

HYPERBOLIC BILLIARDS ON POLYTOPES WITH CONTRACTING REFLECTION LAWS

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ABSTRACT. We study billiards on polytopes in \mathbb{R}^d with contracting reflection laws, i.e. non-standard reflection laws that contract the reflection angle towards the normal. We prove that billiards on generic polytopes are uniformly hyperbolic provided there exists a positive integer k such that for any k consecutive collisions, the corresponding normals of the faces of the polytope where the collisions took place generate \mathbb{R}^d . As an application of our main result we prove that billiards on generic polytopes are uniformly hyperbolic if either the contracting reflection law is sufficiently close to the specular or the polytope is obtuse. Finally, we study in detail the billiard on a family of 3-dimensional simplexes.

1. Introduction. Given a d -dimensional polytope P , a billiard trajectory inside P is a polygonal path described by a point particle moving with uniform motion in the interior of P . When the particle hits the interior of the faces of P , it bounces back according to a reflection law. Therefore, a billiard trajectory is determined by a sequence of reflections on the faces of P . Any reflection can be represented by a pair $x = (p, v)$ where p is a point belonging to a face of P and v is a unit velocity vector pointing inside P . We denote by M the set of reflections. The map $\Phi : M \rightarrow M, x \mapsto x'$ that takes a reflection x to the next reflection x' is called the *billiard map*. The dynamics of billiards on polytopes has been mostly studied considering the specular reflection law. More recently, in the case of polygonal billiards, a new class of reflection laws has been introduced that contract the reflection angle towards the normal of the faces of the polygon [1, 8, 2, 4]. These are called *contracting reflection laws*. A billiard map with a contracting reflection law is called a *contracting billiard map*. It is known that strongly contracting billiard maps on

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generic convex polygons are uniformly hyperbolic and have finite number of ergodic SRB measures [5]. Recently, it has been proved that the same conclusion holds for contracting billiard maps on polygons with no parallel sides facing each other (even for contracting reflection laws close to the specular and for non-convex polygons) [7].

In this paper we extend some of the previous results to contracting billiard maps on polytopes. It is known that the contracting billiard map of any polygon has dominated splitting [5, Proposition 3.1]. In this direction we show in Proposition 4 that the contracting billiard map of any polytope is always (uniformly) partially hyperbolic, i.e. there is a continuous and invariant splitting $E^s \oplus E^{cu}$ of the tangent bundle of M into subbundles of the same dimension such that $D\Phi$ uniformly contracts vectors in the *stable* subbundle E^s and has neutral or expanding action on vectors belonging to the *centre-unstable* subbundle E^{cu} .

There are essentially two obstructions for the uniform expansion in the centre-unstable subbundle E^{cu} . The first obstruction is caused by the billiard orbits that get trapped in a subset of faces of P whose normals do not span the ambient space \mathbb{R}^d . When P is a polygon ($d = 2$), those orbits are exactly the periodic orbits of period two, i.e. orbits bouncing between parallel sides of P . In fact, when P has no parallel sides the contracting billiard map is uniformly hyperbolic [5, Proposition 3.3]. As another example let P be a 3-dimensional prism and consider a billiard orbit unfolding in some plane parallel to the prism's base. The normals to the faces along this orbit will span a plane and the billiard map behaviour transversal to this plane is neutral. This leads to an expansion failure in E^{cu} .

In order to circumvent this obstruction we had to consider a class of polytopes which have the property that for any subset of d faces of P the corresponding normals span \mathbb{R}^d . A polytope with this property is called *spanning* (see Definition 2.9). In addition to being spanning, we suppose that the normals to the $(d-1)$ -faces incident with any given vertex are linearly independent (see Definition 3.1). Spanning polytopes with these properties are *generic*. In fact they form an open and dense subset having full Lebesgue measure in the set of all polytopes.

The second obstruction to uniform expansion corresponds to the billiard orbits that spend a significant amount of time bouncing near the skeleton of P . To control the time spent near the skeleton we introduced the notion of *escaping time*. Roughly speaking, the escaping time of $x \in M$ is the least positive integer $T = T(x) \in \mathbb{N} \cup \{\infty\}$ such that the number of iterates it takes for the billiard orbit of x to leave a neighbourhood of the skeleton of P is less than T (see Definition 2.11).

