DYNAMICS ON THE ATTRACTOR OF THE LOTKA-VOLTERRA EQUATIONS *

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We show that for stable dissipative Lotka-Volterra systems the dynamics on the attractor are hamiltonian and we argue that complex dynamics can occur.

1 Introduction

In his famous monograph "Leçons sur la Théorie Mathématique de la Lutte pour la Vie" ([25]) Volterra introduced the system of differential equations

$$\dot{x}_j = \varepsilon_j x_j + \sum_{k=1}^{n} a_{jk} x_j x_k \qquad (j = 1, \ldots, n) \tag{1}$$

as a model for the competition of n biological species. In this model, x_j represents the number of individuals of species j (so one assumes $x_j > 0$), the a_{jk}'s are the interaction coefficients, and the ε_j's are parameters that depend on the environment. For example, $\varepsilon_j > 0$ means that species j is able to increase with food from the

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environment, while $\varepsilon_j < 0$ means that it cannot survive when left alone in the environment. One can also have $\varepsilon_j = 0$ which means that the population stays constant if the species does not interact.

The dynamics of general systems of type (1) are far from understood, although special classes of Lotka-Volterra systems have been studied. We distinguish the following classes of systems of type (1):

Definition 1.1. A Lotka-Volterra system with interaction matrix $A = (a_{ij})$ is called

(i) COOPERATIVE (resp. COMPETITIVE) if $a_{jk} \geq 0$ (resp. $a_{jk} \leq 0$) for all $j \neq k$;

(ii) CONSERVATIVE if there exists a diagonal matrix $D > 0$ such that AD is skew-symmetric;

(iii) DISSIPATIVE if there exists a diagonal matrix $D > 0$ such that $AD \leq 0$;

Competitive systems and dissipative systems are mutually exclusive classes, except for the trivial case where $a_{jk} = 0$. General results concerning competitive or cooperative systems were obtained by Smale [24] and Hirsch [9, 10] (for recent results see [26] and references therein). These systems typically have a global attractor consisting of equilibria and connections between them (see e. g. [9] theorem 1.7).

Dissipative systems have been less studied than competitive systems, although this class of systems goes back to the pioneer work of Volterra, who introduced them as a natural generalization of predator-prey systems (see [25], chp. III). For systems where predators and preys coexist there is empirical and numerical evidence that periodic oscillations occur. In fact, as is well known, for any two dimensional predator-prey system, the orbits are periodic. But for higher dimensional systems the topology of orbits in phase space is much more complex, and understanding this topology is a challenging problem. The following theorem, to be proved in this paper, is perhaps the first result in this direction.

Theorem 1.2. *Consider a Lotka-Volterra system (1) restricted to the flow invariant set* $\mathbb{R}^n_+ \equiv \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_j > 0, j = 1, \ldots, n\}$, *and assume that (i) the system has a singular point, and (ii) is stably dissipative. Then there exists a global attractor and the dynamics on the attractor are hamiltonian.*

By "stably dissipative" we mean that the system is dissipative and every system close to it is also dissipative. As we mentioned before, the notion of dissipative system is due to Volterra. Stable dissipative systems where first studied by Redheffer et al. ([18, 19, 20, 21, 22]) under the name "stable admissible". They gave a beautiful description of the attractor (see section 4 below) which we will use to prove theorem 1.2. The hypothesis on the existence of a singular point is equivalent to the assumption that some orbit has a α- or ω-limit point in $\mathbb{R}^n_+$.

One of Volterra's main goals in introducing these equations was the "mechanization" of biology, and he made quite an effort in trying to pursue this program. While seeking a variational principle for the system, he was successful in finding a hamiltonian formulation in the case where the interaction matrix is skew-symmetric, at the expense of doubling the number of dimensions (see section 2 for details). Along the way, a polemic with Levi-Civita arose, an account of which can be found in [8]. In this paper we shall give a different solution to the problem of putting system (1) into a hamiltonian frame. In modern language, our approach is related to Volterra's approach by a reduction procedure. This hamiltonian frame is the basis for the hamiltonian structure refered to in theorem 1.2.

Once the hamiltonian character of the dynamics is established, one would like to understand (i) what type of attractors one can get and (ii) what kind of hamiltonian dynamics one can have on the attractor. It will follow from our work that this amounts to classify the dynamics of Lotka-Volterra systems with skew-symmetric matrix whose associated graph is a forest. We do not know of such classification but we shall argue that these dynamics can be rather complex.

In the simplest situation, the attractor will consist of the unique fixed point in $\mathbb{R}^n_+$ and the dynamics will be trivial. It was already observed in [20] that there may exist periodic orbits on (non-trivial) attractors. On the other hand, if the attractor is an integrable Hamiltonian system then one can expect the orbits to be almost periodic. We will show through a detailed study of a 4-dimensional chain of predator-prey systems, that non-integrable hamiltonian system can indeed occur. Therefore, typically, the dynamics of dissipative Lotka-Volterra systems are extremely complex. This is related with a famous conjecture in the theory of Hamiltonian systems which can be stated as follows.

> *Typically, dynamics on the common level sets of the hamiltonian and the Casimirs are ergodic.*

This paper is organized in two parts. In the first part we deal with general systems and prove theorem 1.2. In the second part, we give a detail analysis of a 4-dimensional predator-prey chain. This is an extremely interesting system for which we show, among other properties, that

- the system is non-integrable in the sense of Arnol'd-Liouville;
- the dynamics of the system is equivalent to the dynamics of a homeomorphism of a sphere;
- the system has families of periodic orbits whose stability is determined by an associated Sturm-Liouville problem;
- one can find regions in the space of parameters where periodic orbits are strongly hyperbolic;

We believe that both this system and higher dimension generalizations deserve further study, and can help understanding the conjecture above.

Part I. General Theory

2 Basic Notions

Here, we will recall some basic notions and facts concerning general Lotka-Volterra systems which will be useful in the next sections. All of these notions can be traced back to Volterra. For a more detailed account of general properties of Lotka-Volterra systems we refer to the book by Hofbauer and Sigmund [7].

For fixed $d_j \neq 0$, the transformation

$$y_j = \frac{1}{d_j} x_j, \qquad (j = 1, \ldots, n) \tag{2}$$

takes the Volterra system (1) with interaction matrix A, into a new Volterra system with interaction matrix AD

$$\dot{y}_j = \varepsilon_j y_j + \sum_{k=1}^{n} d_k a_{jk} y_j y_k \qquad (j = 1, \ldots, n). \tag{3}$$

We can therefore think of (2) as a *gauge symmetry* of the system. A choice of representative (a_{jk}) in a class of equivalence under gauge transformations will be called a *choice of gauge*. Since will often take as phase space $\mathbb{R}^n_+$, we consider only gauge transformations with $d_j > 0$ in order to preserve phase space. Note also that the classes of Lotka-Volterra systems introduced in definition 1.1 above are all gauge invariant.

Many properties of a Lotka-Volterra system can be expressed geometrically in terms of its *associated graph* $G(A, \varepsilon)$. These is the labeled graph, where with each species j we associate a vertex $\bigcirc$ labeled with ε_j and we draw an edge connecting vertex j to vertex k whenever $a_{jk} \neq 0$.

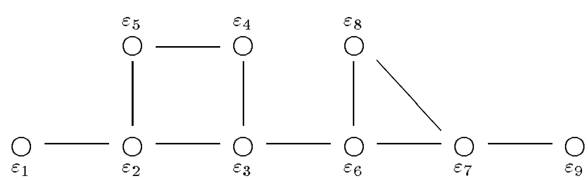

Figure 1: Graph $G(A, \varepsilon)$ associated with a system of type (1).

For example, if two systems are gauge equivalent, they have the same unlabeled graph (but not conversely). Also, conservative systems can be caracterized in terms of its graph as it follows from the following proposition also due to Volterra (cf. [25], chp. III §12):

Proposition 2.1. *A Lotka-Volterra system with interaction matrix $A = (a_{jk})$ is conservative if, and only if, $a_{jj} = 0$,*

$$a_{jk} \neq 0 \Longrightarrow a_{jk}a_{kj} < 0 \qquad (j \neq k), \tag{4}$$

and

$$a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_s i_1} = (-1)^s a_{i_s i_{s-1}} \cdots a_{i_2 i_1} a_{i_1 i_s} \tag{5}$$

for every finite sequence of integers $(i_1, \ldots, i_s)$, with $i_r \in \{1, \ldots, n\}$ for $r = 1, \ldots, s$.

In other words a system is conservative if, and only if, (i) the conditions $a_{jj} = 0$ and $a_{jk} \neq 0 \Rightarrow a_{kj} \neq 0$ are satisfied and (ii) for each closed path in the diagram with a even (respectively odd) number of vertices the product of the coefficients when we go around in one direction is equal to the product (resp. minus the product) of the coefficients when we go around in the opposite direction. Hence, for example, a system with associated graph as in fig. 1 is conservative if, and only if,

$$\begin{aligned} a_{67}a_{78}a_{86} &= -a_{68}a_{87}a_{76}, \\ a_{23}a_{34}a_{45}a_{52} &= a_{25}a_{54}a_{43}a_{32}, \end{aligned}$$

and moreover the conditions $a_{jj} = 0$ and $a_{jk} \neq 0 \Rightarrow a_{jk}a_{kj} < 0$ are satisfied.

The most trivial solutions of system (1) are, of course, the fixed points. The fixed points $\mathbf{q} = (q_1, \ldots, q_n)$ in $\mathbb{R}^n_+$ of system (1) are the solutions of the linear system

$$\varepsilon_j + \sum_{k=1}^{n} a_{jk} q_k = 0 \qquad (j = 1, \ldots, n). \tag{6}$$

The existence of a fixed point in $\mathbb{R}^n_+$ is related with the behavior of the orbits in $\mathbb{R}^n_+$, as it is clear from the following result (see [7], section 9.2).

Proposition 2.2. *There exists a fixed point $\mathbf{q} = (q_1, \ldots, q_n)$ in $\mathbb{R}^n_+$ of system (1) if, and only if, $\mathbb{R}^n_+$ contains some α- or ω-limit point.*

Proof. In one direction the result is clear. On the other hand, assume that there exists no fixed point in $\mathbb{R}^n_+$ so that for the affine operator $L : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$L(\mathbf{x})_j = \varepsilon_j + \sum_{k=1}^{n} a_{jk} x_k$$

one has $0 \notin K = L(\mathbb{R}^n_+)$. Then there exists a hyperplane H through the origin disjoint from the convex set K, and one can choose $\mathbf{c} = (c_1, \ldots, c_n) \in H^\perp$ such that

$$\mathbf{c} \cdot \mathbf{y} > 0, \qquad \forall \mathbf{y} \in K. \tag{7}$$

Consider now the function $V : \mathbb{R}^n_+ \to \mathbb{R}$ given by

$$V(\mathbf{x}) = \sum_{j=1}^{n} c_j \log(x_j). \tag{8}$$

If $\mathbf{x}(t)$ is a solution of (1) in $\mathbb{R}^n_+$ then we compute

$$\frac{d}{dt}V(\mathbf{x}(t)) = \sum_{j=1}^{n} c_j \frac{\dot{x}_j}{x_j} = \mathbf{c}\cdot L(\mathbf{x}(t)) > 0,$$

where we used (7). Hence, V is a Liapounov function and there can be no ω-limit points since for these one must have $\dot{V} = 0$. Similarly, to exclude α-limit points one uses the Liapounov function $-V$. □

We have just seen that the *limit behavior* of the orbits is related to the existence of fixed points. On the other hand, the following result shows that the *average behavior* of the orbits is related to the values of the fixed points (see [4]).

Proposition 2.3. *Suppose that* $\mathbf{x}(t)$ *is an orbit in* $\mathbb{R}^n_+$ *of system (1) satisfying* $0 < m \leq x_j(t) \leq M$. *Then there is a sequence* $\{T_k\}$ *such that* $T_k \to +\infty$ *and a fixed point* $\mathbf{q} \in \mathbb{R}^n_+$ *such that*

$$\lim_{k\to+\infty} \frac{1}{T_k}\int_0^{T_k} \mathbf{x}(t)dt = \mathbf{q}. \tag{9}$$

Moreover, if system (1) has a unique fixed point $\mathbf{q} \in \mathbb{R}^n_+$ *then*

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^{T} \mathbf{x}(t)dt = \mathbf{q}. \tag{10}$$

Proof. Since we have $x_j(t) \leq M$, the function

$$\mathbf{z}(T) = \frac{1}{T}\int_0^{T} \mathbf{x}(t)dt$$

is bounded, and there exists a sequence $\{T_k\}$ such that $T_k \to +\infty$ and the limit

$$\lim_{k\to+\infty} \frac{1}{T_k}\int_0^{T_k} \mathbf{x}(t)dt = \mathbf{q} \tag{11}$$

exists. Since $0 < m \leq x_j(t)$ it is clear that $\mathbf{q} \in \mathbb{R}^n_+$. Now, if we integrate (1) along the orbit $\mathbf{x}(t)$ we obtain

$$\frac{1}{T_k}\left(\log(x_j(T_k)) - \log(x_j(0))\right) = \varepsilon_j + \frac{1}{T_k}\int_0^{T_k} \sum_{k=1}^{n} a_{jk}x_k(t)dt. \tag{12}$$

The left-hand side of this equation converges to zero. For the right-hand side we use (11) to comput the limit so we conclude that

$$0 = \varepsilon_j + \sum_{k=1}^{n} a_{jk}q_k \qquad (j = 1, \ldots, n),$$

i. e., $\mathbf{q}$ is a fixed point.

