constant independent of (δ,μ). On the other hand for g_1 and the ∂x derivatives we can get much better estimates using the mean value theorem,

$$\begin{aligned} Dg_1(\delta,\mu,x,\tilde\lambda^{-2n}y) &= Dg_1(\delta,\mu,x,0) + \frac{\partial}{\partial y}\,(Dg_1)\,(\cdots)\tilde\lambda^{-2n}y \\ &= Dg_1(\delta,\mu,x,0) + O\left(\tilde\lambda^{-2n}\right) \\ &= Dg_1(\delta,\mu,x,0) + O\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)\,, \end{aligned} \tag{14}$$

and the correspondent estimates of lemma 6.2. In particular we see at once that item 1a must hold. To estimate higher derivatives note that, using lemmas 6.3 and 7.1, we have

$$\left|\tilde\lambda'_{\delta,\mu}(s)\right| = \left|\lambda'_{\delta,\mu}(s)\,t'(s)\right| = O(\mu\,\theta_\delta\,\log^{3/2}\lambda) \qquad \text{and} \tag{15}$$

$$\left|\tilde\lambda''_{\delta,\mu}(s)\right| \le \left|\lambda''_{\delta,\mu}(s)\right|\,t'(s)^2 + \left|\lambda'(t(s))\,t''(s)\right| = O(\mu\,\theta_\delta\,\log^{3/2}\lambda)\,. \tag{16}$$

Therefor

$$\begin{aligned} \frac{\partial p}{\partial x}(x,y) &= \frac{\partial g_1}{\partial x}(x,\tilde\lambda^{-2n}y)\;g_2(x,\tilde\lambda^{-2n}y) + \cdots \\ &\sim \frac{\partial g_1}{\partial x}(x,0) + \cdots = O(\mu\,\delta)\,, \end{aligned}$$

where above, and below during this proof, the dots mean terms of lesser order.

$$\begin{aligned} \frac{\partial p}{\partial y}(x,y) &= \frac{\partial g_1}{\partial y}(x,\tilde\lambda^{-2n}y)\;g_2(x,\tilde\lambda^{-2n}y)\,\frac{\partial}{\partial y}\left(\tilde\lambda^{-2n}y\right) + \cdots \\ &\sim \frac{\partial}{\partial y}\left(\tilde\lambda^{-2n}y\right) + \cdots = \tilde\lambda^{-2n}\left(1 - 2\,n\,y\,\frac{\tilde\lambda'}{\tilde\lambda}\right) + \cdots \\ &= O(\tilde\lambda^{-2n}) = O(\mu\,\theta_\delta\,\log^{3/2}\lambda) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 p}{\partial x^2}(x,y) &= \frac{\partial^2 g_1}{\partial x^2}(x,\tilde\lambda^{-2n}y)\;g_2(x,\tilde\lambda^{-2n}y) + \cdots \\ &\sim \frac{\partial^2 g_1}{\partial x^2}(x,0) + \cdots = O\left(\frac{\mu\,\theta_\delta}{\log\lambda}\right) \end{aligned}$$

We have been using above the fact that $O(\tilde\lambda^{-2n}) = O\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)$ which follows from item 1 of lemma 6.4. We have also used also the relations $g_2 \sim 1$ and $\frac{\partial g_1}{\partial y} \sim 1$ which follow from items 1b and 1d of lemma 6.1.

□

8. **The Basic Set**

The goal of this section is to prove that the return map $\tilde T_{\delta,\mu}$ belongs to some class $\mathcal{F}\,(\,\epsilon(\delta,\mu),\,\gamma(\delta,\mu)\,)$, see definition 4.1, where $\epsilon(\delta,\mu)$ and $\gamma(\delta,\mu)$ tend to zero as $(\delta,\mu)\to(0,0)$. Notice that since both branches of the return map $\tilde T_{\delta,\mu}$ are defined as rescaled iterates of $f_{\delta,\mu}$, the corresponding basic set $\Lambda_{\delta,\mu}$, viewed in the coordinates $\Phi_{\delta,\mu}$, is also part of a basic set of the diffeomorphism $f_{\delta,\mu}$, namely $\cup_{i=0}^{2\,n} f^{\,i}_{\delta,\mu}\,(\Lambda_{\delta,\mu})$, where $n = n(\delta,\mu)$. Using the distortion bounds of theorem 2, we prove that $\lim_{(\delta,\mu)\to(0,0)} \tau_{LR}(\Lambda_{\delta,\mu}) = +\infty$.

Let us import the notation used in definition 4.1 and write the Jacobian matrices of $\tilde{L}_{\delta,\mu}$ and its inverse $\tilde{L}_{\delta,\mu}^{-1}$ as

$$D\tilde{L}_{\delta,\mu} = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \quad \text{and} \quad D\tilde{L}_{\delta,\mu}^{-1} = \begin{pmatrix} \tilde{d}_0 & -\tilde{b}_0 \\ -\tilde{c}_0 & \tilde{a}_0 \end{pmatrix} ,$$

over the square $\tilde{S}_0 = [0,2]^2$.

Lemma 8.1. *There is $N \in \mathcal{N}$ such that for all $(\delta,\mu) \in N$ and $(x,y) \in [0,2]^2$, all components of the difference matrices*

$$D\tilde{L}_{\delta,\mu}(x,y) - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \textit{and} \quad D\tilde{L}_{\delta,\mu}^{-1}(x,y) - \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} ,$$

with $\lambda = \lambda_{\delta,\mu}(0)$, are of order $O\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)$.

Proof. Differentiating (10) we obtain the expressions,

$$\begin{aligned}
a_0(x,y) &= \tilde{a}_0(x,y) = \tilde{\lambda}(xy) + \tilde{\lambda}'(xy)\,x\,y = \tilde{\lambda}(xy) + \cdots \\
b_0(x,y) &= \tilde{\lambda}'(xy)\,x^2 = \cdots \\
\tilde{b}_0(x,y) &= \frac{\tilde{\lambda}'(xy)}{\tilde{\lambda}(xy)}\,x^2 = \cdots \\
c_0(x,y) &= -\frac{\tilde{\lambda}'(xy)}{\tilde{\lambda}(xy)}\,y^2 = \cdots \\
\tilde{c}_0(x,y) &= -\tilde{\lambda}'(xy)x^2 = \cdots \\
d_0(x,y) &= \tilde{d}_0(x,y) = \tilde{\lambda}(xy)^{-1} - \frac{\tilde{\lambda}'(xy)}{\tilde{\lambda}(xy)^2}\,x\,y = \tilde{\lambda}(xy)^{-1} + \cdots \quad .
\end{aligned}$$

Using the estimates (15) and (16) in the proof of lemma 7.2 we see that all terms involving $\tilde{\lambda}'(xy)$ are of order $O\left(\mu\,\theta_\delta\,log^{3/2}\lambda\right)$. Since, by the mean value theorem, the difference $\tilde{\lambda}_{\delta,\mu}(xy) - \lambda_{\delta,\mu}(0) = \tilde{\lambda}_{\delta,\mu}'(c)\,x\,y$ is of the same order $O\left(\mu\,\theta_\delta\,log^{3/2}\lambda\right)$ the lemma follows. □

Lemma 8.2. *There is $N \in \mathcal{N}$ such that, for $(\delta,\mu) \in N$ and $(x,y) \in [0,2]^2$, all second order partial derivatives of a_0 , b_0 , c_0 , d_0 , $\tilde{a}_0$, $\tilde{b}_0$, $\tilde{c}_0$ and $\tilde{d}_0$, are of order $O\left(\mu\,\theta_\delta\,\log^{3/2}\lambda\right)$.*

Proof. All second derivatives of a_0 , b_0 , c_0 , d_0 , $\tilde{a}_0$, $\tilde{b}_0$, $\tilde{c}_0$ and $\tilde{d}_0$ are sums of terms which involve one of the derivatives $\tilde{\lambda}'(xy)$ or $\tilde{\lambda}''(xy)$. Therefor we just have to apply the estimates (15) and (16) in the proof of lemma 7.2. □