With these notions we prove that the contracting billiard map has non-zero Lyapunov exponents for almost every point with respect to any given ergodic invariant measure. More precisely we prove:

Theorem 1.1. *If the contracting billiard map Φ of a generic polytope has an ergodic invariant probability measure μ such that T is integrable with respect to μ , then μ is hyperbolic.*

When the contracting billiard map Φ has bounded escaping time, then Φ is uniformly hyperbolic.

Theorem 1.2. *If the contracting billiard map Φ of a generic polytope has an invariant set Λ such that T is bounded on Λ , then $\Phi|_{\Lambda}$ is uniformly hyperbolic.*

Theorems 1.1 and 1.2 follow from Theorem 2.10 which gives a uniform estimate on the expansion along the orbit of every point which is k -generating (see Definition 2.8). Being k -generating simply means that the face normals along any orbit segment of length k span \mathbb{R}^d .

The strategy to prove Theorem 2.10 is the following. Consider the billiard orbit $x_n = (p_n, v_n)$, $n \geq 0$ of a k -generating point $x_0 = (p_0, v_0) \in M$. Denote by η_n the inward unit normal of the face of P where the reflection x_n takes place. In some appropriate coordinates, known as Jacobi coordinates, the unstable space $E^u(x_0)$ is represented by the orthogonal hyperplane v_0^\perp . If the velocity v_1 is collinear with the normal η_1 , then the action of the derivative $D\Phi$ on $E^u(x_0)$ is neutral. Otherwise, the map $D\Phi$ expands the direction $v_0^\perp \cap V_1$ where V_1 denotes the plane spanned by the velocities v_0 and v_1 . Similarly, $D\Phi^2$ expands the directions contained in $v_0^\perp \cap V_2$ where now V_2 is generated by the velocities v_0, v_1 and v_2 . However, it may happen that the plane spanned by the velocities v_1 and v_2 is the same obtained from the span by the normals η_1 and η_2 , thus implying that $\dim(v_0^\perp \cap V_2) = 1$. This coincidence of the *velocity front* with the *normal front* is called a collinearity (see Definition 5.6).

If a collinearity never occurs and x_0 is k -generating then the map $D\Phi^k$ expands $d - 1$ distinct directions in v_0^\perp . Although collinearities prevent full expansion of the iterates $D\Phi^n(x_0)$ they have the good trait of synchronizing the velocity front with the normal front. After a collinearity every time a new face is visited the angle between the new velocity and the previous velocity front is always bounded away from zero. This happens because this velocity angle is related to the angle between the new normal and the previous normal front, and also because we assume the polytope to be spanning. Consider now the velocity front V at some collinearity moment. The previous property implies expansion of $D\Phi^n(x_0)$ transversal to $v_0^\perp \cap V$ after the collinearity moment. Choosing a minimal collinearity (see Definition 5.7) in the orbit of x_0 we can also ensure the expansion of $D\Phi^n(x_0)$ along the velocity front up the collinearity moment. Putting these facts together, if at some instant $t < k$ a minimal collinearity occurs on the orbit of x_0 then for $n \geq t + k$ we have full expansion of $D\Phi^n(x_0)$ on E^u .

Because we seek uniform expansion, one has to deal with δ -collinearities instead (see Definition 5.9). Moreover, since the set of orbits in M is not compact (one has to remove from M the orbits which hit the skeleton of P), δ -collinearities are more easily handled in a bigger set called the *trajectory space*. The trajectory space is compact and defined in a symbolic space which only retains the velocities and the normals of the faces of P where the reflections take place (see Definition 5.1). Finally, using compactness and continuity arguments we derive Theorem 5.3 which gives a uniform estimate on the expansion along an orbit segment of length $2k$ of any k -generating point. Then Theorem 2.10 follows immediately from Theorem 5.3. The crucial tool to prove Theorem 5.3 is Lemma 5.14 which gives a uniform expansion estimate on compositions of linear maps. Since this lemma is formulated in more conceptual terms, we believe that the ideas therein might be of independent interest.

In section 4 we show that contracting billiards on polytopes have finite escaping time if either the contracting law is close to the specular or the polytope is obtuse. This together with Theorem 1.2 prove the following corollaries.

Corollary 1. *The contractive billiard map of a generic polytope with a contracting reflection law sufficiently close to the specular one is uniformly hyperbolic.*

Corollary 2. *The contracting billiard map of a generic obtuse polytope is uniformly hyperbolic.*

The rest of the paper is organized as follows. In section 2 we introduce some notation and define the contracting billiard on a polytope. We also derive several properties of contracting billiards maps and rigorously state our main result. In section 3 we show that polytopes on general position are generic and in section 4 we study the escaping time on polyhedral cones. Technical results concerning the expansion of composition of linear maps are proved in section 5. In section 6 we prove our main results. Finally, in section 7 we study in detail the contracting billiard of a family of 3-dimensional simplexes.