Now if system (1) has a unique fixed point $\mathbf{q} \in \mathbb{R}^n_+$ then the linear system (6) has a isolated solution, so the matrix (a_{jk}) must be non-degenerate. In this case, let us consider any $T \geq 0$ and integrate (1) along the orbit $\mathbf{x}(t)$ from 0 to T:

$$\frac{1}{T}\left(\log(x_j(T)) - \log(x_j(0))\right) = \varepsilon_j + \frac{1}{T}\int_0^T \sum_{k=1}^n a_{jk} x_k(t)\,dt. \qquad (13)$$

Solving this equation for the averages we obtain

$$\frac{1}{T}\int_0^T x_j(t)\,dt = \sum_{k=1}^n b_{jk}\left(\frac{1}{T}\left(\log(x_k(T)) - \log(x_k(0))\right) - \varepsilon_k\right),$$

where (b_{jk}) is the inverse of (a_{jk}). By letting $T \to +\infty$, and using the fact that the $x_j(t)$ are bounded, we obtain

$$\lim_{T\to+\infty} \frac{1}{T}\int_0^T x_j(t)\,dt = -\sum_{k=1}^n b_{jk}\varepsilon_k = q_j.$$

□

In the case where the interaction matrix (a_{jk}) is not invertible it is not clear to which fixed point $\mathbf{q}$ does the time average of the orbit converges.

3 Conservative Systems

In the case were system (1) is conservative Volterra was able to introduce a hamiltonian structure for the system by doubling the number of variables. We recall now Volterra's construction, so we assume that system (1) is conservative and a choice of gauge has been made so that the matrix (a_{jk}) is skew-symmetric. Volterra introduces new variables Q_j (which he calls *quantity of life*) through the formula():

$$Q_j = \int_0^t x_j(\tau)\,d\tau \qquad (j = 1, \ldots, n) \qquad (14)$$

and rewrites system (1) as a second order o.d.e.:

$$\ddot{Q}_j = \varepsilon_j \dot{Q}_j + \sum_{k=1}^n a_{jk}\dot{Q}_j\dot{Q}_k \qquad (j = 1, \ldots, n). \qquad (15)$$

Then he observes that the function $H = \sum_{j=1}^n (\varepsilon_j Q_j - \dot{Q}_j)$ is a first integral of the system because, on account of skew-symmetry, one has

$$\dot{H} = -\sum_{j,k=1}^n a_{jk}\dot{Q}_j\dot{Q}_k = 0.$$

One might argue about the "definition" of the Q_j's. The full justification of this procedure will be given later in the section.

Now, if one introduces another set of variables P_j by the formula

$$P_j = \log \dot{Q}_j - \frac{1}{2}\sum_{k=1}^{n} a_{jk} Q_k \qquad (j = 1, \ldots, n) \tag{16}$$

(which are well defined when we restrict the original system to $\mathbb{R}^n_+$), then, in the coordinates (Q_j, P_j), the function H is expressed as

$$H = \sum_{j=1}^{n} \varepsilon_j Q_j - \sum_{j=1}^{n} e^{(P_j + \frac{1}{2}\sum_{k=1}^{n} a_{jk} Q_k)}. \tag{17}$$

A simple computation shows that system (15) can be rewritten in the following hamiltonian form

$$\begin{cases} \dot{P}_j & = & \frac{\partial H}{\partial Q_j} \\ \dot{Q}_j & = & -\frac{\partial H}{\partial P_j} \end{cases} \qquad (j = 1, \ldots, n). \tag{18}$$

We shall now reverse the all procedure and reformulate it in the language of Poisson manifolds(). Recall that the modern approach to hamiltonian systems is based on the following generalization of the notion of a Poisson bracket (see for example [15]).

Definition 3.1. A Poisson bracket on a smooth manifold M is a bilinear operation $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ on the space of smooth functions satisfying the following properties:

i) $\{f_1, f_2\} = -\{f_2, f_1\}$ (skew-symmetry);

ii) $\{f_1 f_2, f\} = f_1\{f_2, f\} + \{f_1, f\} f_2$ (Leibnitz's identity);

iii) $\{f_1, \{f_2, f_3\}\} + \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_1, f_2\}\} = 0$ (Jacobi's identity);

A hamiltonian system on a Poisson manifold M is defined by a choice of a function $h \in C^\infty(M)$, namely, the defining equations for the flow are

$$\dot{x} = X_h(x), \tag{19}$$

where the hamiltonian vector field X_h is the vector field on M defined by

$$X_h(f) = \{f, h\}, \qquad \forall f \in C^\infty(M).$$

One needs here the more general concept of Poisson manifold rather than symplectic manifold since, as we shall see shortly, the Poisson bracket associated with the original system is, in general, degenerate.

For system (18) $M = \mathbb{R}^{2n}$ and the Poisson bracket in question is, of course, the classical Poisson bracket associated with the standard symplectic structure $\omega_s = \sum_{j=1}^n dQ_j \wedge dP_j$:

$$\{f_1, f_2\}_s = \sum_{j=1}^{n} \left(\frac{\partial f_1}{\partial P_j} \frac{\partial f_2}{\partial Q_j} - \frac{\partial f_2}{\partial P_j} \frac{\partial f_1}{\partial Q_j} \right). \tag{20}$$

When we take the function H given by (17) as the hamiltonian function, it is clear that system (18) takes the canonical form

$$\dot{x}_i = \{x_i, H\}_s, \qquad (i = 1, \ldots, 2n).$$

The key remark to reverse Volterra's procedure is the following: system (18) has n, time-dependent (if $\varepsilon_j \neq 0$), first integrals given by the formulas

$$I_j(Q_j, P_j, t) = P_j - \frac{1}{2} \sum_{k=1}^{n} a_{jk} Q_k - \varepsilon_j t \qquad (j = 1, \ldots, n). \tag{21}$$

In fact, one checks easily that

$$\frac{\partial I_j}{\partial t} + \{I_j, H\}_s = 0.$$

Moreover, the first integrals I_j satisfy the following commutatio relation

$$\{I_j, I_k\}_s = a_{jk}. \tag{22}$$

A standard result (see [15]) in the theory of hamiltonian systems says that a family of r-independent, Poisson commuting integrals, allows one to reduce the dimension of the system by $2r$. Hence, if the n integrals I_j had vanishing Poisson bracket, we would be able to reduce the dimension of the system by $2n$, and the equations would be integrable by quadratures. Condition (22) of course does not give such a complete integrability, but it is enough to guarantee that the corresponding hamiltonian vector fields commute

$$\left[X_{I_j}, X_{I_k}\right] = 0, \qquad (j, k = 1, \ldots, n). \tag{23}$$

This allow us to perform a standard (non-hamiltonian) symmetry reduction and reduce the dimension of the system by n.

Theorem 3.2. *The map $\Psi : (Q_i, P_i) \mapsto x_j$ defined by*

$$x_j = e^{\left(P_j + \frac{1}{2}\sum_{k=1}^n a_{jk} Q_k\right)} \qquad \forall (Q, P) \in \mathbb{R}^{2n}$$

is a Poisson map from $\mathbb{R}^{2n}$ with the canonical Poisson bracket (20) to $\mathbb{R}^n_+$ with bracket

$$\{f_1, f_2\} = \sum_{j<k} a_{jk} x_j x_k \left(\frac{\partial f_1}{\partial x_j} \frac{\partial f_2}{\partial x_k} - \frac{\partial f_2}{\partial x_j} \frac{\partial f_1}{\partial x_k} \right). \tag{24}$$

If $(q_1, \ldots, q_n) \in \mathbb{R}^n_+$ is a fixed point of (1), this map reduces the enlarged system (18) to the Volterra system (1).

Proof. One readily verifies that (24) satisfies the conditions of the definition 3.1. It is also a routine calculation to check that the map $\Psi : (Q_i, P_i) \mapsto x_j$ satisfies

$$\{f \circ \Psi, g \circ \Psi\}_s = \{f, g\} \circ \Psi, \qquad \forall f, g \in C^\infty(\mathbb{R}^n_+).$$

If there is an equilibrium and we let

$$h = \sum_{j=1}^{n} (x_j - q_j \log x_j), \tag{25}$$

we check that $H = h \circ \Psi$, and that system (1) can be written in the form

$$\dot{x}_j = \{x_j, h\} \qquad (j = 1, \ldots, n).$$

Hence Ψ reduces the enlarged system (18) to the Volterra system (1). ☐

We leave it to the reader to check that if one considers the action on $\mathbb{R}^{2n}$ of the (abelian) group of symmetries G generated by the hamiltonian vector fields X_{I_j}, then the map $\Psi : \mathbb{R}^{2n} \to \mathbb{R}^n_+$ is exactly the quocient map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}/G$. Therefore the reduction given in theorem 3.2 is in fact a *symmetry reduction.*

Remarks.

(i) In general, one cannot get way without some assumption of the type of (6) and so it is not possible to give a hamiltonian formulation without introducing new variables (if, for example, $(a_{jk}) = 0$ and $\varepsilon_j > 0$ then the origin is a source and system (1) cannot be hamiltonian).

(ii) In [16] the hamiltonian structure (24) is also introduced, along with other hamiltonian formulations valid for particular classes of interaction matrices. However, there is no reference to its relation to the Volterra hamiltonian formulation.

When we combine these ideas with Volterra's criteria for a system to be conservative we obtain

Corollary 3.3. *Assume system (1) has a fixed point in $\mathbb{R}^n_+$. If the matrix associated with the system satisfies $a_{jj} = 0$,*

$$a_{jk} \neq 0 \Longrightarrow a_{jk} a_{kj} < 0 \tag{26}$$

and the graph is a forest, then the system has a direct hamiltonian formulation.

Remark. *If we do not allow the sign change in condition (5) then we obtain a necessary and sufficient condition for the matrix to be symmetrizable. In this case, the system is gradient with respect to the "metric" $ds^2 = \sum_{jk} (d_j a_{jk} x_j x_k) dx_j dx_k$.*

4 Dissipative Systems

We now turn to the study of dissipative systems. Since we want our results to persist under small perturbation we introduce the following definition.

Definition 4.1. A PERTURBATION of a Lotka-Volterra system with interaction matrix A is any Lotka-Volterra system with interaction matrix $\tilde{A}$ such that

$$\tilde{a}_{jk} = 0 \Leftrightarrow a_{jk} = 0.$$

A Lotka-Volterra system with interaction matrix A is called STABLY DISSIPATIVE if every sufficiently small perturbation is dissipative:

$$\exists \delta > 0 \ : \max_{jk} |a_{jk} - \tilde{a}_{jk}| < \delta \Longrightarrow \tilde{A} \text{ is dissipative.}$$

Note that we only allow perturbations that have the same graph as the original system. The notion of stably dissipative system is due to Redheffer*et al.* who in a series of papers [18, 19, 20, 21, 22] have studied the asymptotic stability of this class of systems. Also they use instead the name *stably admissible*. Since what they call *admissible* is called by Volterra *dissipative* ([25], chp. III), we prefer the term *stably dissipative*. For conditions for a matrix to be stably dissipative we refer to [19].

Let us start then with a stably dissipative Lotka-Volterra system having a fixed point $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{R}^n_+$:

$$\begin{cases} \dot{x}_j = \varepsilon_j x_j + \sum_{k=1}^n a_{jk} x_j x_k, \\ \qquad \varepsilon_j + \sum_{k=1}^n a_{jk} q_k = 0 \end{cases} \qquad (j = 1, \ldots, n). \tag{27}$$

The system is dissipative so we can choose a diagonal matrix $D > 0$ such that $AD \leq 0$. For stably dissipative systems this choice can be improved ([21]):

Lemma 4.2. *One can choose a positive matrix $D = diag\,(d_1, \ldots, d_n)$ such that $AD \leq 0$ and the following condition holds*

$$\sum_{j,k=1}^n d_k a_{jk} w_i w_j = 0 \Longrightarrow a_{jj} w_j = 0, (j = 1, \ldots, n).$$

Proof. Given $A = (a_{ij})$ such that the associated system is stably dissipative we consider the perturbation $\tilde{A} = (\tilde{a}_{jk})$ given by

$$\tilde{a}_{jk} = a_{jk} \qquad (j \neq k), \qquad \tilde{a}_{jj} = (1 - \delta) a_{jj}.$$

Also, choose $D > 0$ such that $\tilde{A}D \leq 0$. Since $a_{jj} \leq 0$ and

$$\sum_{j,k=1}^n d_k a_{jk} w_i w_j = \sum_{j,k=1}^n d_k \tilde{a}_{jk} w_j w_k + \delta \sum_{j=1}^n d_j a_{jj} w_j^2,$$

we see that $AD \leq 0$ and

$$\sum_{j,k=1}^{n} d_k a_{jk} w_i w_j = 0 \Longrightarrow a_{jj} w_j = 0, (j = 1, \ldots, n).$$

□

If $D = \text{diag}\ (d_1, \ldots, d_n)$ is a matrix as in the previous lemma, we perform the change of gauge $x_j \mapsto \frac{1}{d_j} x_j$ so we can assume that $A \leq 0$ and

$$\sum_{j,k=1}^{n} a_{jk} w_i w_j = 0 \Longrightarrow a_{jj} w_j = 0, (j = 1, \ldots, n). \tag{28}$$

Then we have a Liapounov function given by

$$V = \sum_{j=1}^{n} (x_j - q_j \log x_j). \tag{29}$$

In fact, we find that

$$\dot{V} = \sum_{j,k=1}^{n} a_{jk} (x_j - q_j)(x_k - q_k) \leq 0.$$

By La Salle's theorem[13], the solutions exist for all $t \geq 0$ and the set $\dot{V} = 0$ contains an attractor. Therefore one would like to understand the set $\dot{V} = 0$.

We shall now recall Redheffer's beautiful description of the attractor in terms of the *reduced graph* of the system. Notice that by (27), (28) and (29) solutions on the set $\dot{V} = 0$ satisfy

$$\begin{cases} \dot{x}_j = x_j \sum_{k=1}^{n} a_{jk} (x_k - q_k), \\ \\ a_{jj} (x_j - q_j) = 0 \qquad (j = 1, \ldots, n). \end{cases} \tag{30}$$

Therefore, one has either $a_{jj} = 0$ or $a_{jj} < 0$, and in the later case we have $x_j = q_j$ on the attractor.

It will be convenient to modify slightly the notion of graph associated with the system we introduced above. One now draws a black dot ● at vertex j if either $a_{jj} < 0$, or $a_{jj} = 0$ and somehow we have shown that $x_j = q_j$ on the attractor. Otherwise, one draws an open circle ○ at vertex j. It is also convenient to put a $\oplus$ at vertex j if one can show that x_j is constant on the attractor (an intermediate stage between black dots and open circles).