Write the Jacobian matrices of $\tilde{G}_{\delta,\mu}$ and $\tilde{G}_{\delta,\mu}^{-1}$ as

$$D\tilde{G}_{\delta,\mu} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad D\tilde{G}_{\delta,\mu}^{-1} = \begin{pmatrix} \tilde{d}_1 & -\tilde{b}_1 \\ -\tilde{c}_1 & \tilde{a}_1 \end{pmatrix} .$$

Lemma 8.3. *There is some small enough* $N \in \mathcal{N}$ *such that for all* $(\delta,\mu) \in N$ *and all* $(x,y) \in \tilde{S}_1$ *,*

$$D\tilde{G}_{\delta,\mu}(x,y) = \begin{pmatrix} -\log^{-3/2}\lambda & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} O(\log^{-1}\lambda) & o(1) \\ o(1) & o(1) \end{pmatrix} \quad ,$$

and for all $(x,y) \in \tilde{S}_1'$ *,*

$$D\tilde{G}^{-1}_{\delta,\mu}(x,y) = \begin{pmatrix} 0 & 1 \\ -1 & -\log^{-3/2}\lambda \end{pmatrix} + \begin{pmatrix} o(1) & o(1) \\ o(1) & O(\log^{-1}\lambda) \end{pmatrix} \quad .$$

where $\lambda = \lambda_{\delta,\mu}(0)$ *, and all the* $o(1)$ *components tend to zero as* $(\delta,\mu) \to (0,0)$ *.*

Proof. In this proof the dots "$\cdots$" will stand for any term, or sum of terms, converging to zero as $(\delta,\mu) \to (0,0)$. The first component of $\tilde{G}_{\delta,\mu}$, see (11), is

$$\tilde{g}_1(\delta,\mu,x,y) = \lambda(p(x,y))^{2\,n}\, g_1\left(x, \tilde{\lambda}(xy)^{-2\,n} y\right) \; .$$

Combining lemma 7.2 with the estimate for $g_1(x,\tilde{\lambda}(xy)^{-2n}y)$, obtained in (14) during the proof of the same lemma, we see that both

$$\frac{\partial}{\partial x}\left(\lambda(p(x,y))^{2n}\right) g_1\left(x, \tilde{\lambda}^{-2\,n} y\right) \quad \text{and} \quad \frac{\partial}{\partial y}\left(\lambda(p(x,y))^{2n}\right) g_1\left(x, \tilde{\lambda}^{-2\,n} y\right)$$

converge to zero as $(\delta,\mu) \to (0,0)$. From the estimates (15) and (16), obtained in the proof of lemma 7.2, we get that all partial derivatives of $\tilde{\lambda}(xy)^{-2n}y$ converge to zero as $(\delta,\mu) \to (0,0)$. Thus,

$$\begin{aligned}
\frac{\partial \tilde{g}_1}{\partial x}(x,y) &= \lambda(p(x,y))^{2n}\,\frac{\partial g_1}{\partial x}(x,\tilde{\lambda}^{-2n}y) + \cdots \\
&= \lambda(p(x,y))^{2n}\left(\frac{\partial g_1}{\partial x}(x,0) + O(\tilde{\lambda}^{-2n})\right) + \cdots \\
&= \lambda(p(x,y))^{2n}\left(\frac{\partial g_1}{\partial x}(1,0) + \frac{\partial^2 g_1}{\partial x^2}(x^*,0)\,(x-1)\right) + O(1) + \cdots \\
&= \lambda_{\delta,\mu}(0)^{2n}\left(\frac{\partial g_1}{\partial x}(1,0) + O\left(\mu\,\theta_\delta\,\log^{1/2}\lambda\right)\right) + O(1) + \cdots \\
&= \lambda_{\delta,\mu}(0)^{2n}\,\frac{\partial g_1}{\partial x}(1,0) + O\left(\frac{1}{\log\lambda}\right) + \cdots \quad .
\end{aligned}$$

We have used above the estimate (14) in the proof of lemma 7.2. We did also use the definition of $\tilde{S}_1$ and item 3a of lemma 6.2 which give us that $(x-1) = O(\log^{3/2}\lambda)$ and $\frac{\partial^2 g_1}{\partial x^2}(x^*,0) = O\left(\frac{\mu\,\theta_\delta}{\log\lambda}\right)$.

On the other hand

$$\begin{aligned}
\frac{\partial g_1}{\partial x}(\delta,\mu,1,0) &= \frac{\partial g_1}{\partial x}(\delta,0,1,0) + \frac{\partial^2 g_1}{\partial\mu\partial x}(\delta,0,1,0)\,\mu + O(\mu^2) \\
&= -\mu\,\theta_\delta + O(\mu^2)
\end{aligned}$$

and if we assume that $N \in \mathcal{N}$ is such that $0 < \mu \le \theta_\delta\, \log^{1/2}\lambda$ for all $(\delta,\mu) \in N$, then

$$\frac{\partial \tilde{g}_1}{\partial x}(x,y) = -\frac{1}{\log^{3/2}\lambda} + O\left(\frac{1}{\log\lambda}\right) \; .$$

For the other first order derivatives we combine lemma 6.1 with (14). The second derivative of $\tilde{g}_1(x,y)$ is

$$\begin{aligned}\frac{\partial \tilde{g}_1}{\partial y}(x,y) &= \lambda(p(x,y))^{2n}\,\frac{\partial g_1}{\partial y}(x,\tilde{\lambda}^{-2n}y)\,\tilde{\lambda}^{-2n}+\cdots\\ &= \frac{\partial g_1}{\partial y}(x,\tilde{\lambda}^{-2n}y)+\cdots\\ &= \frac{\partial g_1}{\partial y}(x,0)+O(\tilde{\lambda}^{-2n})+\cdots\\ &= 1+\cdots\ .\end{aligned}$$

Now the second component of $\tilde{G}_{\delta,\mu}$ is, see (11),

$$\tilde{g}_2(\delta,\mu,x,y)=g_2\left(x,\tilde{\lambda}(xy)^{-2\,n}y\right)\ .$$

Thus

$$\begin{aligned}\frac{\partial \tilde{g}_2}{\partial x}(x,y) &= \frac{\partial g_2}{\partial x}(x,\tilde{\lambda}^{-2n}y)+\cdots\\ &= \frac{\partial g_2}{\partial x}(x,0)+\cdots\\ &= -1+\cdots\ .\end{aligned}$$

and

$$\begin{aligned}\frac{\partial \tilde{g}_2}{\partial y}(x,y) &= \frac{\partial g_2}{\partial x}(x,\tilde{\lambda}^{-2n}y)\,\tilde{\lambda}^{-2n}+\cdots\\ &= O(\tilde{\lambda}^{-2n})+\cdots=\cdots\ .\end{aligned}$$

The derivatives of $\tilde{G}_{\delta,\mu}^{-1}$ are worked in the same way. ☐

Lemma 8.4. *There is some open set $N\in\mathcal{N}$ and some constant $C>0$ such that, for $(\delta,\mu)\in N$, all second order partial derivatives of a_1, b_1, c_1, d_1, except the one below, are bounded over $\tilde{S}_1$, in absolute value, by C. Similarly all second order partial derivatives of $\tilde{a}_1$, $\tilde{b}_1$, $\tilde{c}_1$ and $\tilde{d}_1$ except the one below, are bounded over $\tilde{S}_1'$, in absolute value, by C. The exceptional derivatives satisfy:*
$\left|\frac{\partial a_1}{\partial x}\right|\le\frac{C}{\log^{5/2}\lambda_{\delta,\mu}(0)}$ *over* $\tilde{S}_1$, *and* $\left|\frac{\partial \tilde{a}_1}{\partial x}\right|\le\frac{C}{\log^{5/2}\lambda_{\delta,\mu}(0)}$ *over* $\tilde{S}_1'$.

Proof. In this proof the dots "$\cdots$" will stand for any term, or sum of terms, bounded as $(\delta,\mu)\to(0,0)$. It is clear that all second derivatives of the second component $\tilde{g}_2(x,y)=g_2(x,\tilde{\lambda}^{-2n}y)$ are bounded, or converge to zero. Thus we only have to deal with the first component $\tilde{g}_1(x,y)=\lambda(p(x,y))^{2n}\,g_1(x,\tilde{\lambda}(xy)^{-2\,n}y)$.