2. Definitions and Statements. A half-space in \mathbb{R}^d ($d \geq 2$) is any set of the form $\{x \in \mathbb{R}^d : \langle x, v \rangle \leq c\}$, for some non-zero vector $v \in \mathbb{R}^d$ and some real number $c \in \mathbb{R}$. A polyhedron is any finite intersection of half-spaces in \mathbb{R}^d . A polytope is a compact polyhedron. We call dimension of a polyhedron to the dimension of the affine subspace that it spans. Let $P \subset \mathbb{R}^d$ be a d -dimensional polytope.

The billiard on P is a dynamical system describing the linear motion of a point particle inside P . When the particle hits the boundary of P , it gets reflected according to a reflection law, usually the specular reflection law. In the following we rigorously define the billiard map Φ_P with the specular reflection law. But first, let us introduce some notation.

2.1. Basic Euclidean Geometry. Let V and V' be Euclidean spaces with $\dim V = \dim V' = d$. Given a linear map $L: V \rightarrow V'$, the maximum expansion of L is the operator norm

$$\|L\| := \max\{\|L(v)\| : v \in V, \|v\| = 1\}$$

while the minimum expansion of L , defined by

$$\mathfrak{m}(L) := \min\{\|L(v)\| : v \in V, \|v\| = 1\}$$

is either 0, when L is non invertible, or else $\mathfrak{m}(L) = \|L^{-1}\|^{-1}$.

We denote by $L^*: V' \rightarrow V$ the adjoint operator of $L: V \rightarrow V'$. Recall that the *singular values* of L are the eigenvalues of the conjugate positive semi-definite symmetric operator $\sqrt{L^*L}$. Being real, and non negative, the singular values of L can be ordered as follows

$$s_1(L) \geq s_2(L) \geq \dots \geq s_d(L) \geq 0.$$

The top singular value is $s_1(L) = \|L\|$, while the last singular value is the minimum expansion $s_d(L) = \mathfrak{m}(L)$. The product of all singular values of L will be referred as the *determinant* of L

$$\det(L) := \prod_{j=1}^d s_j(L).$$

This determinant is the factor by which L expands d -volumes.

Given $\lambda > 0$ we denote by $\mathfrak{v}_\lambda^\geq(L)$ the direct sum of all singular directions of L (eigen-directions of L^*L) associated with singular values $\mu \geq \lambda$. Likewise, we denote by $\mathfrak{v}_\lambda^\leq(L)$ the direct sum of all singular directions of L associated with singular values $\mu < \lambda$. It follows from these definitions that

$$V = \mathfrak{v}_\lambda^\geq(L) \oplus \mathfrak{v}_\lambda^\leq(L), \quad L(\mathfrak{v}_\lambda^\geq(L)) = \mathfrak{v}_\lambda^\geq(L^*) \quad \text{and} \quad L(\mathfrak{v}_\lambda^\leq(L)) = \mathfrak{v}_\lambda^\leq(L^*)$$

and similar relations hold for L^* . To shorten notations we will simply write $\mathbf{v}(L)$ instead of $\mathbf{v}_{\|\cdot\|}^{\geq}(L)$. This subspace will be referred to as the *most expanding direction* of L .

Given vectors $v_1, \dots, v_n \in \mathbb{R}^d$, the linear subspace spanned by the vectors v_1, \dots, v_n is denoted by $\langle\langle v_1, \dots, v_n \rangle\rangle$. Let \mathbb{S} denote the unit sphere in \mathbb{R}^d , i.e. $\mathbb{S} = \{v \in \mathbb{R}^d : \|v\| = 1\}$. Let $v, \eta \in \mathbb{S}$ be unit vectors and $u \in \mathbb{R}^d$. We denote by \mathbb{S}_η^+ the *hemisphere associated with η* ,

$$\mathbb{S}_\eta^+ := \{v \in \mathbb{S} : \langle v, \eta \rangle > 0\}.$$

Let η^\perp denote the orthogonal hyperplane to η . The *orthogonal projection of u onto the hyperplane η^\perp* is,

$$P_{\eta^\perp}(u) := u - \langle u, \eta \rangle \eta = u - P_\eta(u),$$

where $P_\eta(u) = \langle u, \eta \rangle \eta$, is the *orthogonal projection of u onto the line spanned by η* . The *reflection of u about the hyperplane η^\perp* is defined by,

$$R_\eta(u) := u - 2\langle u, \eta \rangle \eta.$$

Finally, the *parallel projection of u along v onto the hyperplane η^\perp* is

$$P_{v, \eta^\perp}(u) := u - \frac{\langle u, \eta \rangle}{\langle v, \eta \rangle} v.$$