We have ([20]):

Lemma 4.3. *The following propagation rules are valid:*

(a) If there is a ● or ⊕ at vertex j and ● at all neighbors of j except one vertex l, then we can put a ● at vertex l;

(b) If there is a ● or ⊕ at vertex j, and a ● or ⊕ at all neighbors of j except one vertex l, then we can put a ⊕ at vertex l;

(c) If there is ○ at vertex j, and ● or ⊕ at all neighbors of j, then we can put ⊕ at vertex j;

Proof. The proof is a straightforward application of (30).

□

One calls the *reduced graph* $R(A)$ of the system, the graph obtained by repeated use of the rules of reduction (a), (b) and (c). Fig. 2 gives an example of a graph and its reduced graph obtained by successive application of these rules. For more on the reduced graph we refer to [20]. Here we shall only need the following fact which follows from the results in [21].

Figure 2: A graph $G(A)$ and its reduced form $R(A)$.

Proposition 4.4. *Let K denote the subgraph of the reduced graph of a stably dissipative Lotka-Volterra system formed by vertices with ○ or ⊕ and connections between them. Then K is a forest, i.e., $K = K_1 \bigcup \cdots \bigcup K_r$ (disjoint) where each K_i is a tree.*

Proof. We have to rule out the existence of a closed path whose vertices are all of type ○ or ⊕. Assume we had such a closed path and label its vertices from 1 to m. Then one has $a_{jj} = 0$ for each $1 \leq j \leq m$, so given two adjacent vertices j and k in this closed closed path we must have

$$a_{jk} + a_{kj} = 0,$$

on account of the condition $A \leq 0$. In other words the reduced system whose graph is the closed path is conservative. By proposition 2.1, this can happen if, and only if

$$a_{12} \cdots a_{m\perp 1m} a_{m1} = (-1)^m a_{1m} a_{mm\perp 1} \cdots a_{21}.$$

Clearly, this condition cannot hold for all small perturbations. Hence the original system would not be stably dissipative. □

We are now in condition to prove theorem 1.2 which we state as follows:

Theorem 4.5. *Consider a Lotka-Volterra system (1) restricted to the flow invariant set* $\mathbb{R}^n_+ \equiv \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \; x_j > 0, j = 1, \ldots, n\}$*, and assume that (i) the system has a singular point* $\mathbf{q} \in \mathbb{R}^n_+$*, and (ii) is stably dissipative. Then the dynamics on the set* $\dot{V} = 0$ *are hamiltonian. Moreover, they can be described by a Lotka-Volterra system of dimension* $m \leq n$.

Proof. Consider the system restricted to $\dot{V} = 0$. We split the variables x_j into two groups labeled by sets J_o and $J_\bullet$. In the first group $\{x_j\}_{j \in J_o}$ we have all the x_j's corresponding to vertices with open circles ○ or ⊕ in $R(A)$, while the second group $\{x_j\}_{j \in J_\bullet}$ we have all the x_j's corresponding to vertices with black circles ● in $R(A)$. For $j \in J_\bullet$ we have $x_j = q_j$, hence the restricted system satisfies

$$\begin{cases} \dot{x}_j &= (\varepsilon_j + \sum_{k \in J_\bullet} a_{jk} q_k) x_j + \sum_{k \in J_o} a_{jk} x_j x_k \qquad \text{if } j \in J_o \\ \\ x_j &= q_j \qquad \text{if } j \in J_\bullet \end{cases} \tag{31}$$

Therefore if we define $\tilde{\varepsilon}_j = \varepsilon_j + \sum_{k \in J_\bullet} a_{jk} q_k$, $\tilde{a}_{jk} = a_{jk}$ $(j, k \in J_o)$, we obtain a new Volterra type system:

$$\dot{x}_j = \tilde{\varepsilon}_j x_j + \sum_{k \in J_o} \tilde{a}_{jk} x_j x_k \qquad (j \in J_o) \tag{32}$$

where the graph associated with the matrix $\tilde{A} = (\tilde{a}_{jk})_{j,k \in J_o}$ is precisely the subgraph K of the reduced graph $R(A)$ formed by vertices with ○ or ⊕ and connections between them. Note that this matrix satisfies $\tilde{a}_{jj} = 0$, and that there exists a diagonal matrix $D > 0$ such that $\tilde{A}D \leq 0$. But this implies

$$d_j \tilde{a}_{jk} + d_k \tilde{a}_{kj} = 0,$$

which shows that

$$a_{jk} \neq 0 \Longrightarrow a_{jk} a_{kj} < 0.$$

Note also that the $(q_j)_{j \in J_o}$ form a solution of the system

$$\tilde{\varepsilon}_j + \sum_{k \in J_o} \tilde{a}_{jk} q_k = 0 \qquad (j \in J_o).$$

By proposition 4.4, we are in the conditions of corollary 3.3, so system (32) has a hamiltonian formulation. □

The proof shows that the dynamics on the attractor can be described by a Lotka-Volterra system of dimension $m \leq n$ whose associated graph is a tree, which is conservative and has a fixed point in $\mathbb{R}^m_+$. Conversely, any such system describes an attractor, since any system whose associated graph is a tree is stably dissipative.

Part II. Behavior of Solutions on the Attractor

5 A Toy Model

One would like to describe the qualitative dynamics on the attractor of the Lotka-Volterra equations. This amounts to classify the dynamics of n-dimensional Lotka-Volterra systems, with skew-symmetric matrix whose associated graph is a forest, and a fixed point $\mathbf{q} \in \mathbb{R}^n_+$. We do not know of such classification, but we shall see by looking at a 4 dimensional linear chain that these dynamics can be rather complex.

First we make some general remarks. If n is odd the Poisson bracket has rank $\leq n-1$ and there exist Casimirs(). In general (any dimension), the Casimirs take the form

$$C(x_1, \ldots, x_n) = \sum_{j=1}^{n} b_j \log x_j,$$

where $(b_1, \ldots, b_n)$ is any vector in the kernel of (a_{ij}). It follows that the dynamics take place on the level sets of these Casimirs, and in the presence of Casimirs we have effectively reduce the dimension. On the other hand, if n is even and the Poisson bracket is non-degenerate then there are no such Casimirs. In fact, apart from these Casimirs, one should expect that generically there should be no other first integrals besides the hamiltonian function h.

Another general remark is that the level sets of h, given by (25), are $n-1$ dimensional spheres $\mathbb{S}^{n\perp 1}$. Locally, in a neighborhood of the fixed point $\mathbf{q}$, this follows from the relations

$$\begin{aligned}
\left(\frac{\partial h}{\partial x_j}\right)_{\mathbf{q}} &= \left(1 - \frac{q_j}{x_j}\right)_{\mathbf{q}} = 0,\\
\left(\frac{\partial^2 h}{\partial {x_j}^2}\right)_{\mathbf{q}} &= \left(\frac{q_j}{x_j^2}\right)_{\mathbf{q}} = \frac{1}{q_j} > 0,\\
\left(\frac{\partial^2 h}{\partial x_j \partial x_k}\right)_{\mathbf{q}} &= 0 \qquad (j \neq k),
\end{aligned}$$

and the Morse lemma. On the other hand, using the flow of $\frac{\text{grad}\, h}{\|\text{grad}\, h\|^2}$, we see that we can deform isotopically each level set onto any other (see [14]). Therefore the level sets are isomorphic to $\mathbb{S}^{n\perp 1}$. Alternatively, we could use the fact that h is a convex function in $\mathbb{R}^n_+$.

In order to illustrate the complexity of the dynamics that can occur on the

A Casimir is a function that Poisson commutes with any other function (see [15]).

attractor we will consider the following 4-dimensional Lotka-Volterra system:

$$\left\{\begin{array}{rcl} \dot{x}_1 &=& -x_1 + x_1 x_2, \\ \dot{x}_2 &=& +x_2 - x_2(x_1 - \delta x_3), \\ \dot{x}_3 &=& -x_3 + x_3(x_4 - \delta x_2), \\ \dot{x}_4 &=& +x_4 - x_4 x_3. \end{array}\right. \tag{33}$$

We have included a parameter δ which must be restricted to $]-1, +\infty[$ since we need the fixed point $\mathbf{q} = (1+\delta, 1, 1, 1+\delta)$ to belong to $\mathbb{R}^4_+$. The interaction matrix is skew-symmetric

$$(a_{ij}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \delta & 0 \\ 0 & -\delta & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{34}$$

and its graph is a linear chain:

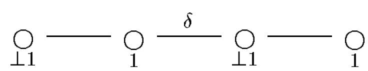

Figure 3: Graph $G(A, \varepsilon)$ associated with system (33).

If $\delta = 0$ the system is separable and hence is completely integrable. Two independent analytic integrals in involution are

$$\begin{array}{rcl} I_1(\mathbf{x}) &=& x_1 + x_2 - \log x_1 - \log x_2, \\ I_2(\mathbf{x}) &=& x_3 + x_4 - \log x_3 - \log x_4. \end{array}$$

The common level sets of these integrals are 2-dimensional tori $\mathbb{S}^1 \times \mathbb{S}^1$, and the solutions are almost periodic. We are interested in investigating what happens when $\delta \neq 0$.

6 Periodic Orbits

For any δ system (33) has a time reversing symmetry. To see this let $\sigma : \mathbb{R}^4_+ \to \mathbb{R}^4_+$ be the involution

$$\sigma(x_1, x_2, x_3, x_4) = (x_4, x_3, x_2, x_1).$$

The hamiltonian vector field X_h associated with the system satisfies $\sigma_* X_h = -X_h$. Therefore, if $\mathbf{x}(t)$ is a solution of (33) so is $\sigma(\mathbf{x}(-t))$, and we see that σ defines a time reversing symmetry of the system. In particular, it follows that any solution crossing twice the set $\text{Fix}(\sigma) = \{x_1 = x_4, x_2 = x_3\}$ is a periodic solution. Using this method we can find the following family of periodic orbits.

Lemma 6.1. *For any δ, the 2-plane*

$$\Pi = \{(x_1, x_2, x_3, x_4) : x_1 = (1+\delta)x_3, x_4 = (1+\delta)x_2\}$$

is formed by periodic orbits of system (33).

Proof. If we look for solutions of the form

$$\begin{cases} x_1 &= (1+\delta)u, \\ x_2 &= v, \\ x_3 &= u, \\ x_4 &= (1+\delta)v, \end{cases} \tag{35}$$

we see that u and v satisfy the predator-prey system

$$\begin{cases} \dot{u} &= -u + uv, \\ \dot{v} &= +v - uv. \end{cases} \tag{36}$$

This system has a fixed point $(1,1) \in \mathbb{R}^2_+$ and, moreover, all its solutions are periodic. These in turn give periodic solutions of the original system.

□

If $\mathbf{x}(\neq \mathbf{q})$ belongs to the 2-plane Π of periodic solutions, then

$$dh(\mathbf{x})(\mathbf{x} - \mathbf{q}) = \frac{(x_2 - 1)^2(2+\delta)}{x_2} + \frac{(x_3 - 1)^2(2+\delta)}{x_3} \neq 0, \tag{37}$$

so the level sets of h and the 2-plane Π intersect transversely along the periodic orbit Γ. Therefore, we have the following figure.

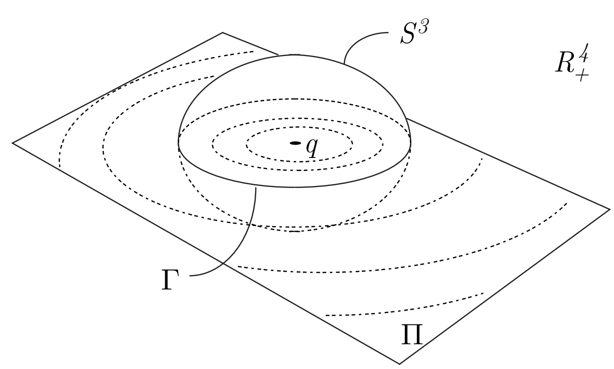

Figure 4: Energy levels and periodic orbits of system (33)

Let us consider now the 1-parameter family of 3-planes

$$\{\mathbf{x} \in \mathbb{R}^4_+ : \phi_\theta(\mathbf{x}) \equiv \cos\theta(x_1 - (1+\delta)x_3) + \sin\theta(x_4 - (1+\delta)x_2) = 0\}, \tag{38}$$

where the parameter θ varies in $[0, \pi[$. Each plane in this family intersects a fixed energy level set $\{h = E\} \simeq \mathbb{S}^3$ along a 2-sphere since h is convex:

$$\mathbb{S}^2_\theta = \{\mathbf{x} \in \mathbb{R}^4_+ : h(\mathbf{x}) = E, \phi_\theta(\mathbf{x}) = 0\}. \tag{39}$$

We shall show that for each sphere $\mathbb{S}^2_\theta$, with $\theta \in [0, \pi/2]$, the flow induces a return map $f_\theta : \mathbb{S}^2_\theta \to \mathbb{S}^2_\theta$ of the sphere. This map codifies all the dynamics since every orbit of the system in the fixed energy level set intersects the sphere.

First we need the following

Lemma 6.2. *As θ varies in $[0, \pi[$ the family of spheres $\mathbb{S}_\theta$ covers the energy level set $\{h = E\} \simeq \mathbb{S}^3$:*

$$\mathbb{S}^3 = \bigcup_{\theta \in [0,\pi[} \mathbb{S}^2_\theta.$$

Moreover, they intersect along the unique periodic orbit Γ of the family Π which lies inside $\mathbb{S}^3$:

$$\Gamma = \bigcap_{\theta \in [0,\pi[} \mathbb{S}^2_\theta.$$

Proof. If $\mathbf{x} \in \mathbb{R}^4_+$ we can choose $\theta \in [0, \pi[$ such that $\mathbf{x}$ lies in the 3-plane (38) by setting

$$\theta = -\arctan \frac{x_1 - (1+\delta)x_3}{x_4 - (1+\delta)x_2}$$

(if $x_4 - (1+\delta)x_2 = 0$ we take $\theta = \pi/2$). Therefore, the family of spheres $\mathbb{S}^2_\theta$ covers the energy level set.