All derivatives of the form

$$D\left(\lambda(p(x,y))^{2n}\right)\ g_1(x,\tilde{\lambda}(xy)^{-2\,n}y)$$

where $D=\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$ or $\frac{\partial^2}{\partial x\partial y}$ are bounded. Up to some multiplicative constant they can be dominated by $\lambda(p(x,y))^{2n}\,g_1(x,\tilde{\lambda}(xy)^{-2\,n}y)$ and this is easily seen to be bounded from item 1a of lemma 6.2, lemma 6.4, and (14) in the proof of lemma 7.2.

Also all derivatives of the form

$$D_1\ \left(\lambda(p(x,y))^{2n}\right)\ D_2\,\frac{\partial}{\partial y}\ \left(g_1(x,\tilde{\lambda}(xy)^{-2\,n}y)\right)\ ,$$

where D_1 and D_2 is any of Id, $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$, are bounded because the function $\frac{\partial}{\partial y}\left(\tilde\lambda(xy)^{-2n}\right)\,\lambda(p(x,y))^{2n}$ is bounded. Therefor all partial derivatives of $\tilde g_1$ with second order involving ∂y are bounded. It remains to analyze the following one

$$\begin{aligned}\frac{\partial^2\tilde g_1}{\partial x^2}(x,y) &= \lambda(p(x,y))^{2n}\,\frac{\partial^2 g_1}{\partial x^2}(x,\tilde\lambda^{-2n}y)+\cdots\\ &= \lambda_{\delta,\mu}(0)^{2n}\,\left(\frac{\partial^2 g_1}{\partial x^2}(x,0)+O(\tilde\lambda^{-2n})\right)+\cdots\\ &= \frac{1}{\mu\,\theta_\delta\,\log^{3/2}\lambda}\,O\left(\frac{\mu\,\theta_\delta}{\log\lambda}\right)+O(1)+\cdots\\ &= O\left(\frac{1}{\log^{5/2}\lambda}\right)\quad .\end{aligned}$$

The proof for the second derivatives of $\tilde G_{\delta,\mu}^{-1}$ is similar. $\square$

Lemma 8.5. *There is some $N\in\mathcal{N}$ and some constant $C>2$ such that defining $\epsilon(\delta,\mu)=\frac{3}{2}\log^{3/2}\lambda$ and $\gamma(\delta,\mu)=C\,\log^{1/2}\lambda$, where $\lambda=\lambda_{\delta,\mu}(0)$, then for all $(\delta,\mu)\in N$ the maps $\tilde L_{\delta,\mu}$ and $\tilde L_{\delta,\mu}^{-1}$ defined in $\tilde S_0$, the map $\tilde G_{\delta,\mu}$ defined in $\tilde S_1$, and the map $\tilde G_{\delta,\mu}^{-1}$ defined in $\tilde S_1'$ they all satisfy the conditions 2, 3 and 4 in the definition 4.1 of class $\mathcal{F}\,(\,\epsilon(\delta,\mu),\,\gamma(\delta,\mu)\,)$.*

Proof. Choose a constant $C_0>2$ according to lemma 8.4 and set $\gamma=3\,C_0\,\log^{1/2}\lambda$.

2. (b) On the first branch of $\tilde T$ combine lemma 8.1 with the inequality

$$\epsilon\,|a_0|\le 2\,\epsilon=3\,\log^{3/2}\lambda\ll 2\,.$$

On the second branch combine lemma 8.3 with the inequality

$$\epsilon\,|a_1|=\frac{3}{2}+O\left(\log^{1/2}\lambda\right)<2\,.$$

2. (c) On the first branch of $\tilde T$ combine lemma 8.1 with the inequality

$$\epsilon\,(|a_0|-1)\ge C^{-1}\log^{5/2}\lambda\gg\mu\,\theta_\delta\,\log^{3/2}\lambda\,.$$

On the second branch combine lemma 8.3 with the inequality

$$\epsilon\,(|a_1|-1)=\frac{3}{2}+O\left(\log^{1/2}\lambda\right)>1\,.$$

3. On the first branch of $\tilde T$ combine lemma 8.2 with the inequality

$$\gamma\,(|a_0|-1)\ge C^{-1}\log^{3/2}\lambda\gg\mu\,\theta_\delta\,\log^{3/2}\lambda\,.$$

On the second branch combine lemma 8.4 with the inequality

$$\gamma\,(|a_1|-1)=3\,C_0\,(\log^{1/2}\lambda)\,O\left(\frac{1}{\log^{3/2}\lambda}\right)=O\left(\frac{1}{\log\lambda}\right)\gg C_0\,,$$

for the subitems (a) and (b), or with the inequality

$$\gamma\,|a_1|\,(|a_1|-1) = \frac{3\,C_0}{\log^{5/2}\lambda} + O\left(\frac{1}{\log^2\lambda}\right) > \frac{C_0}{\log^{5/2}\lambda}\,,$$

for the subitems (c) and (d) .

4. On the first branch of $\tilde{T}$, if $\alpha_0 = \max\left\{|a_0(x,y)| \,:\, (x,y)\in[0,2]^2\right\}$, we have $\alpha_0 = \lambda + O(\mu\,\theta_\delta\,\log^{3/2}\lambda)$. Then for some constant $C>0$,

$$1-\alpha_0^{-1} \geq C^{-1}(\lambda-1) \geq C^{-1}\log\lambda \quad \text{and} \quad \gamma\left(1-\alpha_0^{-1}\right) \geq C^{-1}\log^{3/2}\lambda\,.$$

Given (x_1,y_1) and (x_2,y_2) in $[0,2]^2$, we get from lemma 8.2

$$\begin{aligned}\left|\log\frac{a_0(x_2,y_2)}{a_0(x_1,y_1)}\right| &\leq \max\left\{\left|\frac{\partial a_0}{\partial x}\right| + \left|\frac{\partial a_0}{\partial y}\right|\right\}\underbrace{\|(x_2-x_1,y_2-y_1)\|}_{\leq 2}\\ &\leq C\,\mu\,\theta_\delta\,\log^{3/2}\lambda \ll C^{-1}\,\log^{3/2}\lambda \leq \gamma\,\left(1-\alpha_0^{-1}\right)\,.\end{aligned}$$

On the second branch, setting $\alpha_1 = \max\left\{|a_1(x,y)| \,:\, (x,y)\in[0,2]^2\right\}$, we have $\alpha_1 = O\left(\log^{-3/2}\lambda\right)$ and therefore $\gamma\,(1-\alpha_1^{-1}) = \gamma\,\left(1-O(\log^{3/2}\lambda)\right) = 3\,C_0\,\log^{1/2}\lambda + O(\log^2\lambda)$. Take now two points $(x_1,y_1),(x_2,y_2)\in\tilde{S}_1$. By definition of $\tilde{S}_1$ we have $|x_1-x_2|\leq 2\,\log^{3/2}\lambda$ and $|y_1-y_2|\leq 2$, and applying the mean value theorem,

$$\begin{aligned}\left|\log\frac{a_1(x_2,y_2)}{a_1(x_1,y_1)}\right| &\leq \frac{1}{|a_1|}\left|\frac{\partial a_1}{\partial x}\right|\,|x_1-x_2| + \frac{1}{|a_1|}\left|\frac{\partial a_1}{\partial y}\right|\,|y_1-y_2|\\ &\leq \frac{1}{\frac{1}{\log^{3/2}\lambda} - O\left(\frac{1}{\log\lambda}\right)}\,\frac{C_0}{\log^{5/2}\lambda}\,2\,\log^{3/2}\lambda + O(\log^{3/2}\lambda)\\ &\leq \frac{2\,C_0\,\log^{1/2}\lambda}{1-O(\log^{1/2}\lambda)} + O(\log^{3/2}\lambda)\\ &< 3\,C_0\,\log^{1/2}\lambda + O(\log^2\lambda) = \gamma\,(1-\alpha_1^{-1})\,.\end{aligned}$$