Denote by $\angle(v, w)$ the angle between two non-zero vectors in \mathbb{R}^d , defined as

$$\angle(v, w) := \arccos\left(\frac{\langle v, w \rangle}{\|v\| \|w\|}\right).$$

The angle between a non-zero vector $v \in \mathbb{R}^d$ and a linear subspace $E \subseteq \mathbb{R}^d$ is defined to be

$$\angle(v, E) := \min_{u \in E \setminus \{0\}} \angle(v, u).$$

The angle between two linear subspaces E and F of \mathbb{R}^d of the same dimension is defined as

$$\angle(E, F) := \max\left\{\max_{u \in E \setminus \{0\}} \angle(u, F), \max_{v \in F \setminus \{0\}} \angle(v, E)\right\}.$$

This angle defines a metric on the Grassmann manifold $\text{Gr}_k(\mathbb{R}^d)$ of all k -dimensional linear subspaces $E \subseteq \mathbb{R}^d$.

Given two linear subspaces $E, F \subseteq \mathbb{R}^d$, with $\dim E \leq \dim(F^\perp)$, we define the minimum angle

$$\angle_{\min}(E, F) := \min\{\angle(e, f) : e \in E \setminus \{0\}, f \in F \setminus \{0\}\}.$$

Unlike the previous angle, this minimum angle is not even a pseudo-metric on $\text{Gr}(\mathbb{R}^d) = \cup_{0 \leq k \leq d} \text{Gr}_k(\mathbb{R}^d)$. Notice that $\angle_{\min}(E, F) > 0$ if and only if $E \cap F = \{0\}$. The minimum angle $\angle_{\min}(E, F)$ quantifies the ‘transversality’ on the intersection $E \cap F$.

We denote by $\pi_{E, F^\perp} : E \rightarrow F^\perp$ the restriction to E of the orthogonal projection to F^\perp .

Lemma 2.1. *Given linear subspaces $E, F \subseteq \mathbb{R}^d$ with $\dim E = \dim F$,*

$$\sin \angle(E, F) = \|\pi_{E, F^\perp}\| = \|\pi_{F, E^\perp}\|.$$

Proof. Given $u \in E \setminus \{0\}$ and $v \in F \setminus \{0\}$, we have

1. $\|\pi_{E,F^\perp}(u)\|/\|u\| = d(u, F)/\|u\| = \sin(\angle(u, F)),$
2. $\|\pi_{F,E^\perp}(v)\|/\|v\| = d(v, E)/\|v\| = \sin(\angle(v, E)).$

Since $\dim E = \dim F$, there is an orthogonal linear map $\Phi: \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that $\Phi(F) = E$. By orthogonality one has $\Phi(F^\perp) = E^\perp$. Hence $\pi_{F,E^\perp} = \Phi^{-1} \circ \pi_{E,F^\perp} \circ \Phi$, which implies that $\|\pi_{E,F^\perp}\| = \|\pi_{F,E^\perp}\|$. Thus the sine of the maxima in the definition of $\angle(E, F)$ coincides with this common norm. \square

Lemma 2.2. *Let E, E' and H be linear subspaces of \mathbb{R}^d such that*

1. $\dim(E) = \dim(E'),$
2. $\angle(h, E) \geq \varepsilon,$ for all $h \in H \setminus \{0\}.$

Then

$$\sin(\angle(E + H, E' + H)) \leq \frac{\sin(\angle(E, E'))}{\sin \varepsilon}$$

Proof. First notice that

$$\angle(E + H, E' + H) = \angle((E + H) \cap H^\perp, (E' + H) \cap H^\perp).$$

Given $u \in (E + H) \cap H^\perp$ we can write $u = v - h$ with $v \in E$ and $h \in H$. Hence, since $u \in H^\perp$,

$$\begin{aligned} \frac{d(u, (E' + H) \cap H^\perp)}{\|u\|} &= \frac{d(u, E' + H)}{\|u\|} = \frac{d(v, E' + H)}{\|u\|} \\ &\leq \frac{d(v, E')}{\|u\|} = \frac{\|v\|}{\|u\|} \frac{d(v, E')}{\|v\|} \leq \frac{\|v\|}{\|u\|} \sin(\angle(E, E')) \\ &= \frac{\sin(\angle(E, E'))}{\sin(\angle(v, h))} \leq \frac{\sin(\angle(E, E'))}{\sin \varepsilon}. \end{aligned}$$