If $\theta_1 \neq \theta_2$ and $\mathbf{x} \in \mathbb{S}^2_{\theta_1} \cap \mathbb{S}^2_{\theta_2}$ then $\mathbf{x}$ is a solution of the system

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 \\ \cos\theta_2 & \sin\theta_2 \end{pmatrix} \begin{pmatrix} x_1 - (1+\delta)x_3 \\ x_4 - (1+\delta)x_2 \end{pmatrix} = 0.$$

Since the determinant of the matrix of this system is $2\sin(\theta_2 - \theta_1) \neq 0$, we see that $\mathbf{x}$ must satisfy the system

$$\begin{cases} x_1 - (1+\delta)x_3 = 0, \\ x_4 - (1+\delta)x_2 = 0, \end{cases}$$

i. e., it belongs to Γ. □

Volterra's time average principle (proposition 2.3) states that

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \mathbf{x}(t)dt = \mathbf{q}.$$

This average behavior *suggests* that an orbit starting at a sphere $\mathbb{S}^2_\theta$ should eventually return to the sphere. This however does not follow from Volterra's principle. What we can say is that every orbit of system (33) must visit every neighborhood of a sphere $\mathbb{S}^2_\theta$. In fact, we find

$$\begin{aligned}\lim_{T\to+\infty}\frac{1}{T}\int_0^T \phi_\theta(\mathbf{x}(t))dt &= \lim_{T\to+\infty}\frac{1}{T}\int_0^T \cos\theta(x_1-(1+\delta)x_3)+\sin\theta(x_4-(1+\delta)x_2)dt \\ &= \cos\theta(q_1-(1+\delta)q_3)+\sin\theta(q_4-(1+\delta)q_2)=0,\end{aligned}$$

for any orbit $\mathbf{x}(t)$. So we see that for every $\varepsilon > 0$ there exists a time t_ε such that

$$|\phi_\theta(\mathbf{x}(t_\varepsilon))| < \varepsilon.$$

This of course *does not* mean that the orbit actually returns to the sphere. It could, for example, approach the orbit Γ always from the same side of the sphere $\mathbb{S}_\theta$. The fact that this does not happen is a consequence of the following proposition.

Proposition 6.3. *Let* $\mathbf{x}_0 \in \mathbb{S}^2_0 - \Gamma$ *and denote by* $\mathbf{x}(t)$ *the orbit of system (33) with initial condition* $\mathbf{x}_0$*. Then there are times* $0 < t_0 < t_1$ *such that:*

(i) $\mathbf{x}(t_0)$ *lies in a different connected component of* $\mathbb{S}^2_0 - \Gamma$ *than* $\mathbf{x}_0$*;*

(ii) $\mathbf{x}(t_1)$ *lies in the same connected component of* $\mathbb{S}^2_0 - \Gamma$ *as* $\mathbf{x}_0$*.*

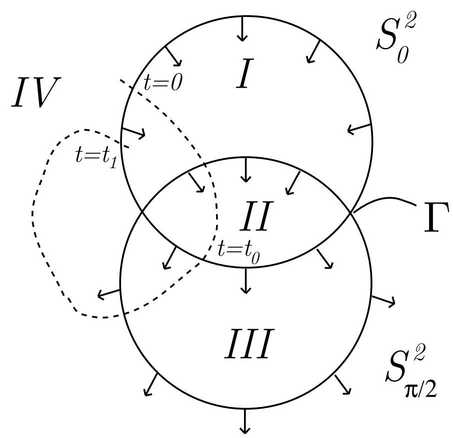

Figure 5: Flow of system (33) on an energy level set.

Proof. The spheres $\mathbb{S}^2_0$ and $\mathbb{S}^2_{\pi/2}$ are the boundaries of four connected regions in the energy level set $\{h = E\} \simeq \mathbb{S}^3$. These regions are determined by the signs of the functions $\phi_0(\mathbf{x}) = x_1 - (1+\delta)x_3$ and $\phi_{\pi/2}(\mathbf{x}) = x_4 - (1+\delta)x_2$ as described in the following table:

	Region			
	I	II	III	IV
$\phi_0(\mathbf{x})$	$-$	$-$	$+$	$+$
$\phi_{\pi/2}(\mathbf{x})$	$+$	$-$	$-$	$+$

Now we observe that $V_0(\mathbf{x}) = \log(\frac{x_1}{x_3})$ and $V_{\pi/2}(\mathbf{x}) = \log(\frac{x_4}{x_2})$ are local Liapounov functions in the regions I–IV. In fact, we compute along an orbit of system (33)

$$\dot{V}_0 = \frac{d}{dt}\log(\frac{x_1}{x_3}) = -x_4 + (1+\delta)x_2,$$
$$\dot{V}_{\pi/2} = \frac{d}{dt}\log(\frac{x_4}{x_2}) = x_1 - (1+\delta)x_3.$$

This gives the following behavior for the signs

	Region			
	I	II	III	IV
$\dot{V}_0$	$-$	$+$	$+$	$-$
$\dot{V}_{\pi/2}$	$-$	$-$	$+$	$+$

Also, we have

$$\mathbb{S}^2_0 = \{V_0 = \log(1+\delta)\} = \{\dot{V}_{\pi/2} = 0\},$$
$$\mathbb{S}^2_{\pi/2} = \{V_{\pi/2} = \log(1+\delta)\} = \{\dot{V}_0 = 0\}.$$

It follows that

$$\dot{V}_0 = dV_0(X_h) \neq 0 \qquad \text{on} \qquad \mathbb{S}^2_0 - \mathbb{S}^2_{\pi/2} = \mathbb{S}^2_0 - \Gamma,$$
$$\dot{V}_{\pi/2} = dV_{\pi/2}(X_h) \neq 0 \qquad \text{on} \qquad \mathbb{S}^2_{\pi/2} - \mathbb{S}^2_0 = \mathbb{S}^2_{\pi/2} - \Gamma$$

Therefore, X_h is transversal to $\mathbb{S}^2_0 - \Gamma$ and to $\mathbb{S}^2_{\pi/2} - \Gamma$.

We now claim that if $\mathbf{x}(t)$ is a solution of system (33) which at time t_i is in the interior of some region R ($R = I, II, III, IV$) then the solution must leave region R, so there exists some later time $t_l > t_i$ for which $\mathbf{x}(t_l)$ is in the interior of region $R + I \pmod 4$.

Assume for example $\mathbf{x}(t_i)$ is in the interior of region I. Then we have

$$\phi_0(\mathbf{x}(t_i)) < 0 \Longrightarrow \frac{\mathbf{x}_1(t_i)}{\mathbf{x}_3(t_i)} < 1 + \delta.$$

If $\mathbf{x}(t)$ stayed for ever in region I then its ω-limit set would be in

$$\{\dot{V}_0 = 0\} \cap \{\dot{V}_{\pi/2} = 0\} = \mathbb{S}^2_0 \cap \mathbb{S}^2_{\pi/2} = \Gamma.$$

This means that the ratio $\frac{\mathbf{x}_1}{\mathbf{x}_3}$ should approach $1+\delta$, which contradicts the fact that in region I we have $V_0 = \log(\frac{\mathbf{x}_1}{\mathbf{x}_3})$ strictly decreasing. Therefore, $\mathbf{x}(t)$ must leave region I. The transvesality condition on the boundaries guarantees that there exists $t_l > t_i$ for which $\mathbf{x}(t_l)$ is inside region II.

The reasoning for the other regions is similar, so the proposition follows.

□

We have seen that for the spheres $\mathbb{S}_0^2$ and $\mathbb{S}_{\pi/2}^2$ the flow is transversal except at points of Γ. This also holds for any sphere $\mathbb{S}_\theta^2$ with $\theta \in [0, \pi/2]$, and we obtain

Theorem 6.4. *For any $\theta \in [0, \pi/2]$, the flow of system (33) induces a homeomorphism of the sphere $f_\theta : \mathbb{S}_\theta^2 \to \mathbb{S}_\theta^2$. The periodic orbit Γ divides de sphere $\mathbb{S}_\theta^2$ into two open hemispheres, and f_θ fixes Γ and maps each open hemisphere diffeomorphically onto the other.*

Proof. For any sphere $\mathbb{S}_\theta^2$ with $\theta \in [0, \pi/2]$, we observe that X_h is transversal to $\mathbb{S}_\theta^2 - \Gamma$. In fact, we compute

$$\begin{aligned} d\phi_\theta \cdot X_h =&(\cos\theta + (1+\delta)\sin\theta)x_1x_2 + \delta(1+\delta)(\cos\theta - \sin\theta)x_2x_3 \\ &- ((1+\delta)\cos\theta + \sin\theta)x_3x_4. \end{aligned}$$

If $\mathbf{x} \in \mathbb{R}_+^4$ satisfies $\phi_\theta(\mathbf{x}) = 0$ we have

$$\mathbf{x} = a_1(1+\delta, 0, 1, 0) + a_2(0, 1, 0, 1+\delta) + a_3(\sin\theta, 0, 0, -\cos\theta),$$

for some real numbers $a_1 > 0$, $a_2 > 0$ and a_3. It follows that

$$\begin{aligned} d\phi_\theta \cdot X_h|_{\mathbb{S}_\theta^2} =&a_1a_3(\cos\theta\sin\theta + (1+\delta)\cos^2\theta) \\ &+ a_2a_3(\cos\theta\sin\theta + (1+\delta)\sin^2\theta). \end{aligned} \tag{40}$$

Therefore, if $\theta = 0, \pi/2$ the hamiltonian vector field X_h is transversal to the the sphere $\mathbb{S}_\theta^2$ except at those points where $a_3 = 0$, i. e., except for those $\mathbf{x} \in \Gamma$.

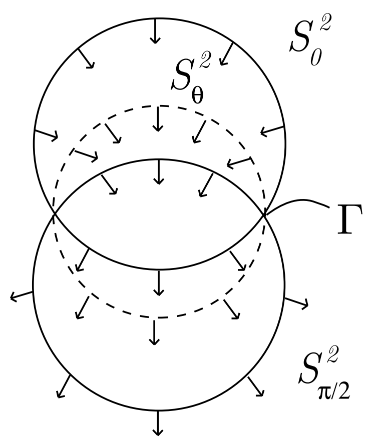

Figure 6: Spheres $\mathbb{S}_\theta^2$ for $\theta \in [0, \pi/2]$.

It is clear from the proof of the previous proposition that an orbit starting on a open hemisphere of $\mathbb{S}_\theta^2 - \Gamma$ will hit first the other open hemisphere before returning. The theorem then follows from standard results on continuity and differentiability of solutions o.d.e.'s with respect to the initial conditions. $\square$

We remark that for $\pi/2 < \theta < \pi$ there are points in $\mathbb{S}^2_\theta - \Gamma$ where the flow is tangential, so for these spheres theorem 6.4 fails.

Note also that the spheres $\mathbb{S}^2_\theta$ and $\mathbb{S}^2_{\pi/2 \perp \theta}$ are conjugate under the involution σ. Therefore, although all spheres $\mathbb{S}^2_\theta$, with $\theta \in [0, \pi/2]$, give the full description of the dynamics, it is very natural to consider the "symmetric" sphere $\mathbb{S}^2_{\pi/4}$. For this sphere the map $f_{\pi/4}$ is conjugated to its inverse $f^{\perp 1}_{\pi/4}$ through the involution σ.

In the integrable case $\delta = 0$ it is not hard to figure out the phase portrait of the map f_θ. The orbits of this map are the intersection of the cilinders given by the level sets of the integral I_1 (or I_2) with the sphere $\mathbb{S}_\theta$ (see section 5).

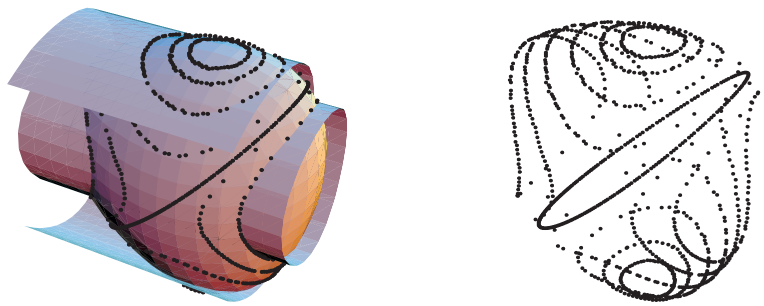

Figure 7: Intersections of $I_1 = const.$ with the sphere $\mathbb{S}_\theta$.

These levels sets consist of

- Two circles of degenerate fixed points corresponding to $I_1 = I_2 = h/2$ (one circle is Γ);
- Two elliptic fixed points corresponding to the periodic orbits with the fixed energy and satisfying, respectively, $x_1 = x_2 = 1$ and $x_3 = x_4 = 1$;
- periodic orbits around the two elliptic fixed points;

In the following picture we show the phase portrait of $f_{\pi/4}$ on the sphere $\mathbb{S}^2_{\pi/4}$ for $\delta = 0$. Note that we only need the portrait of one of the hemispheres, the other one being homeomorphic.

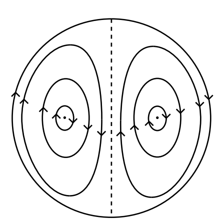

Figure 8: Phase portrait of $f_{\pi/4}$ for $\delta = 0$.

The phase portrait of the map f for $\delta \neq 0$ is much more complex. To have some insight we now turn to the study of the stability of periodic orbits.

7 Stability of Periodic Orbits

We shall now consider the stability of the periodic orbits of the family Π. Recall from lemma 6.1 that these orbits are parametrized by the solutions of the two dimensional Volterra system

$$\begin{cases} \dot{u} &= -u + uv, \\ \dot{v} &= +v - uv. \end{cases} \tag{41}$$

This system is hamiltonian with $h_0 = u + v - \log(uv)$.

For each value $h_0 = E > 2$, the periodic solution $(u(t, E), v(t, E))$ of system (41) determines a periodic solution $\Gamma = \Gamma(\delta, E)$ of the original system. Therefore, the energy parametrizes the orbits in the family Π, and we have()

Lemma 7.1. *The period $T = T(E)$ of the orbits lying in Π is a strictly increasing function of the energy. In fact, $\frac{dT}{dE} > 0$.*

Proof. See [23] for a proof. □

Later, in theorem 9.4, we will derive an asymptotic formula for the period $T(E)$ as $E \to \infty$.