□

Lemma 8.6. *For all (δ,μ) in some small $N\in\mathcal{N}$ the map $\tilde{G}_{\delta,\mu}$ has a fixed point $Q_{\delta,\mu} = (x_1,y_1)$ with both coordinates satisfying $x_1 = 1 + O(\mu\,\log^{3/2}\lambda)$, $y_1 = 1 + O(\mu\,\log^{3/2}\lambda)$. This fixed point is hyperbolic and there are C^1 functions $\Gamma^s,\Gamma^u:[0,2]\to\mathbb{R}$ such that:*

(1) *the graph $\{(x,\Gamma^u(x)) \,:\, x\in[0,2]\}$ is a local unstable manifold of Q and for all $x\in[0,2]$, $-\frac{3}{2}\,\log^{3/2}\lambda \leq \frac{d}{dx}\Gamma^u(x) \leq -\frac{2}{3}\,\log^{3/2}\lambda$.*

(2) *the graph $\{(\Gamma^s(y),y) \,:\, y\in[0,2]\}$ is a local stable manifold of Q and for all $y\in[0,2]$, $-\frac{3}{2}\,\log^{3/2}\lambda \leq \frac{d}{dy}\Gamma^s(y) \leq -\frac{2}{3}\,\log^{3/2}\lambda$.*

Proof. Let us begin by proving that

$$\tilde{G}_{\delta,\mu}(1,1) = \left(1 + O(\mu)\,,\, 1 + O(\mu\,\theta_\delta\,\log^{3/2}\lambda)\right)\,, \tag{1a}$$

and

$$\tilde{G}_{\delta,\mu}^{-1}(1,1) = \left(1 + O(\mu\,\theta_\delta\,\log^{3/2}\lambda)\,,\, 1 + O(\mu)\right)\,. \tag{1b}$$

Since these facts are proved in the same way we will restrict to (1a). The first component of $\tilde{G}_{\delta,\mu}(1,1)$ is

$$\begin{aligned}
\tilde{g}_1(1,1) &= \lambda(p(1,1))^{2n}\, g_1(1,\tilde{\lambda}(1)^{-2n}) \\
&= \lambda(p(1,1))^{2n}\left(\underbrace{g_1(1,0)}_{=0} + \frac{\partial g_1}{\partial y}(1,0)\,\tilde{\lambda}^{-2n} + O\left(\tilde{\lambda}^{-4n}\right)\right) \\
&= \left(\frac{\lambda(p(1,1))}{\tilde{\lambda}(1)}\right)^{2n}\frac{\partial g_1}{\partial y}(1,0) + O(\mu\,\theta_\delta\,\log^{3/2}\lambda) \\
&= \frac{\partial g_1}{\partial y}(1,0) + O(\sqrt{\mu\,\theta_\delta})\,.
\end{aligned}$$

In the last step we have used item 3. of lemma 6.3. But by item 1d. of lemma 6.1,

$$\frac{\partial g_1}{\partial y}(\delta,\mu,1,0) = \frac{\partial g_1}{\partial y}(\delta,0,1,0) + O(\mu) = 1 + O(\mu)\,.$$

Therefor the first component of $\tilde{G}_{\delta,\mu}(1,1)$ is of order $1+O(\mu)$. The second component of $\tilde{G}_{\delta,\mu}(1,1)$ is

$$\tilde{g}_2(1,1) = g_2(1,\tilde{\lambda}(1)^{-2n}) = g_2(1,0) + O\left(\tilde{\lambda}^{-2n}\right) = 1 + O\left(\tilde{\lambda}^{-2n}\right)\,,$$

where, again, we have used item 1b. of lemma 6.1, and item 1. of lemma 6.3. Notice that $G_{\delta,\mu}(1,0) = (0,1)$ for all (δ,μ).

The strong hyperbolicity of the map $\tilde{G}_{\delta,\mu}$, proved in lemma 8.3, together with (1a) and (1b), above, shows that we can apply a standard Graph Transform argument to prove the existence of a fixed point Q with stable and unstable manifolds written as graphs of C^1 functions. Let us assume the existence of such objects: the fixed point Q and the C^1 functions $\Gamma^u(x)$ and $\Gamma^s(y)$ whose graphs lay in the invariant manifolds of Q, and proceed with the estimations.

Now denote by $\ell^u_{\delta,\mu}$, respectively $\ell^s_{\delta,\mu}$, the intersection of the line through $(1,1)$ with slope $-\log^{3/2}\lambda$, respectively $-\log^{-3/2}\lambda$, with the rectangle $\tilde{S}_1$, respectively $\tilde{S}_1'$. Let us prove that

$$\operatorname{dist}\left(\tilde{G}_{\delta,\mu}(\ell^u_{\delta,\mu}),\ell^u_{\delta,\mu}\right) = O\left(\mu\,\log^{3/2}\lambda\right) \tag{2a}$$

and

$$\operatorname{dist}\left(\tilde{G}^{-1}_{\delta,\mu}(\ell^s_{\delta,\mu}),\ell^s_{\delta,\mu}\right) = O\left(\mu\,\log^{3/2}\lambda\right)\,. \tag{2b}$$

Writte $(\tilde{x}_1,\tilde{y}_1) = \tilde{G}_{\delta,\mu}(1,1)$ and take the unique point $(1,\hat{y}_1) \in \tilde{G}_{\delta,\mu}(\ell^u_{\delta,\mu})$ in this line with abcissa equal to 1. Then

$$\begin{aligned}
\operatorname{dist}\left(\tilde{G}_{\delta,\mu}(\ell^u_{\delta,\mu}),\ell^u_{\delta,\mu}\right) &\leq |\hat{y}_1 - 1| \leq |\hat{y}_1 - \tilde{y}_1| + |\tilde{y}_1 - 1| \\
&\leq O\left(\log^{3/2}\lambda\right)\,|1-\tilde{x}_1| + O\left(\mu\,\log^{3/2}\lambda\right) \\
&= O\left(\mu\,\log^{3/2}\lambda\right)\,.
\end{aligned}$$

Remark that $|1-\tilde{x}_1| = |1-\tilde{g}_1(1,1)| = O(\mu)$, and $|\tilde{y}_1 - 1| = |\tilde{g}_2(1,1)-1| = O(\mu \log^{3/2}\lambda)$, and finally observe lemma 8.3 implies that $\tilde{G}_{\delta,\mu}(\ell^u_{\delta,\mu})$ is a curve with tangent slope of order $O(\log^{3/2}\lambda)$. (2b) is proved in the same way.

Because the hyperbolicity granted by lemma 8.3 is strong enough we get from (2a) and (2b),

$$\operatorname{dist}\left(W^u_{\text{loc}}(Q), \ell^u_{\delta,\mu}\right) \leq 2 \operatorname{dist}\left(\tilde{G}_{\delta,\mu}(\ell^u_{\delta,\mu}), \ell^u_{\delta,\mu}\right) = O\left(\mu \log^{3/2}\lambda\right) ,$$

and analogously

$$\operatorname{dist}\left(W^s_{\text{loc}}(Q), \ell^s_{\delta,\mu}\right) \leq 2 \operatorname{dist}\left(\tilde{G}_{\delta,\mu}(\ell^s_{\delta,\mu}), \ell^s_{\delta,\mu}\right) = O\left(\mu \log^{3/2}\lambda\right) .$$

But this implies that $\operatorname{dist}(Q,(1,1)) = O\left(\mu \log^{3/2}\lambda\right)$.

Consider now the map

$$\rho(x,y,s) = \frac{\tilde{c}_1(x,y) + \tilde{d}_1(x,y)\, s}{\tilde{a}_1(x,y) + \tilde{b}_1(x,y)\, s} ,$$

defined as in the proof of lemma 4.1. A simple computation using lemma 8.3 shows that for all (δ,μ) in some small $N \in \mathcal{N}$, all $(x,y) \in [0,2]^2$ and all $|s| \leq 1$

$$\rho(x,y,s) = -\log^{3/2}\lambda + O\left(\log^3\lambda\right) .$$

By the definition of ρ, if $\tilde{G}\,(x,\Gamma^u(x)) = (\tilde{x},\Gamma^u(\tilde{x}))$ then

$$\frac{d}{dx}\Gamma^u(\tilde{x}) = \rho\left(x, \Gamma^u(x), \frac{d}{dx}\Gamma^u(x)\right) ,$$

Therefor $\frac{d}{dx}\Gamma^u(\tilde{x}) = -\log^{3/2}\lambda + O\left(\log^3\lambda\right)$ for all $x \in [0,2]$. The stable manifold is worked analogously.