On the last equality we use that $v = h + u$ is an orthogonal decomposition with $h \in H$ and $u \in H^\perp$. Thus taking the sup in $u \in (E + H) \cap H^\perp \setminus \{0\}$ we get

$$\sin(\angle((E + H) \cap H^\perp, (E' + H) \cap H^\perp)) \leq \frac{\sin(\angle(E, E'))}{\sin \varepsilon}.$$

\square

Lemma 2.3. *Given linear subspaces $E, F \subseteq \mathbb{R}^d$ with $\dim E \leq \dim(F^\perp),$*

$$\det(\pi_{E,F^\perp}) \leq \mathbf{m}(\pi_{E,F^\perp}).$$

Equality holds when $\dim E = 1.$

Proof. Just notice that all singular values of π_{E,F^\perp} are in the range $[0, 1]$ because π_{E,F^\perp} is the restriction of an orthogonal projection. \square

Given an integer $k \in \mathbb{N}$ and a linear subspace $E \subseteq \mathbb{R}^d$, the *Grassmann space* of k -vectors in E will be denoted by $\wedge_k(E)$. This space inherits a natural Euclidean structure from E (see [10]).

Lemma 2.4. *Let $E, F \subseteq \mathbb{R}^d$ be linear subspaces with orthonormal basis $\{e_1, \dots, e_k\}$ and $\{f_1, \dots, f_r\}$ respectively such that $k \leq d - r.$ Let $e = e_1 \wedge \dots \wedge e_k \in \wedge_k(E)$ and $f = f_1 \wedge \dots \wedge f_r \in \wedge_r(F).$ Then*

$$\sin \angle_{\min}(E, F) = \mathbf{m}(\pi_{E,F^\perp}) \geq \det(\pi_{E,F^\perp}) = \|e \wedge f\|.$$

Proof. Given a unit vector $v \in E$, by the proof of Lemma 2.1 we have $\sin \angle(v, F) = \|\pi_{E, F^\perp}(v)\|$ which implies that

$$\sin \angle_{\min}(E, F) = \mathbf{m}(\pi_{E, F^\perp}).$$

On the other hand

$$\begin{aligned} \|e \wedge f\| &= \|(e_1 \wedge \dots \wedge e_k) \wedge (f_1 \wedge \dots \wedge f_r)\| \\ &= \|(\pi_{E, F^\perp}(e_1) \wedge \dots \wedge \pi_{E, F^\perp}(e_k)) \wedge (f_1 \wedge \dots \wedge f_r)\| \\ &= \|\pi_{E, F^\perp}(e_1) \wedge \dots \wedge \pi_{E, F^\perp}(e_k)\| \|f_1 \wedge \dots \wedge f_r\| \\ &= \|\wedge_k \pi_{E, F^\perp}(e)\| \|f\| = \det(\pi_{E, F^\perp}) \end{aligned}$$

because $\|e\| = \|f\| = 1$. The middle inequality follows from Lemma 2.3. \square

Lemma 2.5. *Let $E \subseteq \mathbb{R}^d$ be a linear space and $\{v_1, \dots, v_k\}$ be a family of unit vectors such that for all $1 \leq i \leq k$,*

$$\angle_{\min}(\langle v_i \rangle, E \oplus \langle v_1, \dots, v_{i-1} \rangle) \geq \varepsilon > 0.$$

Then

$$\sin \angle_{\min}(E, \langle v_1, \dots, v_k \rangle) \geq (\sin \varepsilon)^k.$$