Next we will show that the stability of the orbits in the family Π can be reduced to a Sturm-Liouville problem. First we look at the linearization around a periodic orbit $\Gamma(\delta, E) \subset \Pi$.

Proposition 7.2. *Let $\Gamma = \Gamma(\delta, E) \subset \Pi$ be the periodic orbit of system (33) associated with a solution $(u, v) = (u(t, E), v(t, E))$ of system (41). Then Γ has one characteristic multiplier equal to 1, and the other two multipliers $\sigma_1(\delta, E)$ and $\sigma_2(\delta, E)$ coincide with the Floquet multipliers of the linear system with periodic coefficients*

$$\dot{\mathbf{w}} = (1 + \delta) \begin{pmatrix} 0 & u(t, E) \\ -v(t, E) & 0 \end{pmatrix} \mathbf{w}. \tag{42}$$

They satisfy the hamiltonian symmetry

$$\sigma_1(\delta, E)\sigma_2(\delta, E) = 1. \tag{43}$$

Proof. Let $\Gamma = \{\mathbf{x}_0(t) : t \in [0, T]\}$ be a T-periodic solution of system (33) associated with a T-periodic solution (u, v) of system (41). We linearize the system around this T-periodic orbit and obtain the linear system with periodic coefficients:

$$\dot{\mathbf{z}} = L(t, \delta)\mathbf{z}, \tag{44}$$

Note that the energy of an orbit of system (41) and the energy of the corresponding orbit in Π of system (33) are related by a multiplicative factor of $(2 + \delta)$.

where $L = L(t, \delta)$ is given by

$$L = \left(\frac{\partial X_i}{\partial x_j}\right)_{\mathbf{x}_0(t)} = \begin{pmatrix} -1+v & (1+\delta)u & 0 & 0 \\ -v & 1-u & +\delta v & 0 \\ 0 & -\delta u & -1+v & u \\ 0 & 0 & -(1+\delta)v & 1-u \end{pmatrix}. \tag{45}$$

Now consider the linear change of coordinates given by

$$\widehat{\mathbf{w}} = \begin{pmatrix} 0 & -(1+\delta) & 0 & 1 \\ 1 & 0 & -(1+\delta) & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \mathbf{z}.$$

Then equation (44) is transformed into

$$\dot{\widehat{\mathbf{w}}} = \begin{pmatrix} 1-u & (1+\delta)\,v & 0 & 0 \\ -\,(1+\delta)\,u & -1+v & 0 & 0 \\ 0 & 0 & 1-u & -v \\ 0 & 0 & u & -1+v \end{pmatrix} \widehat{\mathbf{w}}.$$

If we make the time-dependent change of variables:

$$\widehat{w}_1 = v(t)w_1,\ \widehat{w}_2 = u(t)w_2,\ \widehat{w}_3 = v(t)w_3,\ \widehat{w}_4 = u(t)w_4,$$

we find, using (41), that $\mathbf{w}$ satisfies the linear system

$$\dot{\mathbf{w}} = \begin{pmatrix} 0 & (1+\delta)\,u & 0 & 0 \\ -\,(1+\delta)\,v & 0 & 0 & 0 \\ 0 & 0 & 0 & -u \\ 0 & 0 & v & 0 \end{pmatrix} \mathbf{w}.$$

Therefore, we conclude that the linearization around the periodic orbit is equivalent to two, 2-dimensional, linear systems with periodic coefficients. Let $\sigma_1 = \sigma_1(\delta, E)$ and $\sigma_2 = \sigma_2(\delta, E)$ be the Floquet multipliers of the first system

$$\dot{\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}} = (1+\delta) \begin{pmatrix} 0 & u(t) \\ -v(t) & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

and denote by $B(t)$ the matrix of this system. We have

$$\sigma_1\sigma_2 = \exp\left(\int_0^T \mathrm{tr}B(s)ds\right) = 1.$$

On the other hand, the Floquet multipliers of the second system

$$\dot{\begin{pmatrix} w_3 \\ w_4 \end{pmatrix}} = \begin{pmatrix} 0 & -u(t) \\ v(t) & 0 \end{pmatrix} \begin{pmatrix} w_3 \\ w_4 \end{pmatrix}.$$

also have hamiltonian symmetry. Since $\mathbf{w}(t) = (1-u(t), v(t)-1)$ is a T-periodic solution of this system, its Floquet multipliers are equal to 1. □

Note that $u(t,E)$ and $v(t,E)$ are positive, smooth, T-periodic functions. System (42) is then equivalent to the eigenvalue equation

$$L[x] = -\frac{1}{v(t,E)}\left(\frac{x'}{u(t,E)}\right)' = \lambda x \tag{46}$$

where we set $\lambda = \sqrt{1+\delta}$, $w_1 = x$ and $w_2 = \frac{x'}{\sqrt{\lambda}u(t)}$. This remark implies

Proposition 7.3. *Let $\Phi(t,\delta,E)$ $(E>2)$ denote the fundamental matrix solution of (42) with initial value $\Phi(0,\delta,E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and set*

$$f(\delta,E) = tr\,\Phi(T(E),\delta,E) = \sigma_1(\delta,E) + \sigma_2(\delta,E).$$

Then for each value of the energy E there exists a sequence of parameters $\delta_i = \delta_i(E)$ and $\tilde\delta_i = \tilde\delta_i(E)$ such that

$$\delta_0 < \tilde\delta_1 \le \tilde\delta_2 < \delta_1 \le \delta_2 < \tilde\delta_3 \le \tilde\delta_4 < \delta_3 \le \delta_4 < \cdots \tag{47}$$

such that

$$f(\delta_i(E),E) = 2, \qquad \textit{and} \qquad f(\tilde\delta_i(E),E) = -2. \tag{48}$$

Furthermore, we have $\delta_0 = -1$, $\delta_1(E) = 0$ and $\delta_1(E) < \delta_2(E)$.

Proof. The existence of the sequence (47) satisfying (48) follows from standard results in Sturm-Liouville theory (see, e. g., [3] chapter 8 and [6] chapter 5). It is also obvious that $\delta_0 = -1$ and, as in the the proof of proposition 7.2, we see that for $\delta = 0$ we have a T-periodic solution, so $\delta_1 = 0$. It remains to show that $\delta_2 > \delta_1$.

We also know from standard Sturm-Liouville theory that $\delta_1 = \delta_2 = 1$ iff there are two linearly independent periodic solutions of the linear system

$$\dot{\mathbf{w}} = \begin{pmatrix} 0 & -u(t) \\ v(t) & 0 \end{pmatrix} \mathbf{w}. \tag{49}$$

We claim that this is not the case. First we remark that system (49) is (equivalent to) the linearization of system (41) around the periodic orbit $(u(t),v(t))$. This follows from a computation as in the proof of proposition 7.2. We will show now that the fact the period of the orbits is a monotone function of the energy (lemma 7.1) implies that system (49) cannot have two linearly independent periodic solutions.

In fact, we can introduce action-angle variables (s,ϕ) in a neighborhood of the periodic orbit such that system (41) is equivalent to

$$\begin{cases} \dot s = 0, \\ \dot\phi = -\frac{\partial h_0}{\partial s}, \end{cases}$$

where $h_0 = h_0(s)$. The periodic orbit $(u(t), v(t))$ corresponds to some solution $s(t) = c_1$, $\phi(t) = -\frac{\partial h_0}{\partial s}(c)t + c_2$ for some constants c_1 and c_2. If we linearize the system in action-angle variables we obtain the linear system

$$\dot{\mathbf{w}} = \begin{pmatrix} 0 & 0 \\ -\frac{\partial^2 h_0}{\partial s^2}(c) & 0 \end{pmatrix} \mathbf{w}, \tag{50}$$

so all that remains to show is that $-\frac{\partial^2 h_0}{\partial s^2} \neq 0$. Now recall how the action variable s is constructed (see [2], chapter 10): if the original system is written in canonical coordinates (p, q) then $s(E)$ is the area enclosed by the orbit $\{h_0(p, q) = E\}$:

$$s(E) = \int_{h_0(p,q) \leq E} dq \wedge dp.$$

Moreover, the period of the orbits is given by

$$T(E) = \frac{\partial s}{\partial E}.$$

Since we know from lemma 7.1 that $\frac{dT}{dE} \neq 0$, we have $\frac{\partial^2 s}{\partial E^2} \neq 0$ and implicit differentiation gives $-\frac{\partial^2 h_0}{\partial s^2} \neq 0$ as desired.

□

As a corollary we obtain the following bifurcation

Corollary 7.4. *For a fixed energy E, the periodic orbit $\Gamma(\delta, E)$ changes its stability from elliptic to hyperbolic as δ crosses zero.*

The family of periodic orbits Π we have been discussing so far can also be obtained by linearizing the system around the fixed point $\mathbf{q}$. Recall that a fixed point $\mathbf{q}$ of a 4-dimensional hamiltonian system (M^4, ω, h) on a symplectic manifold is a *non-ressonant elliptic singular point* if the eigenvalues $\lambda_1, \lambda_2, \bar{\lambda}_1, \bar{\lambda}_2$ of the linearization of the hamiltonian vector field X_h at $\mathbf{q}$ satisfy:

(i) $\{\lambda_1, \lambda_2\}$ are simple $(\lambda_1 \neq \lambda_2)$;

(ii) each λ_i has real part zero;

(iii) λ_1 and λ_2 are $\mathbb{Z}$-linearly independent;

In this case we have the Liapounov sub-center theorem (see [1], chapter V):

Theorem 7.5. *(*Liapounov*) For each pair $(\lambda_i, \bar{\lambda}_i)$ there exists a 2-dimensional manifold of periodic orbits Π_i through $\mathbf{q}$ such that the tangent space $T_{\mathbf{q}}\Pi_i$ is the eigenspace corresponding to the pair $(\lambda_i, \bar{\lambda}_i)$.*

The eigenvalues of the linearization of system (33) at $\mathbf{q}$ are $\pm i$ and $\pm i(1+\delta)$. If $\delta \neq 0$, the eigenspace corresponding to the pair $(i, -i)$ is

$$T_{\mathbf{q}}\Pi_1 = L(\{(1+\delta, 0, 1, 0), (0, 1, 0, 1+\delta)\},$$

while the eigenspace corresponding to the pair $(i(1+\delta), -i(1+\delta))$ is

$$T_{\mathbf{q}}\Pi_2 = L(\{(-1, 0, 1, 0), (0, 1, 0, -1)\}.$$

Therefore, for $\delta \notin \mathbb{Q}$, $\mathbf{q}$ is non-ressonant and the family Π_1 given by Liapounov's theorem coincides with the family Π we have studied before.

There are two other families of periodic orbits Π_3 and Π_4 through the fixed point $\mathbf{q}$, at least for small values of δ. In fact, for $\delta = 0$ we have the two families of periodic orbits $\Pi_3 = \{x_3 = x_4 = 1\}$ and $\Pi_4 = \{x_1 = x_2 = 1\}$. Moreover, it is easy to check that these orbits are elliptic, and hence must persist for small δ. Note also that these orbits are conjugated by the involution σ.

The stability of the family of periodic orbits Π_2 is harder to obtain, but we conjecture that as δ crosses zero these orbits change from hyperbolic to elliptic. This would mean a change of stability between Π_1 and Π_2.

	$\delta < 0$	$\delta = 0$	$\delta > 0$	
Π_1	elliptic	parabolic	hyperbolic	
Π_2	hyperbolic	parabolic	elliptic	*(conjectured)*
Π_3	elliptic			
Π_4	elliptic			

From this table we obtain the following sketch for the phase portraits of the map $f_{\pi/4} : \mathbb{S}^2_{\pi/4} \to \mathbb{S}^2_{\pi/4}$ as δ crosses zero (again we consider only one hemisphere, the other one being homeomorphic):

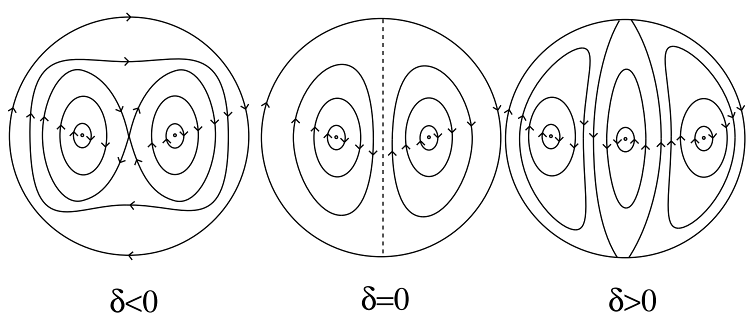

Figure 9: Bifurcation of $f_{\pi/4}$ as δ crosses zero.

In fact, there is much more to this phase portrait as we will show in the next section that for $\delta \neq 0$ the system is non-integrable. Also, simple numerical integration schemes show the appearance of elliptic isles.

8 Non-integrability

Poincaré [17] observed that the existence of independent integrals in a neighborhood of a periodic orbit forces some of its characteristic multipliers to be 1. This remark can be explored to look for integrals in a neighborhood of a periodic solution. In this section we carry through with this idea to show that the dynamics described by the 4-dimensional hamiltonian system (33) are non-integrable.

The key result is the following theorem due to Poincaré on the relationship between integrals in involution and characteristic multipliers. For a proof and a complete discussion we refer to [12].

Theorem 8.1. *(Poincaré) Suppose a Hamiltonian system $(M, \{\ ,\ \}, h)$ admits k-integrals $I_1, \ldots, I_k$ in involution*

$$\{I_j, I_l\} = 0, \qquad (j, l = 1, \ldots, k),$$

independent at some point $\mathbf{x}_0$ of a periodic solution Γ

$$\mathbf{x}_0 \in \Gamma,\ dI_1 \wedge \cdots \wedge dI_k(\mathbf{x}_0) \neq 0.$$

Then Γ has $2k-1$ characteristic multipliers equal to 1.