□

In particular $W^s_{\text{loc}}(Q)$ intersects transversally $W^u(P) = \{(x,0) : x \in \mathbb{R}\}$ at the heteroclinic point $(x_s,0) = (\Gamma^s(0),0)$. Similarly, $W^u_{\text{loc}}(Q)$ intersects transversally $W^s(P) = \{(0,y) : y \in \mathbb{R}\}$ at another heteroclinic point $(0,y_u) = (0,\Gamma^u(0))$. Denote the arcs of stable and unstable manifolds that connect the fixed points P and Q with these heteroclinic points by,

$$\gamma^u_0(P) = \{(x,0) \,:\, x \in [0,x_s]\,\} ,$$

$$\gamma^s_0(P) = \{(0,y) \,:\, y \in [0,y_u]\,\} ,$$

$$\gamma^u_0(Q) = \{(x,\Gamma^u(x)) \,:\, x \in [0,x_1]\,\} \quad \text{and}$$

$$\gamma^s_0(Q) = \{(\Gamma^s(y),y) \,:\, y \in [0,y_1]\,\} .$$

Lemma 8.7. *For all (δ,μ) in some small $N \in \mathcal{N}$,*

(1) $|\Gamma^u(x) - 1| \leq \frac{7}{4}\log^{3/2}\lambda$ *for all* $0 \leq x \leq x_1$, *and*
(2) $|\Gamma^s(y) - 1| \leq \frac{7}{4}\log^{3/2}\lambda$ *for all* $0 \leq y \leq y_1$.

In particular $\gamma^s_0(Q) \subseteq \tilde{S}_1$ *and* $\gamma^u_0(Q) \subseteq \tilde{S}'_1$.

Proof.

$$
\begin{aligned}
|\Gamma^u(x) - 1| &\leq |\Gamma^u(x) - y_1| + |y_1 - 1| \\
&= |\Gamma^u(x) - \Gamma^u(x_1)| + O\left(\mu \log^{3/2} \lambda\right) \\
&\leq \frac{3}{2} (\log^{3/2} \lambda)\ |x_1| + O\left(\mu \log^{3/2} \lambda\right) \\
&< \frac{7}{4} \log^{3/2} \lambda\ .
\end{aligned}
$$

The second inequality is analogous. □

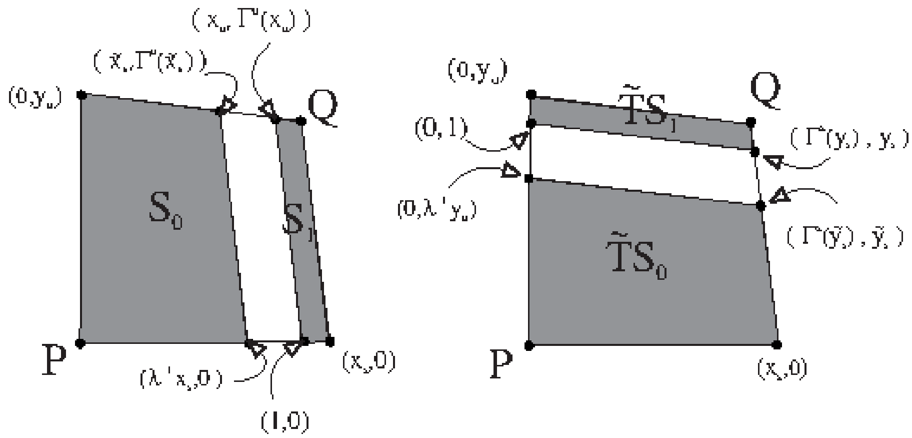

FIGURE 5. The Markov Partition

Let S be the square with the following

- edges: $\gamma_0^s(P)$, $\gamma_0^u(Q)$, $\gamma_0^s(Q)$ and $\gamma_0^u(P)$, and
- vertices: $P = (0,0)$, $(0, y_u)$, $Q = (x_1, y_1)$ and $(x_s, 0)$.

The sets $S_0 = S \cap \tilde{L}^{-1}(S)$ and $\tilde{T}S_0 = \tilde{L}(S) \cap S$ are rectangles bounded between the following edges and vertices.

- Vertical edges of S_0: $\gamma_0^s(P)$ and $\gamma_-^s(Q) = S \cap \tilde{L}^{-1}\gamma_0^s(Q)$.
- Horizontal edges of S_0: pieces of $\gamma_0^u(P)$ and $\gamma_0^u(Q)$.
- Vertical edges of $\tilde{T}S_0$: pieces of $\gamma_0^s(P)$ and $\gamma_0^s(Q)$.
- Horizontal edges of $\tilde{T}S_0$: $\gamma_0^u(P)$ and $\gamma_+^u(Q) = S \cap \tilde{L}\gamma_0^u(Q)$.
- Vertices of S_0: $(0,0)$, $(0, y_u)$, $(\tilde{x}_u, \Gamma^u(\tilde{x}_u))$ and $(\lambda^{-1}x_s, 0)$. The arc $\gamma_-^s(Q)$ is bounded between the heteroclinic point $\tilde{L}^{-1}(x_s, 0) = (\lambda^{-1}x_s, 0)$ and some homoclinic point in $\gamma_0^u(Q)$ that we denote as $(\tilde{x}_u, \Gamma^u(\tilde{x}_u))$.
- Vertices of $\tilde{T}S_0$: $(0,0)$, $(x_s, 0)$, $(\Gamma^s(\tilde{y}_s), \tilde{y}_s)$ and $(0, \lambda^{-1}y_u)$. $\gamma_+^s(Q)$ is bounded between the heteroclinic point $\tilde{L}(0, y_u) = (0, \lambda^{-1}y_u)$ and some homoclinic point in $\gamma_0^s(Q)$ that we denote by $(\Gamma^s(\tilde{y}_s), \tilde{y}_s)$.

The sets $S_1 = S \cap \tilde{G}^{-1}(S)$ and $\tilde{T}S_1 = \tilde{G}(S) \cap S$ are rectangles bounded between the following edges and vertices. Remember the relation (13) satisfied by the homoclinic points $(1,0)$ and $(0,1)$.

- Vertical edges of S_1: $\gamma_-^s(P) = \left\{\tilde{G}^{-1}(0, y) \,:\, 1 \leq y \leq y_u\right\}$ and $\gamma_0^s(Q)$.

- Horizontal edges of S_1 : pieces of $\gamma_0^u(P)$ and $\gamma_0^u(Q)$.
- Vertical edges of $\tilde{T}S_1$: pieces of $\gamma_0^s(P)$ and $\gamma_0^s(Q)$.
- Horizontal edges of $\tilde{T}S_1$: $\gamma_0^u(Q)$ and $\gamma_+^u(P) = \left\{\tilde{G}(x,0)\ :\ 1 \le x \le x_s\right\}$.
- Vertices of S_1 : $(1,0)$, $(x_u, \Gamma^u(x_u)) = \tilde{G}^{-1}(0, y_u)$, (x_1, y_1) , and $(0, x_s)$.
- Vertices of $\tilde{T}S_1$: $(0,1)$, $(\Gamma^s(y_s), y_s) = \tilde{G}(x_s, 0)$, (x_1, y_1) , and $(y_u, 0)$.