Proof. Let $\{e_1, \dots, e_r\}$ be an orthonormal basis of E . We apply Lemma 2.4 to the subspaces $\langle v_i \rangle$ and $E \oplus \langle v_1, \dots, v_{i-1} \rangle$. Since the first subspace has dimension 1 the inequality in this lemma becomes an equality. Hence, because $\|v_i\| = 1$ we have

$$\frac{\|e_1 \wedge \dots \wedge e_r \wedge v_1 \wedge \dots \wedge v_i\|}{\|e_1 \wedge \dots \wedge e_r \wedge v_1 \wedge \dots \wedge v_{i-1}\|} \geq \sin \varepsilon$$

for all $1 \leq i \leq k$. Multiplying these inequalities and using Lemma 2.4 again we obtain

$$\begin{aligned} \sin \angle_{\min}(E, \langle v_1, \dots, v_k \rangle) &\geq \frac{\|e_1 \wedge \dots \wedge e_r \wedge v_1 \wedge \dots \wedge v_k\|}{\|e_1 \wedge \dots \wedge e_r\| \|v_1 \wedge \dots \wedge v_k\|} \\ &\geq \frac{\|e_1 \wedge \dots \wedge e_r \wedge v_1 \wedge \dots \wedge v_k\|}{\|e_1 \wedge \dots \wedge e_r\|} \\ &= \prod_{i=1}^k \frac{\|e_1 \wedge \dots \wedge e_r \wedge v_1 \wedge \dots \wedge v_i\|}{\|e_1 \wedge \dots \wedge e_r \wedge v_1 \wedge \dots \wedge v_{i-1}\|} \geq (\sin \varepsilon)^k. \end{aligned}$$

We have used above that $\|v_1 \wedge \dots \wedge v_k\| \leq \|v_1\| \cdots \|v_k\| = 1$. \square

2.2. Billiard map. Suppose that P has N faces (of dimension $d - 1$) which we denote by F_1, \dots, F_N . For each $i = 1, \dots, N$, denote by η_i the interior unit normal vector to the face F_i . Also denote by Π_i the hyperplane that supports the face F_i . We write the interior of F_i as F_i° , and its $(d - 2)$ -dimensional boundary as ∂F_i . Define $\partial P = \bigcup_{i=1}^N F_i$, and the $(d - 2)$ -skeleton $\Sigma P = \bigcup_{i=1}^N \partial F_i$. Finally define

$$M := \bigcup_{i=1}^N F_i^\circ \times \mathbb{S}_{\eta_i}^+.$$

The domain of the billiard map Φ_P is the set of points $(p, v) \in M$ such that the half-line $\{p + tv : t \geq 0\}$ does not intersect the skeleton ΣP . We denote this set by M' . Clearly, M' is the complement of a co-dimension two subset of M .

Now the billiard map $\Phi_P : M' \rightarrow M$ is defined as follows. Given $x = (p, v) \in M'$, let $\tau = \tau(p, v) > 0$ be minimum $t > 0$ such that $p + tv \in F_j'$ for some $j = 1, \dots, N$.

The real number τ is called the *flight time* of (p, v) . Then the billiard map is defined by

$$\Phi_P(x) = (p + \tau v, R_{\eta_j}(v)).$$

Note that the billiard map Φ_P is a piecewise smooth map and it has finitely many domains of continuity. The number of domains of continuity is at most $N(N-1)$, which is the number of 2-permutations of N faces. If P is convex, then all permutations define a branch map.

Let $(p', v') = \Phi_P(p, v)$ for $(p, v) \in M'$. It is easy to obtain a formula for the branch maps and its derivatives.

Proposition 1. *Suppose that $(p'_i, v'_i) = \Phi_P(p_i, v_i)$ for some $p_i \in F_i^\circ$ such that $p'_i \in F'_j$ with $i \neq j$. For every $x = (p, v) \in F_i^\circ \times \mathbb{S}_{\eta_i}^+$ such that $p' \in F'_j$ we have*

$$\Phi_P(x) = \left(p_j + P_{v, \eta_j^\perp}(p - p_j), R_{\eta_j}(v) \right).$$

Moreover

$$D\Phi_P(x)(u, w) = \left(P_{v, \eta_j^\perp}(u + \gamma(x)w), R_{\eta_j}(w) \right),$$

where

$$\gamma(x) = \frac{\langle p - p_j, \eta_j \rangle}{\langle v, \eta_j \rangle}.$$

Proof. Recall that $p' = p + \tau(p, v)v$ where $\tau(p, v)$ is the length of the vector $p' - p$. Taking the inner product with η_j in both sides of the equation and noting that $\langle p' - p_j, \eta_j \rangle = 0$, we get

$$\tau(p, v) = \frac{\langle p' - p, \eta_j \rangle}{\langle v, \eta_j \rangle} = \frac{\langle p_j - p, \eta_j \rangle}{\langle v, \eta_j \rangle}.$$