Consider now a 4-dimensional hamiltonian system (M^4, ω, h) on a 4-dimensional symplectic manifold, and assume that the system has a non-ressonant elliptic singular point $\mathbf{q} \in M$. We shall say that the system is *completely integrable* in a neighborhood U of $\mathbf{q}$ if there exists a second first integral I such that

$$dI \wedge dh(\mathbf{x}) \neq 0, \qquad \forall \mathbf{x} \in O,$$

where O is some open dense set in U. Using the results of Ito [11] and Eliasson [5] on Birkhoff canonical forms, one can prove the following criteria.

Theorem 8.2. *If a hamiltonian system (M^4, ω, h) is completely integrable in a neighborhood of a non-ressonant elliptic singular point $\mathbf{q} \in M$ then the only non-degenerate families of periodic orbits through $\mathbf{q}$ are the ones given by the Liapounov theorem.*

Proof. Assume that the system is completely integrable. Then ([11, 5]) there exist canonical coordinates $(\xi_1, \xi_2, \eta_1, \eta_2)$ defined in a neighborhood of $\mathbf{q}$ such that

$$h = h\left(\frac{\xi_1^2+\eta_1^2}{2}, \frac{\xi_2^2+\eta_2^2}{2}\right) \qquad \text{and} \qquad I = I\left(\frac{\xi_1^2+\eta_1^2}{2}, \frac{\xi_2^2+\eta_2^2}{2}\right).$$

If we let

$$\xi_i = \sqrt{2s_i}\cos\phi_i, \qquad \eta_i = \sqrt{2s_i}\sin\phi_i, \qquad (i = 1, 2)$$

we obtain action-angle variables in a neighborhood of $\mathbf{q}$:

$$h = h(s_1, s_2), \qquad \omega = \sum_i d\xi_i \wedge d\eta_i = \sum_i ds_i \wedge d\phi_i.$$

Now, by Poincaré theorem, if Π is any family of non-degenerate periodic orbits through $\mathbf{q}$ we must have

$$\left\{\begin{array}{l} dh \wedge ds_1|_{\Pi} = 0, \\ dh \wedge ds_2|_{\Pi} = 0, \end{array}\right. \Longrightarrow ds_1 \wedge ds_2|_{\Pi} = 0.$$

Since Π is a smooth 2-dimensional manifold, we check easily that this condition gives $\Pi = \{s_1 = 0\}$ or $\Pi = \{s_2 = 0\}$, i. e., Π is one of the families given by Liapounov theorem. □

We have seen in the previous section that, for $\delta \neq 0$ small, system (33) has at least 3 families of nondegenerate periodic orbits through $\mathbf{q}$. Hence this criteria can be applied to system (33) and we obtain

Corollary 8.3. *For sufficiently small $\delta \neq 0$ system (33) is non-integrable.*

9 Strong Hyperbolicity

The results we have obtain so far for system (33) deal mostly with small values of the parameter δ and small values of the energy E (i. e., a neighborhood of $\mathbf{q}$). In this section we consider other regions of these parameters, and we show that we can find regions of strong hyperbolicity.

Let us consider again the linearization of system (33) around a periodic orbit $\Gamma(\delta, E) \subset \Pi$ which, according to proposition 7.2, can be reduced to the linear system

$$\dot{\mathbf{w}} = (1+\delta)\begin{pmatrix} 0 & u \\ -v & 0 \end{pmatrix}\mathbf{w}. \tag{51}$$

In polar coordinates, $w_1 = r\cos\theta$, $w_2 = r\sin\theta$, this system is equivalent to

$$\left\{\begin{array}{rl} \dot{r} = & (1+\delta)(u-v)r\sin\theta\cos\theta \\ \dot{\theta} = & -(1+\delta)(v\cos^2\theta + u\sin^2\theta) \end{array}\right. \tag{52}$$

The second equation defines a flow on $\mathbb{R}/2\pi\mathbb{Z}$ with rotation number

$$\rho(\delta, E) = -\frac{T}{\pi}\lim_{t\to\infty}\frac{\theta(t,\delta,E) - \theta(0,\delta,E)}{t},$$

where $\theta(t,\delta,E)$ denotes any solution of the equation. The number $\rho(\delta, E)$ measures (counterclockwisely) the average number of half turns per period of a vector $\Phi(t,\delta,E)\mathbf{v}_0$ when t runs from 0 to $+\infty$. It is easily checked that $\rho(\delta, E)$ is an even integer if and only if $\Phi(T,\delta,E)$ has positive eigenvalues which is equivalent to say that $f(\delta, E) \geq 2$. Similarly $\rho(\delta, E)$ is an odd integer if and only if $\Phi(T,\delta,E)$ has negative eigenvalues or equivalently that $f(\delta, E) \leq -2$ (see proposition 7.3). This implies that $\rho(\delta, E)$ is constant in each unstability interval. More precisely, we have

$$\rho(\delta, E) = \left\{\begin{array}{l} 2i+1 \text{ for } \delta \in]\tilde{\delta}_{2i+1}, \tilde{\delta}_{2i+2}[, \\ \\ 2i+2 \text{ for } \delta \in]\delta_{2i+1}, \delta_{2i+2}[, \end{array}\right. \quad (i = 0, 1, 2, \cdots).$$

In the intervals where (51) is elliptic one has

$$f(\delta, E) = 2\cos(\pi\rho(\delta, E)),$$

and since $f(\delta, E)$ is a strictly monotone function of δ the rotation number $\rho(\delta, E)$ is also strictly increasing in these intervals.

It is shown in [3] that the values of δ for which $\Phi(T, \delta, E)$ is a diagonal matrix, form a discrete sequence $(\mu_i(E))$ satisfying

$$\mu_1 < \mu_2 < \cdots < \mu_n < \cdots \to +\infty.$$

Moreover, each unstability interval contains exactly one μ_i so the sequence (47) can be completed to

$$\begin{aligned}\delta_0 = -1 < \tilde{\delta}_1(E) \le \mu_1(E) \le \tilde{\delta}_2(E) < \delta_1(E) = 0 \le \mu_2(E) \le \delta_2(E) < \\ < \tilde{\delta}_3(E) \le \mu_3(E) \le \tilde{\delta}_4(E) < \delta_3(E) \le \mu_4(E) \le \delta_4(E) < \cdots\end{aligned}$$

and for each $k = 1, 2, \ldots,$ we have

$$\Phi(T, \mu_k(E), E) = \begin{pmatrix} a_k(E) & 0 \\ 0 & a_k(E)^{\perp 1} \end{pmatrix} \qquad \text{with } \rho(\mu_k, E) = k.$$

Based on numerical evidence we conjecture that for a fixed energy $E \in]2, +\infty[$, the amplitude of oscillations of the function $\delta \mapsto f(\delta, E)$ converges to 4, with oscillations between -2 and 2, as $\delta \to +\infty$, while the length of the unstability intervals decreases to 0 as δ goes to $+\infty$. This would imply that for a fixed (low) energy level E there are no parameters with simultaneously high rotation number and strong hyperbolicity. On the other hand, for large energy levels we have:

Theorem 9.1. *For each $k = 2, 3, \ldots,$ one has*

$$\lim_{E \to \infty} |f(\mu_k(E), E)| = +\infty$$

In other words, given $k \ge 2$, for all large enough E and δ sufficiently close to $\mu_k(E)$, $\Gamma(\delta, E)$ is strongly hyperbolic with rotation number k.

The proof of this theorem requires studying in detail the asymptotics of system (41). This study will be done in the next subsection. After that we return to the proof of the theorem.

9.1 Asymptotics of system (41)

We are interested in understanding what happens to the solutions of system 41 when $E \to \infty$. Let $\alpha = \alpha(E)$ and $\beta = \beta(E)$, with $0 < \alpha < 1 < \beta$, be the two unique solutions of $x - \log x = \frac{E}{2}$. The points (α, α) and (β, β) lie in the intersection of the energy level

$$h_0(u, v) = u + v - \log(uv) = E \tag{53}$$

with the diagonal $u = v$. Let $(u(t, E), v(t, E))$ be the periodic orbit of system (41) with initial condition $(u(0, E), v(0, E)) = (\alpha, \alpha)$, whose period we denote by $T = T(E)$. From the reversing symmetry $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma(u, v) = (v, u)$, of this system, which fixes the initial condition (α, α), it follows easily that

$$\begin{cases} u(T - t, E) = & v(t, E), \\ v(T - t, E) = & u(t, E). \end{cases} \tag{54}$$

In particular we get $(u(T/2, E), v(T/2, E)) = (\beta, \beta)$.

It will be convenient to reparametrize the orbits of system (41). For each $x \in [\alpha, \beta]$ let $\tau(x) \in [0, T/2]$ be defined implicitly by

$$\frac{u(\tau(x)) + v(\tau(x))}{2} = x. \tag{55}$$

From (53) we get

$$u(\tau(x))v(\tau(x)) = e^{2x\perp E}.$$

Thus $u(\tau(x))$ and $v(\tau(x))$ are the solutions of a quadratic equation:

$$\begin{cases} u(\tau(x)) = & x - \sqrt{x^2 - e^{2x\perp E}}, \\ \\ v(\tau(x)) = & x + \sqrt{x^2 - e^{2x\perp E}}. \end{cases}$$

The reparametrization $\tau(x)$ satisfies:

$$\begin{aligned} \tau(x) &= \int_0^{\tau(x)} 1dt = \int_0^{\tau(x)} \frac{u'(t) + v'(t)}{v(t) - u(t)} dt = \\ &= \int_\alpha^x \frac{2dy}{v(\tau(y)) - u(\tau(y))} = \int_\alpha^x \frac{1}{\sqrt{y^2 - e^{2y\perp E}}} dy. \end{aligned}$$

Notice that by differentiating (55) we get

$$(u'(t) + v'(t))dt = (u'(\tau(y)) + v'(\tau(y)))\tau'(y)dy = 2dy.$$

Notice also that the radicand $y^2 - e^{2y\perp E}$ has two simple zeros at $y = \alpha$ and $y = \beta$ and is strictly positive in between. This guaranties the convergence, for any $x \in [\alpha, \beta]$, of the improper integral

$$\tau(x) = \int_\alpha^x \frac{1}{\sqrt{y^2 - e^{2y\perp E}}} dy. \tag{56}$$

Now define for $x \in (\alpha, \beta)$,

$$\begin{cases} \tilde{u}(x, E) := & \tau'(x)u(\tau(x)) = \dfrac{1 - \sqrt{1 - \frac{e^{2x-E}}{x^2}}}{\sqrt{1 - \frac{e^{2x-E}}{x^2}}}, \\ \\ \tilde{v}(x, E) := & \tau'(x)v(\tau(x)) = 2 + \tilde{u}(x). \end{cases} \tag{57}$$

The function $\tilde{u}(x,E)$ can be expressed as a composition of two simpler functions

$$\tilde{u}(x,E) = \frac{1-\sqrt{1-\varphi_E(x)}}{\sqrt{1-\varphi_E(x)}} = g(\varphi_E(x)),$$

where:

- $\varphi_E : [\alpha,\beta] \to \mathbb{R}_+$, is the strictly convex function given by

$$\varphi_E(x) = \frac{e^{2x\perp E}}{x^2}.$$

- $g : [0,1[\to \mathbb{R}_+$ is the strictly increasing function given by

$$g(w) = \frac{1-\sqrt{1-w}}{\sqrt{1-w}}.$$

Note also that φ_E takes its minimum value $e^{2\perp E}$ at the point $x = 1$ and satisfies $\varphi_E(\alpha) = \varphi_E(\beta) = 1$.

In the following lemma we enumerate some preliminary estimates.

Lemma 9.2. *Let $E > 2$. Then:*

(i) For $2\alpha \le x \le \beta - 1$,

$$\frac{1}{2}\varphi_E(x) \le \tilde{u}(x,E) \le \varphi_E(x) \le \frac{\sqrt{5}-1}{2}; \tag{58}$$

(ii) If $E \to \infty$,

$$\int_\alpha^2 \tilde{u}(x,E)dx = O\left(Ee^{\perp E/2}\right); \tag{59}$$

(iii) For $2 \le z \le \beta - 1$,

$$\int_\alpha^z \tilde{u}(x,E)dx \le \varphi_E(z). \tag{60}$$

Proof. (i) Just check that $\varphi_E(\beta-1) \le \frac{\sqrt{5}\perp 1}{2}$, $\varphi_E(2\alpha) \le \frac{\sqrt{5}\perp 1}{2}$ and

$$\frac{w}{2} \le g(w) \le w, \quad \text{for all} \quad 0 < w \le \frac{\sqrt{5}-1}{2}.$$

(ii) For all $\alpha \le x \le 1$ we have $e^{2x\perp E} \le \alpha x$. Then, as $E \to \infty$,

$$\int_\alpha^{\sqrt{\alpha}} \tilde{u}(x)dx \le \int_\alpha^{\sqrt{\alpha}} \frac{1}{\sqrt{1-\frac{\alpha}{x}}} - 1dx = O(\alpha\log\alpha) = O\left(Ee^{\perp E/2}\right),$$

$$\int_{\sqrt{\alpha}}^2 \tilde{u}(x)dx \le \int_{\sqrt{\alpha}}^2 \varphi_E(x)dx = O(\varphi_E(\sqrt{\alpha})) = O(e^{\perp E/2}).$$

Addition of these inequalities proves (ii).