Lemma 8.8. *For all* (δ,μ) *in some small* $N \in \mathcal{N}$ *,*

$$S_1 \subseteq \tilde{S}_1 \qquad \text{and} \qquad \tilde{T}S_1 \subseteq \tilde{S}_1'$$

Proof. To prove that $\tilde{T}S_1 \subseteq \tilde{S}_1'$ it is enough to see that $\gamma_+^u(P) = \left\{\tilde{G}(x,0)\ :\ 1 \le x \le x_s\right\}$ is contained in $\tilde{S}_1'$. For that matter take $1 \le x \le x_s$. Using lemma 8.7,

$$\begin{aligned} |g_2(x,0) - 1| &\le \left|\frac{\partial g_2}{\partial x}(x^*,0)\right| |x-1| \le \left|\frac{\partial g_2}{\partial x}(x^*,0)\right| |x_s - 1| \\ &\le \left|\frac{\partial g_2}{\partial x}(x^*,0)\right| \frac{7}{4} \log^{3/2}\lambda < 2 \log^{3/2}\lambda \,. \end{aligned}$$

Notice that, by lemma 6.1, if (δ,μ) is small enough $\frac{\partial g_2}{\partial x} \sim 1$. □

The maximal invariant set of $\tilde{T}$ in S , $\Lambda_{\delta,\mu} = \bigcap_{n\in\mathbb{Z}} \tilde{T}^{-n}(S)$, is a "horse-shoe" type basic set with Markov partition $\mathcal{P} = \{S_0, S_1\}$. The map $\tilde{T} : S_0 \cup S_1 \to S$ belongs to class $\mathcal{F}$ of definition 3.2, and all conditions of definition 4.1, except the fifth one, have already been verified. Let us now check this last condition.

Lemma 8.9. *There is some small* $N \in \mathcal{N}$ *and some constant* $C > 0$ *such that for all* $(\delta,\mu) \in N$ *,*

(1) $dist\,(S_0, S_1) \ge \dfrac{\epsilon}{\gamma}$

(2) $dist\left(\tilde{T}(S_0), \tilde{T}(S_1)\right) \ge \dfrac{\epsilon}{\gamma}$

where $\epsilon = \epsilon(\delta,\mu)$ *and* $\gamma = \gamma(\delta,\mu)$ *were defined in lemma 8.5.*

Proof. Because the right vertical boundary of S_0 , which goes down from $(\tilde{x}_s, \Gamma^u(\tilde{x}_s))$ to $(\lambda^{-1}x_s, 0)$, and the left vertical boundary of S_1 , going down from $(x_u, \Gamma^u(x_u)$ to $(1,0)$, are both graphs of C^1 functions with negative derivatives, the distance between S_0 and S_1 is greater or equal to $x_u - \lambda^{-1}x_s$. A simple application of the mean value theorem shows that

$$0 \le x_s - 1 \le O\left(\log^{3/2}\lambda\right) \qquad \text{and} \qquad 0 \le 1 - x_u \le O\left(\log^{3/2}\lambda\right) \,.$$

Therefor

$$\begin{aligned} x_u - \lambda^{-1}x_s &\ge (x_s - \lambda^{-1}x_s) - (x_s - 1) - (1 - x_u) \\ &\ge x_s\,(1 - \lambda^{-1}) - O\left(\log^{3/2}\lambda\right) \\ &\ge \frac{4}{5}\log\lambda - O\left(\log^{3/2}\lambda\right) \ge \frac{3}{4}\log\lambda \\ &\ge \frac{3}{2\,C}\log\lambda = \frac{\epsilon}{\gamma} \,. \end{aligned}$$

In a similar way one proves that

$$\operatorname{dist}\left(\tilde{T}(S_0), \tilde{T}(S_1)\right) \geq y_s - \lambda^{-1} y_u \geq \frac{\epsilon}{\gamma}\ .$$

□

Define $I_*^s = W^s(P) \cap S = \{0\} \times [0, y_u]$ and $I_*^u = W^u(P) \cap S = [0, x_s] \times \{0\}$. The dynamically defined Cantor sets $(K^s = I_*^u \cap \Lambda, \psi^s)$ and $(K^u = I_*^s \cap \Lambda, \psi^u)$, induced respectively by the action of $\tilde{T}^{-1}$ on the stable leaves and by the action of $\tilde{T}$ on the unstable leaves, have the following Markov partitions

$$\mathcal{P}^s = \{I_*^u \cap S_0, I_*^u \cap S_1\} = \big\{[0, \lambda^{-1} x_s], [1, x_s]\big\}\ ,$$

and

$$\mathcal{P}^u = \left\{I_*^s \cap \tilde{L}(S_0), I_*^s \cap \tilde{G}(S_1)\right\} = \big\{[0, \lambda^{-1} y_u], [1, y_u]\big\}\ .$$

Lemma 8.10. *There is some small enough $N \in \mathcal{N}$ such that for all $(\delta, \mu) \in N$,*

(1) $\tau_L(\mathcal{P}^s)\ ,\quad \tau_L(\mathcal{P}^u)\ \geq \dfrac{1}{\lambda - 1}$

(2) $\tau_R(\mathcal{P}^s)\ ,\quad \tau_R(\mathcal{P}^u)\ \geq \dfrac{1}{4}\sqrt{\lambda - 1}$

where $\lambda = \lambda_{\delta,\mu}(0)$. In particular $\tau_{LR}(\mathcal{P}) \geq \dfrac{1}{4\sqrt{\lambda - 1}}$.

Proof.

$$\tau_L(\mathcal{P}^s) = \frac{\lambda^{-1} x_s}{1 - \lambda^{-1} x_s} \geq \frac{\lambda^{-1} x_s}{x_s - \lambda^{-1} x_s} = \frac{1}{\lambda - 1}\ ,$$

and similarly

$$\tau_L(\mathcal{P}^u) = \frac{\lambda^{-1} y_u}{1 - \lambda^{-1} y_u} \geq \frac{\lambda^{-1} y_u}{y_u - \lambda^{-1} y_u} = \frac{1}{\lambda - 1}\ .$$

Applying the mean value theorem to Γ^s, at the points $y = 0$ and $y = y_1$, we get from lemma 8.6,

$$\frac{1}{4}(\lambda - 1)^{3/2} \leq \frac{1}{2}\log^{3/2}\lambda \leq x_s - 1 \leq O\left(\log^{3/2}\lambda\right) \ll \lambda - 1\ .$$

Therefor,

$$\begin{aligned} \tau_R(\mathcal{P}^s) &= \frac{x_s - 1}{1 - \lambda^{-1} x_s} \geq \frac{x_s - 1}{x_s(1 - \lambda^{-1})} \geq \frac{\lambda}{x_s}\,\frac{x_s - 1}{\lambda - 1} \\ &\geq \frac{1}{4}\,\frac{(\lambda - 1)^{3/2}}{\lambda - 1} = \frac{1}{4}\sqrt{\lambda - 1}\ . \end{aligned}$$

Using Γ^u instead of Γ^s we obtain the same estimates for $y_u - 1$ as we did for $x_s - 1$. Thus,

$$\tau_R(\mathcal{P}^u) = \frac{y_u - 1}{1 - \lambda^{-1} y_u} \geq \frac{1}{4}\sqrt{\lambda - 1}\ .$$

□

Corollary 8.1. *There is some small $N \in \mathcal{N}$ such that for all $(\delta, \mu) \in N$,*

$$\tilde{T}_{\delta,\mu} \in \mathcal{F}\left(\,\epsilon(\delta, \mu),\ \gamma(\delta, \mu)\,\right) \quad \textit{and} \quad \tau_{LR}(\Lambda_{\delta,\mu}) > 1\ .$$

Proof. From lemmas 8.5 and 8.9 we obtain $\tilde{T} \in \mathcal{F}(\epsilon, \gamma)$. Combining theorem 2 with lemma 8.10 we see that for $(\delta, \mu) \in N$ $\lim_{(\delta,\mu)\to(0,0)} \tau_{LR}(\Lambda_{\delta,\mu}) = +\infty$, which proves this corollary. □

9. **Positive Homoclinic Tangencies**

In this final section we find orbits of positive homoclinic tangencies, of the fixed point $P_{\delta,\mu}$, for sequences of parameters $(\delta, \mu_n(\delta))$ accumulating in $(\delta, 0)$ as $n \to +\infty$. Actually, because it is much easier, we find orbits of negative homoclinic tangencies and then use an elementary abstract lemma relating negative with positive homoclinic tangencies.