So

$$p' = p_j + \left((p - p_j) - \frac{\langle p - p_j, \eta_j \rangle}{\langle v, \eta_j \rangle} v \right) = p_j + P_{v, \eta_j^\perp}(p - p_j).$$

To prove the formula for the derivative, define the map $\Psi_\eta : (p, v) \mapsto P_{v, \eta^\perp}(p)$ for any given $\eta \in \mathbb{S}$. The claim follows from the formula

$$D\Psi_\eta(x)(u, w) = P_{v, \eta^\perp}(u) + \frac{\langle p, \eta \rangle}{\langle v, \eta \rangle} P_{v, \eta^\perp}(w).$$

□

2.3. Contracting reflection laws. A *contracting law* is any family $\{C_\eta : \mathbb{S}_\eta^+ \rightarrow \mathbb{S}_\eta^+\}_{\eta \in \mathbb{S}}$ of class C^2 mappings that satisfies for every $\eta \in \mathbb{S}$,

- (a) $C_\eta(\eta) = \eta$,
- (b) there are non-negative C^2 functions $a_\eta, b_\eta : \mathbb{S}_\eta^+ \rightarrow [0, +\infty)$ such that,

$$C_\eta(v) = a_\eta(v)P_\eta(v) + b_\eta(v)P_{\eta^\perp}(v), \quad \forall v \in \mathbb{S}_\eta^+.$$

- (c) $0 < \sup\{\|DC_\eta(x)\| : x \in \mathbb{S}_\eta^+\} < 1$,
- (d) $O \circ C_\eta = C_{O(\eta)} \circ O$, for every rotation $O \in \mathcal{O}(n, \mathbb{R})$.

A contracting law can be uniquely characterized by a single C^2 map of the interval $[0, \frac{\pi}{2})$ as the following proposition shows.

Proposition 2. *Given a contracting law $\{C_\eta : \mathbb{S}_\eta^+ \rightarrow \mathbb{S}_\eta^+\}_{\eta \in \mathbb{S}}$, there is a class C^2 mapping $f : [0, \frac{\pi}{2}) \rightarrow [0, \frac{\pi}{2})$ such that*

- (a) $f(0) = 0$,
- (b) $0 < \sup\{|f'(\theta)| : 0 \leq \theta < \frac{\pi}{2}\} < 1$,

(c) for every $\eta \in \mathbb{S}$, and $v \in \mathbb{S}_\eta^+$,

$$C_\eta(v) = \frac{\cos f(\theta)}{\cos \theta} P_\eta(v) + \frac{\sin f(\theta)}{\sin \theta} P_{\eta^\perp}(v),$$

where $\theta = \arccos\langle v, \eta \rangle$ is the angle between η and v ,

(d) for every $\eta \in \mathbb{S}$,

$$\sup_{x \in \mathbb{S}_\eta^+} \|DC_\eta(x)\| = \sup_{0 \leq \theta < \pi/2} |f'(\theta)|.$$

Proof. Let $\eta \in \mathbb{S}$ and $v \in \mathbb{S}_\eta^+$. By item (b) of the definition of a contracting law we can write

$$C_\eta(v) = a_\eta(v)P_\eta(v) + b_\eta(v)P_{\eta^\perp}(v)$$

where a_η and b_η are non-negative C^2 functions. Taking the inner product with η on both sides of the previous equation we get,

$$a_\eta(v) = \frac{\langle C_\eta v, \eta \rangle}{\cos \theta},$$

where $\theta = \arccos\langle v, \eta \rangle \in [0, \frac{\pi}{2})$ is the angle formed by the vectors v and η . By item (d) we conclude that $\langle C_\eta(v), \eta \rangle = \langle C_{O(\eta)}(O(v)), O(\eta) \rangle$, thus its value depends only on the angle θ . So, there is a C^2 function $f : [0, \frac{\pi}{2}) \rightarrow [0, \frac{\pi}{2})$ such that $\langle C_\eta(v), \eta \rangle = \cos f(\theta)$. Similarly, we conclude that

$$b_\eta(v) = \frac{\sin f(\theta)}{\sin \theta}.$$

This shows (c). The remaining properties follow immediately. \square

A C^2 mapping $f : [0, \frac{\pi}{2}) \rightarrow [0, \frac{\pi}{2})$ satisfying (a)-(d) above is called a *contracting reflection law*. We also define