(iii) From (i) we have

$$\begin{aligned}\int_2^z \tilde{u}(x,E)dx &\le \int_2^z \varphi_E(x)dx \\ &\le \int_2^z 2\varphi_E(x)\left(1-\frac{1}{x}\right)dx \\ &= [\varphi_E(x)]_2^z = \varphi_E(z)-\varphi_E(2) \le \varphi_E(z).\end{aligned}$$

□

Using the lemma we can prove the

Proposition 9.3.

$$\lim_{E\to\infty}\int_{\alpha(E)}^{\beta(E)} \tilde{u}(x,E)dx = \log 2.$$

Proof. Given $\epsilon > 0$, fix $z = \frac{3}{\epsilon}$. Making the substitution $w = \varphi_E(x)$ we have $\frac{1}{2w}dw = \left(1-\frac{1}{x}\right)dx$ and therefore

$$\begin{aligned}\int_z^\beta \tilde{u}(x,E)\left(1-\frac{1}{x}\right)dx &= \int_{\varphi_E(z)}^1 \frac{1-\sqrt{1-w}}{\sqrt{1-w}}\frac{dw}{2w} \\ &= \log\left(1+\sqrt{1-\varphi_E(z)}\right) = \log 2 - O\left(\varphi_E(z)\right),\end{aligned}$$

where the remainder $O\left(\varphi_E(z)\right)$ is positive and converges to zero as $E\to\infty$. Thus

$$\begin{aligned}\int_z^\beta \frac{1}{x}\tilde{u}(x,E)dx &\le \frac{\epsilon}{3}\int_z^\beta \tilde{u}(x,E)dx \le \frac{2\epsilon}{3}\int_z^\beta \tilde{u}(x,E)\left(1-\frac{1}{x}\right)dx \\ &\le \frac{2\epsilon}{3}\log 2 \le \frac{\epsilon}{2}.\end{aligned}$$

Taking E large enough, we obtain

$$\begin{aligned}\left|\log 2 - \int_\alpha^\beta \tilde{u}(x,E)dx\right| &\le \left|\log 2 - \int_z^\beta \tilde{u}(x,E)\left(1-\frac{1}{x}\right)dx\right| \\ &\quad + \int_z^\beta \frac{1}{x}\tilde{u}(x,E)dx + \int_\alpha^z \tilde{u}(x,E)dx \\ &\le O\left(\varphi_E(z)\right) + \frac{\epsilon}{2} + \varphi_E(z) \\ &= \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon.\end{aligned}$$

□

We can state and prove our final estimates for the asymptotic behavior of solutions of system (41).

Theorem 9.4. *Let $u(t,E)$ and $v(t,E)$ be solutions of (41) with energy E and period $T = T(E)$. Then:*

(i) $\lim_{E\to\infty} \int_0^{T/2} u(t,E)\,dt - \log 2 = 0$;

(ii) $\lim_{E\to\infty} \int_0^{T/2} v(t,E)\,dt - 2\beta(E) = 0$;

(iii) $\lim_{E\to\infty} T(E) - 2(\beta(E) + \log 2) = 0$.

Proof. From (57) we get,

$$\int_0^{T/2} u(t,E)\,dt = \int_\alpha^\beta \tilde{u}(x,E)\,dx, \text{ and}$$

$$\begin{aligned}\int_0^{T/2} v(t,E)\,dt &= \int_\alpha^\beta \tilde{v}(x,E)\,dx = \int_\alpha^\beta 2 + \tilde{u}(x,E)\,dx \\ &= 2(\beta-\alpha) + \int_\alpha^\beta \tilde{u}(x,E)\,dx,\end{aligned}$$

so (i) and (ii) follow. To prove (iii) we remark that for all t, $1 = u(t) + \frac{d}{dt}(\log v(t))$. Thus

$$\frac{T}{2} = \int_0^{T/2} u(t)\,dt + \log\beta - \log\alpha = \int_\alpha^\beta \tilde{u}(x)\,dx + \beta - \alpha.$$

□

9.2 Proof of theorem 9.1

The following notation is used throughout this subsection. An integer $k \geq 2$ is fixed and we denote by

$$\mathbf{w}(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} = \begin{bmatrix} r(t)\cos\theta(t) \\ r(t)\sin\theta(t) \end{bmatrix}$$

the first column of the fundamental matrix solution $\Phi(t, \mu_k(E), E)$ of system (42) with $\delta = \mu_k(E)$.

It is clear from (52) that $\theta : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism with $\theta(0) = 0$. Thus given $p \in \mathbb{R}$ there is a unique $t_p \in \mathbb{R}$ such that $\theta(t_p) = -p\pi$. Geometrically, t_p is the time it takes for the vector $\mathbf{w}(t)$ to execute p half-turns (it helps to think about this vector as being attached to the periodic orbit). Of course $t_0 = 0$ and, because $\rho(\mu_k, E) = k$, $t_k = T(E)$.

Since $f(\mu_k(E), E) = a_k(E) + a_k(E)^{\perp 1}$, we only have to show that $|a_k(E)| \to \infty$ as $E \to \infty$. We write

$$|a_k(E)| = |w_1(t_k)| = \frac{|w_1(t_k)|}{|w_1(t_{k\perp 1})|} \frac{|w_1(t_{k\perp 1})|}{|w_1(t_{k\perp 2})|} \cdots \frac{|w_1(t_2)|}{|w_1(t_1)|} \frac{|w_1(t_1)|}{|w_1(t_0)|}.$$

In propositions 9.8 and 9.11 below we show that, for large E, (i) $|w_1(t_j)| \geq |w_1(t_{j\perp 1})|$ for all $2 \leq j \leq k-1$, and (ii) the quotient $\frac{|w_1(t_1)|}{|w_1(t_0)|}$ is very large while $\frac{|w_1(t_k)|}{|w_1(t_{k-1})|}$ has a lower bound close to 1. Therefore, we see that

$$|a_k(E)| \geq \frac{|w_1(t_k)|}{|w_1(t_{k\perp 1})|} \frac{|w_1(t_1)|}{|w_1(t_0)|} \text{ is very large when } E \text{ is large,}$$

so the theorem follows.

We start with a upper bound on the numbers μ_k.

Lemma 9.5. *If $E > 2$ is large enough and $k \geq 2$ then $0 \leq \mu_k(E) \leq 3k - 1$.*

Proof. Using theorem 9.4 and the symmetry relation (54) of the previous subsection,

$$\begin{aligned} k\pi &= |\theta(T) - \theta(T/2)| + |\theta(T/2) - \theta(0)| \\ &= (1+\mu_k) \int_{T/2}^{T} v\cos^2\theta + u\sin^2\theta dt + (1+\mu_k)\int_0^{T/2} v\cos^2\theta + u\sin^2\theta dt \\ &\geq 2(1+\mu_k)\int_0^{T/2} u(t)dt \geq 2(1+\mu_k)\log 2 \geq (1+\mu_k)\log 3, \end{aligned}$$

which implies $1 + \mu_k \leq \frac{k\pi}{\log 3} \leq 3k$. □

Next we show that, for large E, the vector $\mathbf{w}(t)$ executes all half-turns in the region $u + v \gg 1$.

Lemma 9.6. *If $E > 2$ is large enough,*

(i) $\dfrac{u(t_{\frac{1}{2}}) + v(t_{\frac{1}{2}})}{2} \geq \beta - \dfrac{2}{3}\log E$, *and*

(ii) $\dfrac{u(t_{k\perp\frac{1}{2}}) + v(t_{k\perp\frac{1}{2}})}{2} \geq \beta - \dfrac{2}{3}\log E.$

Proof. Defining $\tilde\theta : (\alpha, \beta) \to \mathbb{R}$ as $\tilde\theta(x) = \theta(\tau(x))$, we see that it satisfies

$$\begin{aligned} \tilde\theta'(x) &= -(1+\mu_k)\left(\tilde v(x)\cos^2\tilde\theta(x) + \tilde u(x)\sin^2\tilde\theta(x)\right) \\ &= -(1+\mu_k)\left(2\cos^2\tilde\theta(x) + \tilde u(x)\right). \end{aligned} \tag{61}$$

Similarly, if we define $\hat\theta : (\alpha, \beta) \to \mathbb{R}$ setting $\hat\theta(x) = \theta(T - \tau(x))$, this function solves the equation,

$$\begin{aligned} \hat\theta'(x) &= (1+\mu_k)\left(\tilde u(x)\cos^2\hat\theta(x) + \tilde v(x)\sin^2\hat\theta(x)\right) \\ &= (1+\mu_k)\left(2\sin^2\hat\theta(x) + \tilde u(x)\right). \end{aligned} \tag{62}$$

The proofs of (i) and (ii) run by contradiction. Assume

$$x_{\frac{1}{2}} = \frac{u(t_{\frac{1}{2}}) + v(t_{\frac{1}{2}})}{2} \leq \beta - \frac{2}{3}\log E.$$

and take $x_* < x_{\frac{1}{2}}$ such that $\tilde{\theta}(x_*) = -\frac{\pi}{2} + E^{\perp 7/6}$. The variation of $\tilde{\theta}$ in the interval $[x_*, x_{\frac{1}{2}}]$ is $\left|\tilde{\theta}(x_*) - \tilde{\theta}(x_{\frac{1}{2}})\right| = E^{\perp 7/6}$, which is a small. But from equation (61) we will derive the conclusion that this variation must actually be much smaller, a contradiction. Since $[x_*, x_{\frac{1}{2}}] \subseteq [\alpha, \beta - \frac{2}{3}\log E]$, by inequality (60), we have

$$\int_{x_*}^{x_{\frac{1}{2}}} \tilde{u}(x)dx \leq \varphi_E\left(\beta - \frac{2}{3}\log E\right) \leq 2E^{\perp 4/3},$$

provided E is large enough. On the other hand, we also find

$$\begin{aligned}\int_{x_*}^{x_{\frac{1}{2}}} \cos^2\tilde{\theta}(x)dx &\leq \int_{x_*}^{x_{\frac{1}{2}}} \left(\tilde{\theta}(x) + \frac{\pi}{2}\right)^2 dx \\ &\leq \left(E^{\perp 7/6}\right)^2 \left(x_{\frac{1}{2}} - x_*\right) \\ &\leq E^{\perp 7/3}E = E^{\perp 4/3}.\end{aligned}$$

Thus

$$\begin{aligned}\left|\tilde{\theta}(x_{\frac{1}{2}}) - \tilde{\theta}(x_*)\right| &= \int_{x_*}^{x_{\frac{1}{2}}} (1+\mu_k)\left(\tilde{u}(x) + 2\cos^2\tilde{\theta}(x)\right)dx \\ &\leq 12kE^{\perp 4/3} \ll E^{\perp 7/6}.\end{aligned}$$

This contradiction shows that (i) must hold.

To prove (ii), define $x_* \in (\alpha, \beta)$ such that $\hat{\theta}(x_*) = -k\pi + E^{\perp 7/6}$. Since $E^{\perp 7/6}$ is much smaller than $\pi/2$ we must have

$$\hat{\theta}(x_*) \ll -(k - \frac{1}{2})\pi,$$

and therefore

$$x_* < x_{k\perp\frac{1}{2}} = \frac{u(t_{k\perp\frac{1}{2}}) + v(t_{k\perp\frac{1}{2}})}{2}.$$

We only have to show now that $x_* > \beta - \frac{2}{3}\log E$. Assume, in order to derive a contradiction, that

$$x_* \leq \beta - \frac{2}{3}\log E.$$

The variation of $\hat{\theta}$ in the interval $[\alpha, x_*]$ is $\left|\hat{\theta}(x_*) - \hat{\theta}(\alpha)\right| = E^{\perp 7/6}$, but from equation (62) it will follow that this variation should be much smaller. As before since $[\alpha, x_*] \subseteq [\alpha, \beta - \frac{2}{3}\log E]$, we have

$$\int_{\alpha}^{x_*} \tilde{u}(x)dx \leq \varphi_E\left(\beta - \frac{2}{3}\log E\right) \leq 2E^{\perp 4/3},$$

and

$$\int_{\alpha}^{x_*} \sin^2 \hat{\theta}(x) dx \leq \int_{\alpha}^{x_*} \left(\hat{\theta}(x) + k\pi \right)^2 dx \leq E^{\perp 4/3}.$$

Thus we have the same contradiction as before,

$$E^{\perp 7/6} = \left| \hat{\theta}(x_*) - \hat{\theta}(\alpha) \right| \leq 12kE^{\perp 4/3} \ll E^{\perp 7/6}.$$

□

We also observe that on the region $u+v \gg 1$ the quocient $\frac{v(t)}{u(t)}$ decreases. In fact we have:

Lemma 9.7. *The quotient* $\frac{v(t)}{u(t)}$ *is strictly decreasing if* $u+v \geq 2$*, i. e., inside the interval* $[\tau(1), T - \tau(1)]$*. This is the same as saying that* $\frac{\tilde{v}(x)}{\tilde{u}(x)}$ *decreases in* $[1, \beta]$.

Proof. Just check that

$$\left(\frac{v}{u} \right)' = 2\frac{v}{u} \left(1 - \frac{u+v}{2} \right),$$

and use the definition of $\tau(x)$. □

We can now prove the

Proposition 9.8. *If* $E > 2$ *is large enough,*

$$|w_1(t_{k\perp 1})| \geq |w_1(t_{k\perp 2})| \geq \cdots \geq |w_1(t_2)| \geq |w_1(t_1)|.$$

Proof. From lemma (9.6) we have,

$$\frac{u(t)+v(t)}{2} \geq \beta - \frac{2}{3} \log E \gg 1,$$

for all $t \in [t_{\frac{1}{2}}, t_{k\perp\frac{1}{2}}]$. Thus, from lemma (9.7), the quotient $\frac{v(t)}{u(t)}$ decreases in the interval $[t_1, t_{k\perp 1}]$ and we only have to prove now that $|w_1(t_{j+1})| \geq |w_1(t_j)|$ whenever $\frac{v(t)}{u(t)}$ decreases in $[t_j, t_{j+1}]$. Defining the quadratic form in the (w_1, w_2) plane

$$Q_{j+\frac{1}{2}}(w) = Q_{j+\frac{1}{2}}(w_1, w_2) = \frac{w_1^2}{u(t_{j+\frac{1}{2}})} + \frac{w_2^2}{v(t_{j+\frac{1}{2}})},$$

let us show that we have

$$\frac{d}{dt} Q_{j+\frac{1}{2}}(w(t)) \geq 0, \qquad \forall t \in [t_j, t_{j+1}]. \tag{63}$$

We compute:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}Q_{j+\frac{1}{2}}(w(t)) &= \frac{w_1(t)w_1'(t)}{u(t_{j+\frac{1}{2}})} + \frac{w_2(t)w_2'(t)}{v(t_{j+\frac{1}{2}})} \\ &= (1+\mu_k)\left(\frac{u(t)}{u(t_{j+\frac{1}{2}})} - \frac{v(t)}{v(t_{j+\frac{1}{2}})}\right) w_1(t)w_2(t) \\ &= (1+\mu_k)\frac{u(t)}{v(t_{j+\frac{1}{2}})}\left(\frac{v(t_{j+\frac{1}{2}})}{u(t_{j+\frac{1}{2}})} - \frac{v(t)}{u(t)}\right) w_1(t)w_2(t).\end{aligned}$$

Now, for $t \in [t_j, t_{j+\frac{1}{2}}]$, we have $w_1(t)w_2(t) \leq 0$ and for $t \in [t_{j+\frac{1}{2}}, t_{j+1}]$, we have $w_1(t)w_2(t) \geq 0$. Since $\frac{v(t)}{u(t)}$ decreases in $[t_j, t_{j+1}]$ we conclude in both cases that (63) holds.