First we need some definitions. Let P be a fixed point of a diffeomorphism $\varphi : M^2 \to M^2$, in some oriented surface M^2, having both eigenvalues positive. We orient the stable and unstable branches of $W^s(P) - \{P\}$ and $W^u(P) - \{P\}$ so that orbits increase along them. Homoclinic tangencies of P are called positive, c.f. definition 3.3, if both the orientations, on the stable and unstable branches, agree near the tangency. Given an orbit of transversal homoclinic points $\{\varphi^n(x) : n \in \mathbb{Z}\}$ between two components, $\gamma^s(P)$ of $W^s(P) - \{P\}$, and $\gamma^u(P)$ of $W^u(P) - \{P\}$, consider the family of linear "return maps" to a neighborhood of P,

$$R_{n,m} = D\varphi^{n+m}_{\varphi^{-n}(x)} : T_{\varphi^{-n}(x)}(M^2) \to T_{\varphi^m(x)}(M^2) .$$

For all large enough $n, m \in \mathbb{N}$, identify both domain and target space with $T_P(M^2)$ via some local coordinates. Then $R_{n,m}$ is strongly hyperbolic. Two cases may occur:

Case I The linear map $R_{n,m}$ has two positive eigenvalues, for all large $n, m \in \mathbb{N}$. In this case the pieces of $\gamma^u(P)$ through $\varphi^m(x)$, resp. of $\gamma^s(P)$ through $\varphi^{-n}(x)$, that by the λ-lemma accumulate in $W^u_{\text{loc}}(P)$ as $m \to +\infty$, resp. in $W^s_{\text{loc}}(P)$ as $n \to +\infty$, are oriented in the same way as $\gamma^u_{\text{loc}}(P) = \gamma^u(P) \cap W^u_{\text{loc}}(P)$. In this case we will say that the points of the orbit are *positive transversal homoclinic points*.

Case II The linear map $R_{n,m}$ has two negative eigenvalues, for all large $n, m \in \mathbb{N}$. In this case the pieces of $\gamma^u(P)$ through $\varphi^m(x)$, resp. of $\gamma^s(P)$ through $\varphi^{-n}(x)$, that by the λ-lemma accumulate in $W^u_{\text{loc}}(P)$ as $m \to +\infty$, resp. in $W^s_{\text{loc}}(P)$ as $n \to +\infty$, are oriented in the opposite way of $\gamma^u_{\text{loc}}(P) = \gamma^u(P) \cap W^u_{\text{loc}}(P)$. In this second case we will say that the points $\varphi^n(x)$ are *negative transversal homoclinic points.*

In our working context let $t(\delta)$, $\delta > 0$, be a smooth family of (simple) zeros of the Melnikov function $M_\delta(t)$ and consider the corresponding family $H_{\delta,\mu}$ of transversal homoclinic points in $W^s(P_{\delta,\mu}) \cap W^u(P_{\delta,\mu})$. It is easily seen that

I If for all small $\delta > 0$, $\frac{d}{dt} M_\delta(t(\delta)) < 0$ then $H_{\delta,\mu}$ is a positive transversal homoclinic point for all (δ, μ) in some small enough $N \in \mathcal{N}$,

II If for all small $\delta > 0$, $\frac{d}{dt} M_\delta(t(\delta)) > 0$ then $H_{\delta,\mu}$ is a negative transversal homoclinic point for all (δ, μ) in some small enough $N \in \mathcal{N}$.

In particular, since the Melnikov function must have zeros with positive derivative then there is a family of negative transversal homoclinic points in $W^s(P_{\delta,\mu}) \cap W^u(P_{\delta,\mu})$ for all (δ, μ) in some small enough $N \in \mathcal{N}$.

Lemma 9.1. *Let $\varphi_\mu : M^2 \to M^2$ be a family of C^2 orientation preserving diffeomorphisms with a hyperbolic fixed point P_μ having both eigenvalues positive. Assume there is an orbit of negative transversal homoclinic points between the components $\gamma^u(P_\mu)$ of $W^u(P_\mu) - \{P_\mu\}$ and $\gamma^s(P_\mu)$ of $W^s(P_\mu) - \{P_\mu\}$. If an orbit of negative quadratic homoclinic tangencies, between these components, unfolds generically at $\mu = \mu_0$ then there is a sequence μ_n converging to μ_0, as $n \to +\infty$, of parameters where orbits of positive quadratic homoclinic tangencies, between $\gamma^u(P_\mu)$ and $\gamma^s(P_\mu)$, are generically unfold.*

Proof. Take two homoclinic points in $\gamma^s(P_\mu) \cap \gamma^u(P_\mu)$:

- $x \in \gamma^s_{\text{loc}}(P_\mu)$, close to P_μ, a negative transversal homoclinic point and
- $y \in \gamma^u_{\text{loc}}(P_{\mu_0})$, close to P_{μ_0}, a negative quadratic homoclinic tangency which unfolds generically at $\mu = \mu_0$.

By the λ-lemma there is a sequence of arcs $\sigma^u_n(\mu) \subseteq \gamma^u(P_\mu)$ containing x whose forward iterates $\gamma^u_n(\mu) = \varphi_\mu^{\,n}(\sigma^u_n(\mu))$ converge in the C^1 topology to $\gamma^u_{\text{loc}}(P_\mu)$. Because x is a negative transversal homoclinic point the arcs $\gamma^u_n(\mu)$ are oriented in the opposite direction of their limit $\gamma^u_{\text{loc}}(P_\mu)$. The stable branch $\gamma^s(P_{\mu_0})$ makes a negative tangency with $\gamma^u_{\text{loc}}(P_{\mu_0})$ at point y and locally $\gamma^s(P_\mu)$ moves transversally with "positive velocity" with respect to $\gamma^u_{\text{loc}}(P_\mu)$. Therefor $\gamma^s(P_\mu)$ will also move with "positive velocity" with respect to $\gamma^u_n(\mu)$ and will have a tangencial contact with it for some parameter μ_n close to μ_0. Because of the opposite orientation of $\gamma^u_n(\mu)$, relative to $\gamma^u_{\text{loc}}(P_\mu)$, the new tangency between $\gamma^u_n(\mu) \subseteq \gamma^u(P_\mu)$ and $\gamma^s(P_\mu)$ will be positive. Since the tangency at x is quadratic so is the new one. □

Denote by $\gamma^u(P_{\delta,\mu})$, resp. $\gamma^s(P_{\delta,\mu})$, the component of $W^u(P_{\delta,\mu}) - \{P_{\delta,\mu}\}$, resp. $W^s(P_{\delta,\mu}) - \{P_{\delta,\mu}\}$, depending continuously in (δ,μ) that coincides with the homoclinic connection γ_δ when $\mu = 0$.

Lemma 9.2. *For each $\delta > 0$ there is a sequence $(\delta, \mu_n(\delta))$ converging to $(\delta, 0)$ as $n \to \infty$, at which parameters some orbit of negative quadratic homoclinic tangencies, between $\gamma^u(P_{\delta,\mu})$ and $\gamma^s(P_{\delta,\mu})$, unfolds generically with parameter μ.*

Proof. Denote by Γ^s the arc of stable manifold in $\mathbb{R}^2_+$ connecting the negative transversal homoclinic point $(1,0)$ with the previous (positive) transversal homoclinic point $(x_0, 0)$ with $x_0 < 1$. Similarly denote by Γ^u the arc of stable manifold in $\mathbb{R}^2_+$ connecting the negative transversal homoclinic point $(0,1)$ with the previous (positive) transversal homoclinic point $(0, y_0)$ with $y_0 > 1$. These arcs can be written as graphs of smooth positive functions,

$$\Gamma^s = \Gamma^s(\delta,\mu) = \{(x, g^s(\delta,\mu,x)) \;:\; x_0 \le x \le 1\} \;,$$
$$\Gamma^u = \Gamma^u(\delta,\mu) = \{(g^u(\delta,\mu,y), y) \;:\; 1 \le y \le y_0\} \;.$$

For each $n \in \mathbb{N}$ let us write,

$$\Gamma^s_n(\delta,\mu) = L^{-n}_{\delta,\mu}(\Gamma^s(\delta,\mu)) \quad \text{and} \quad \Gamma^u_n(\delta,\mu) = L^{n}_{\delta,\mu}(\Gamma^u(\delta,\mu)) \;.$$

These arcs are also graphs of smooth functions, respectively g^s_n and g^u_n. From the asymptotic relation,

$$L^n_{\delta,\mu}(x,y) \sim \left(\lambda^n x, \lambda^{-n} y\right) \qquad \lambda = \lambda_{\delta,0}(0)\,, \quad \text{as} \quad \mu \to 0\,,$$

we obtain,

$$(*) \quad g^s_n(\delta,\mu,x) \sim \lambda^n\, g^s(\delta,\mu,\lambda^n x) \quad \text{and} \quad g^u_n(\delta,\mu,y) \sim \lambda^n\, g^u(\delta,\mu,\lambda^n y)\;.$$

Define now

$$\mu_n(\delta) = \sup\{\mu \geq 0 \,:\, \Gamma^s_n(\delta,\mu) \cap \Gamma^u_n(\delta,\mu) = \emptyset\}\ .$$

Then for $\mu < \mu_n(\delta)$, $\Gamma^s_n(\delta,\mu) \cap \Gamma^u_n(\delta,\mu) = \emptyset$. By compactness of the arcs $\Gamma^s(\delta,\mu_n)$ and $\Gamma^u(\delta,\mu_n)$ we have $\Gamma^s_n(\delta,\mu_n) \cap \Gamma^u_n(\delta,\mu_n) \neq \emptyset$. Therefor this is a first intersection between these arcs, and so it must be a tangency. The sequence $\mu_n(\delta)$ converges geometrically to zero. One can easily prove that $0 < \mu_n(\delta) \leq C(\delta)\,\mu\,\lambda^{-2n}$ for all large $n \in \mathbb{N}$ and some constant $C(\delta) > 0$.

We still have to prove that these tangencies are quadratic and unfold generically. For $\mu = \mu_n(\delta)$ denote by $(x_n, y_n) = (x_n(\delta), y_n(\delta)) \in \Gamma^s_n \cap \Gamma^u_n$ the point of tangency between the graphs $(x, g^s_n(\delta,\mu,x))$ and $(g^u_n(\delta,\mu,y), y)$. Then of course

$$\frac{\partial g^s_n}{\partial x}(\delta,\mu_n(\delta),x_n)\,\frac{\partial g^u_n}{\partial y}(\delta,\mu_n(\delta),y_n) = 1\ ,$$

and both these derivatives are positive. The slope of the first tangency must be positive.

To prove that the tangency is quadratic we just outline a qualitative argument that may easily be quantified into a rigorous, but tedious, analytic proof. The argument relies on the following facts, which may seen from $(*)$:

a) If $\frac{\partial g^s_n}{\partial x}(\delta,\mu_n,x_n)$, resp. $\frac{\partial g^u_n}{\partial y}(\delta,\mu_n,y_n)$, is very small then the curvature vector of $\Gamma^s_n(\delta,\mu_n)$, resp. $\Gamma^u_n(\delta,\mu_n)$, at (x_n,y_n) is very large and points inward the domain bounded by Γ^s_n and the x-axis, resp. Γ^u_n and the y-axis.

b) If the curvature vector of $\Gamma^s_n(\delta,\mu_n)$, resp. $\Gamma^u_n(\delta,\mu_n)$, at (x_n,y_n) points outward then it must be small and the tangent slope $\frac{\partial g^s_n}{\partial x}(\delta,\mu_n,x_n)$, resp. $\frac{\partial g^u_n}{\partial y}(\delta,\mu_n,y_n)$, will be large.

We just remark that in the case x_n, or y_n, were close to local minima of g^s_n, or g^u_n, these facts would fail to be true. But this situation is ruled out by the assumption that (x_n,y_n) is a first tangency.

Now, if both curvature vectors of $\Gamma^s_n(\delta,\mu_n)$ and $\Gamma^u_n(\delta,\mu_n)$ at (x_n,y_n) point inward they will have opposite directions and the tangency will, therefore, be quadratic. Assume that one of them, the curvature of $\Gamma^u_n(\delta,\mu_n)$ for instance, points outward. Then by item b) this curvature is small while $\frac{\partial g^u_n}{\partial y}(\delta,\mu_n,y)$ is very large. But this implies that $\frac{\partial g^s_n}{\partial s}(\delta,\mu_n,y)$ is very small and from item a) above it follows that the curvature of $\Gamma^s_n(\delta,\mu_n)$, pointing inward, is very large, much larger then the curvature vector of $\Gamma^s_n(\delta,\mu_n)$ which points in the same direction. And the tangency is again quadratic.

Before establishing the genericity of the unfolding remark that the Melnikov function of the family $f_{\delta,\mu}$ may be expressed as either:

$$M_\delta(t) = \frac{\partial}{\partial\mu}\left(g^s(\delta,\mu,\lambda_{\delta,\mu}(0)^t\right)_{\mu=0} = \frac{\partial g^s}{\partial\mu}(\delta,0,\lambda_{\delta,0}(0)^t)\ ,$$

or

$$M_\delta(t) = \frac{\partial}{\partial\mu}\left(g^u(\delta,\mu,\lambda_{\delta,\mu}(0)^{-t}\right)_{\mu=0} = \frac{\partial g^u}{\partial\mu}(\delta,0,\lambda_{\delta,0}(0)^{-t})\ .$$

Then because the Melnikov function is a Morse function we will have on some small enough $N \in \mathcal{N}$ and for all $(\delta,\mu) \in N$,

$$\frac{\partial g^s_n}{\partial\mu}(\delta,\mu,x) \sim \lambda^n\,\frac{\partial g^s}{\partial\mu}(\delta,\mu,\lambda^n\,x) \sim \lambda^n\,M_\delta\left(\frac{\log(\lambda^n x)}{\log\lambda}\right) > 0$$

for all $x_0 \leq \lambda^n x \leq 1$, and similarly

$$\frac{\partial g^u_n}{\partial\mu}(\delta,\mu,y) \sim \lambda^n\,\frac{\partial g^u}{\partial\mu}(\delta,\mu,\lambda^n\,y) \sim \lambda^n\,M_\delta\left(-\frac{\log(\lambda^n y)}{\log\lambda}\right) > 0$$

for all $1 \leq \lambda^n y \leq y_0$.

Now, consider the vertical line $x = x_n$, which crosses transversally both arcs Γ^s_n and Γ^u_n at the point (x_n, y_n). Let us fix δ and vary μ close to $\mu_n(\delta)$. The intersection of $\Gamma^s_n(\delta,\mu)$ with this line, $(x_n, g^s_n(\delta,\mu,x_n))$, moves upward since $\frac{\partial g^s_n}{\partial \mu}(\delta,\mu,x_n) > 0$. The intersection of $\Gamma^u_n(\delta,\mu)$ with the line $x = x_n$ is $(x_n, \phi(\mu))$, where $\phi(\mu)$ is defined implicitly by $g^u_n(\delta,\mu,\phi(\mu)) = g^u_n(\delta,\mu_n,y_n)$. Thus

$$\frac{\partial \phi}{\partial \mu}(\mu_n) = -\frac{\frac{\partial g^u_n}{\partial \mu}(\delta,\mu_n,y_n)}{\frac{\partial g^u_n}{\partial y}(\delta,\mu_n,y_n)} = -\frac{\partial g^u_n}{\partial \mu}(\delta,\mu_n,y_n)\,\frac{\partial g^s_n}{\partial x}(\delta,\mu_n,x_n) < 0\,,$$

which shows that this intersection point moves downward. Therefor the tangency unfolds generically. □

Corollary 9.1. *For each $\delta > 0$ there is a sequence $(\delta, \mu_n(\delta))$ converging to $(\delta, 0)$ as $n \to \infty$, at which parameters some orbit of positive quadratic homoclinic tangencies, between $\gamma^u(P_{\delta,\mu})$ and $\gamma^s(P_{\delta,\mu})$, unfolds generically with parameter μ.*

Proof. Combine lemmas 9.1 and 9.2. □

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