Finally, we obtain

$$\frac{w_1(t_{j+1})^2}{u(t_{j+\frac{1}{2}})} = Q_{j+\frac{1}{2}}(w(t_{j+1})) \geq Q_{j+\frac{1}{2}}(w(t_j)) = \frac{w_1(t_j)^2}{u(t_{j+\frac{1}{2}})}$$

which gives $|w_1(t_{j+1})| \geq |w_1(t_j)|$.

□

We now look at what happens for t close to $t_k = T$ and $t_0 = 0$. We need two lemmas.

Lemma 9.9. *If $E > 2$ is large enough,*

(i) $\dfrac{u(t_{\frac{1}{2}})}{v(t_{\frac{1}{2}})} \geq \dfrac{1}{6}E^{\perp 4/3}$, *and*

(ii) $\dfrac{u(t_{k\perp\frac{1}{2}})}{v(t_{k\perp\frac{1}{2}})} \leq 6E^{4/3}$.

Proof. Define, as in lemma 9.6,

$$x_{\frac{1}{2}} = \frac{u(t_{\frac{1}{2}}) + v(t_{\frac{1}{2}})}{2} \quad \Leftrightarrow \quad \tau(x_{\frac{1}{2}}) = t_{\frac{1}{2}},$$

and

$$x_{k\perp\frac{1}{2}} = \frac{u(t_{k\perp\frac{1}{2}}) + v(t_{k\perp\frac{1}{2}})}{2} \quad \Leftrightarrow \quad T - \tau(x_{k\perp\frac{1}{2}}) = t_{k\perp\frac{1}{2}}.$$

Then

$$\frac{u(t_{\frac{1}{2}})}{v(t_{\frac{1}{2}})} = \frac{\tilde{u}(x_{\frac{1}{2}})}{\tilde{v}(x_{\frac{1}{2}})} = \frac{\tilde{u}(x_{\frac{1}{2}})}{2+\tilde{u}(x_{\frac{1}{2}})}, \tag{64}$$

and analogously, from (57) and (54), it follows that

$$\frac{v(t_{k\perp\frac{1}{2}})}{u(t_{k\perp\frac{1}{2}})} = \frac{\tilde{u}(x_{k\perp\frac{1}{2}})}{\tilde{v}(x_{k\perp\frac{1}{2}})} = \frac{\tilde{u}(x_{k\perp\frac{1}{2}})}{2+\tilde{u}(x_{k\perp\frac{1}{2}})}. \tag{65}$$

By lemma 9.6, we have $x_{\frac{1}{2}}, x_{k\perp\frac{1}{2}} > \beta - \frac{2}{3}\log E$. Then, using lemma 9.7, we see that both quotients (64) and (65) are greater or equal to $\dfrac{\tilde{u}(x)}{2+\tilde{u}(x)}$ with $x = \beta - \frac{2}{3}\log E$. But from (58) we have

$$\frac{\tilde{u}(x)}{2+\tilde{u}(x)} \geq \frac{\tilde{u}(x)}{3} \geq \frac{\varphi_E(x)}{6} = \frac{1}{6}\varphi_E\left(\beta - \frac{2}{3}\log E\right) \geq \frac{1}{6}E^{\perp 4/3},$$

and this proves both inequalities.

□

Lemma 9.10. *If $E > 2$ is large enough,*

(i) $\left|w_2(t_{\frac{1}{2}})\right| \geq |w_1(t_0)|\, E^{5/6}$, *and*

(ii) $|w_2(T-\tau(1))| \leq |w_1(T)|\, e^{\perp E/3}$.

Proof. We first prove (i). Since the function $|w_2(\tau(x))|$ is strictly increasing in the interval $[\alpha, x_{\frac{1}{2}}]$, where it ranges from 0 to $w_2(t_{\frac{1}{2}})$, and also because

$$0 < \beta - \frac{2}{3}\log E < x_{\frac{1}{2}},$$

(see lemma 9.6), we must have

$$\left|w_2 \circ \tau\left(\beta - \frac{2}{3}\log E\right)\right| < \left|w_2(t_{\frac{1}{2}})\right|.$$

Therefore it is enough to prove now that

$$|w_2(t)| \geq E^{5/6} \quad \text{with} \quad t = \tau\left(\beta - \frac{2}{3}\log E\right).$$

Assume not, i. e., $|w_2(t)| < E^{5/6}$. Then

$$\begin{aligned} |w_1(t) - 1| &= |w_1(t) - w_1(0)| \leq (1+\mu_k)\int_0^t u(s)\,|w_2(s)|\,ds \\ &\leq (1+\mu_k)\,|w_2(t)|\int_\alpha^{\beta\perp\frac{2}{3}\log E} \tilde{u}(x)dx \\ &\leq (1+\mu_k)\,|w_2(t)|\,\varphi_E\left(\beta - \frac{2}{3}\log E\right) \\ &\leq \frac{6kE^{5/6}}{E^{4/3}} = 6kE^{\perp 1/2}, \end{aligned}$$

which implies that $|w_1(s)| \geq \frac{1}{2}$, for any $0 \leq s \leq t$. But then

$$\begin{aligned} |w_2(t)| &= |w_2(t) - w_2(0)| = (1+\mu_k) \int_0^t v(s)\,|w_1(s)|\,ds \\ &\geq \frac{1}{2} \int_\alpha^{\beta \perp \frac{2}{3}\log E} \underbrace{\tilde{v}(x)}_{\geq 2}\,dx \\ &\geq \beta - \frac{2}{3}\log E - \alpha = O(E), \end{aligned}$$

contradicting the assumption $|w_2(t)| < E^{5/6}$. This shows that (i) holds.

To prove (ii), we use the estimate (59):

$$\begin{aligned} |w_2(T-\tau(1))| &= |w_2(T) - w_2(T-\tau(1))| \\ &= (1+\mu_k) \int_{T\perp\tau(1)}^{T} v(t)\,|w_1(t)|\,dt \\ &\leq (1+\mu_k) \int_0^{\tau(1)} u(t)\,|w_1(T)|\,dt \\ &\leq (1+\mu_k)\,|w_1(T)| \int_\alpha^1 \tilde{u}(x)dx \\ &\leq 3kO\left(Ee^{\perp E/2}\right)|w_1(T)| \leq e^{\perp E/3}\,|w_1(T)|\,. \end{aligned}$$

□

We can now show the

Proposition 9.11. *If $E > 2$ is large enough,*

(i) $|w_1(t_1)| \geq \dfrac{1}{\sqrt{6}} E^{1/6} |w_1(t_0)|$, *and*

(ii) $|w_1(t_k)| \geq \left(1 - e^{\perp E/3}\right) |w_1(t_{k\perp 1})|$.

Proof. Consider the quadratic form

$$Q_{\frac{1}{2}}(w) = \frac{w_1^2}{u(t_{\frac{1}{2}})} + \frac{w_2^2}{v(t_{\frac{1}{2}})}.$$

Because $\frac{v}{u}$ decreases in $[t_{\frac{1}{2}}, t_1]$ (see lemmas 9.6 and 9.7) we have, just as in the proof of lemma 9.8, $\frac{d}{dt}Q_{\frac{1}{2}}(w(t)) \geq 0$ for all $t \in [t_{\frac{1}{2}}, t_1]$. From this fact and lemmas 9.9 and 9.10 we obtain

$$\begin{aligned} w_1(t_1)^2 &= u(t_{\frac{1}{2}}) Q_{\frac{1}{2}}(w(t_1)) \\ &\geq u(t_{\frac{1}{2}}) Q_{\frac{1}{2}}(w(t_{\frac{1}{2}})) \\ &= \frac{u(t_{\frac{1}{2}})}{v(t_{\frac{1}{2}})} w_2(t_{\frac{1}{2}})^2 \geq \frac{1}{6} E^{1/3} w_1(t_0)^2, \end{aligned}$$

which proves (i).

Similarly, we consider the quadratic form

$$Q_{k\perp\frac{1}{2}}(w) = \frac{w_1^2}{u(t_{k\perp\frac{1}{2}})} + \frac{w_2^2}{v(t_{k\perp\frac{1}{2}})}.$$

As before, because $\frac{v}{u}$ decreases in $[t_{k\perp 1}, T-\tau(1)]$, we have $\frac{d}{dt}Q_{k\perp\frac{1}{2}}(w(t)) \geq 0$ for $t \in [t_{k\perp 1}, T-\tau(1)]$. Therefore, using again lemmas 9.9 and 9.10, we obtain

$$\begin{aligned}
\frac{w_1(t_{k\perp 1})^2}{u(t_{k\perp\frac{1}{2}})} &= Q_{k\perp\frac{1}{2}}\left(w(t_{k\perp 1})\right) \\
&\leq Q_{k\perp\frac{1}{2}}\left(w(T-\tau(1))\right) \\
&= \frac{w_1(T-\tau(1))^2}{u(t_{k\perp\frac{1}{2}})} + \frac{w_2(T-\tau(1))^2}{v(t_{k\perp\frac{1}{2}})} \\
&\leq \frac{w_1(T)^2}{u(t_{k\perp\frac{1}{2}})} + \frac{e^{\perp 2E/3} w_1(T)^2}{v(t_{k\perp\frac{1}{2}})} \\
&\leq \frac{w_1(T)^2}{u(t_{k\perp\frac{1}{2}})}\left(1 + e^{\perp 2E/3}\frac{u(t_{k\perp\frac{1}{2}})}{v(t_{k\perp\frac{1}{2}})}\right) \leq \frac{w_1(T)^2}{u(t_{k\perp\frac{1}{2}})}\left(1+e^{\perp E/2}\right),
\end{aligned}$$

which implies

$$|w_1(t_{k\perp 1})| \leq |w_1(T)|\left(1+e^{\perp E/2}\right) \leq |w_1(T)|\left(1-e^{\perp E/3}\right)^{\perp 1}.$$

□

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REFERENCES

[1] R. Abraham, J. Marsden, "Foundations of Mechanics", 2nd edition, Addison-Wesley, Reading, Massachusetts, 1985.

[2] V. Arnol'd, "Mathematical Methods of Classical Mechanics", 2nd edition, GTM vol. 60, Springer-Verlag, New-York, 1989.

[3] E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New-York, 1955.

[4] J. Coste, J. Peyraud, P. Coulet, Asymptotic behavior in the dynamics of competing species, *SIAM J. Appl. Math.* **36** (1979), 516–542.

[5] L. H. Eliasson, Normal forms for hamiltonian systems with Poisson commuting integrals – elliptic case, *Comment. Math. Helv.* **65** (1990), 4–35.

[6] J. Hale, "Ordinary Differential Equations", Wiley-Interscience, New York, 1969.

[7] J. Hofbauer, K. Sigmund, "The Theory of Evolution and Dynamical Systems", Cambridge University Press, Cambridge, 1988.

[8] G. Israel, Volterra's 'Analytical Mechanics' of Biological Associations, *Arch. Inter. d'Histoire des Sciences* **41** (1991), 57–104.
[9] M. W. Hirsch, Systems of differential equations that are competitive or cooperative III: competing species, *Nonlinearity* **1** (1988), 117–124.
[10] M. W. Hirsch, Systems of differential equations that are competitive or cooperative I: limit sets, *SIAM J. Math. Anal.***13** (1982), 167–169.
[11] H. Ito, Convergence of Birkhoff normal forms for integrable systems, *Comment. Math. Helv.* **64** (1989), 412–461.
[12] V. V. Kozlov, "Symmetries, Topology and Resonances in Hamiltonian Mechanics", Springer Verlag, Berlin, 1996.
[13] J. La Salle, An invariance principle in the theory of stability, in "Differential Equations and Dynamical Systems", J. Hale and J. La Salle (eds.), Academic Press, New-York (1967).
[14] J. Milnor, "Morse Theory", Princeton University Press, New Jersey, 1969.
[15] P. J. Olver, "Applications of Lie Groups to Differential Equations", GTM vol. 107, Springer-Verlag, New-York, 1986.
[16] M. Plank, Hamiltonian structure for the n-dimensional Lotka-Volterra equations, *J. Math. Phys.* **36** (1995), 3520–3534.
[17] H. Poincaré, "Les Méthodes Nouvelles de la Mécanique Celeste", Gauthier-Villars, Paris, 1889.
[18] R. Redheffer, A new class of Volterra Differential Equations for which the solutions are globally asymptotically stable, *J. Diff. Eqs.* **82** (1989), 251–268.
[19] R. Redheffer, Volterra multipliers I, *SIAM J. Discrete Math.* **6** (1985), 592–611, II, *SIAM J. Discrete Math.* **6** (1985), 612–623.
[20] R. Redheffer, W. Walter, Solution of the stability problem for a class of generalized Volterra prey-predator systems, *J. Diff. Eqs.* **52** (1984), 245–263.
[21] R. Redheffer, Z. Zhiming, A class of matrices connected with Volterra prey-predator equations, *SIAM J. Alg. Disc. Meth.* **3** (1982), 122–134.
[22] R. Redheffer, Z. Zhiming, Global asymptotic stability for a class of many-variable Volterra prey-predator systems, *J. Nonlinear Math. Appl.* **5** (1981), 1309–1329.
[23] F. Rothe, Remarks on the periods of planar Hamiltonian systems, *SIAM J. Math. Anal.* **24** (1993), 129–154.
[24] S. Smale, On the differential equations of species in competition, *J. Math. Biol.* **3** (1976), 5–7.
[25] V. Volterra, "Leçons sur la Théorie Mathématique de la Lutte pour la Vie", Gauthier-Villars et Cie., Paris, 1931.
[26] M. L. Zeeman, Extinction in competitive Lotka-Volterra systems, *Proc. Amer. Math. Soc.* **123** (1995), 87–96.