$$\lambda(f) := \sup_{0 \leq \theta < \pi/2} |f'(\theta)|.$$

2.4. Contracting billiard map. Given a contracting law $\{C_\eta\}$ with contracting reflection law f , define the map $\chi_f : M \rightarrow M$ by $\chi_f(p, v) = (p, C_{\eta(p)}(v))$ where $\eta(p)$ denotes the interior unit normal of the face of the polytope where p lies. The *contracting billiard map* $\Phi_{f,P} : M' \rightarrow M$ is

$$\Phi_{f,P} = \chi_f \circ \Phi_P.$$

There is a system of coordinates which is convenient to represent the derivative of the contracting billiard map. For each $x = (p, v) \in M$ define $\Psi_x : T_x M \rightarrow v^\perp \times v^\perp$ by

$$\Psi_x(u, w) = (P_{v^\perp}(u), w) .$$

The previous linear isomorphism will be referred as *Jacobi coordinates* on the tangent space $T_x M$. We shall use the notation (J, J') to denote an element in $v^\perp \times v^\perp$. The following proposition gives a formula for the derivative of the contracting billiard map in terms of Jacobi co-ordinates.

Proposition 3. *Let $x = (p, v) \in M'$ and suppose that $x' = (p', v') = \Phi_{f,P}(x)$ with $p' \in F'_j$. Then $\Psi_{x'} \circ D\Phi_{f,P}(x) \circ \Psi_x^{-1}$ is given by*

$$(J, J') \mapsto \left(P_{v'^\perp} \circ P_{v, \eta_j^\perp}(J + \tau(p, v) J'), (DC_{\eta_j})_{R_{\eta_j}(v)} R_{\eta_j}(J') \right) .$$

Moreover, if $\theta = \arccos |\langle v, \eta_j \rangle|$, then

$$\left| \frac{\langle v', \eta_j \rangle}{\langle v, \eta_j \rangle} \right| = \frac{\cos f(\theta)}{\cos \theta} > 1.$$

Proof. Immediate from Propositions 1 and 2. \square

2.5. Orbits, invariant sets and hyperbolicity. Denote by M^+ the subset of points in M that can be iterated forward, i.e.

$$M^+ = \{x \in M : \Phi_{f,P}^n(x) \in M' \forall n \geq 0\}.$$

A *billiard orbit* is a sequence $\{x_n\}_{n \geq 0}$ in M' such that $x_{n+1} = \Phi_{f,P}(x_n)$ for every $n \geq 0$. A *billiard path or trajectory* is the polygonal path formed by segments of consecutive points of a billiard orbit.

Define

$$D := \bigcap_{n \geq 0} \Phi_{f,P}^n(M^+).$$

It is easy to see that D is an invariant set and $\Phi_{f,P}$ and its inverse are defined on D . Following Pesin we call the closure of D the *attractor* of $\Phi_{f,P}$. We say that $\Lambda \subset M$ is an *invariant set* if $\Lambda \subset D$ and $\Phi_{f,P}^{-1}(\Lambda) = \Lambda$.

To simplify the notation let us write $\Phi = \Phi_{f,P}$.

Definition 2.6. Given an invariant set Λ of Φ , we say that Φ is *uniformly partially hyperbolic on Λ* if for every $x \in \Lambda$ there exists a continuous splitting

$$T_x M = E^s(x) \oplus E^{cu}(x),$$

which is $D\Phi$ -invariant,

$$D\Phi(E^s(x)) = E^s(\Phi(x)), \quad D\Phi(E^{cu}(x)) = E^{cu}(\Phi(x)), \quad \forall x \in \Lambda$$

and there are constants $\lambda < 1$, $\sigma \geq 1$ and $C > 0$ such that for every $n \geq 1$ we have

$$\|D\Phi^n|_{E^s}\| \leq C\lambda^n \quad \text{and} \quad \|D\Phi^{-n}|_{E^{cu}}\| \leq C\sigma^{-n}.$$

If $\sigma > 1$, then we say that Φ is *uniformly hyperbolic on Λ* and write E^u for the subbundle E^{cu} . When $\Lambda = D$, then we simply say that Φ is *uniformly partially hyperbolic*.

We denote by

$$\chi(x, v) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \|D\Phi^n(x)v\|$$

the *Lyapunov exponent* of a non-zero tangent vector $v \in T_x M$ at $x \in D$.

Definition 2.7. A Φ -invariant Borel probability measure μ supported on D is called *hyperbolic* if μ -almost every $x \in D$ satisfies $\chi(x, v) \neq 0$ for every non-zero $v \in T_x M$.

The proof of the following result is an adaptation of [5, Proposition 3.1].

Proposition 4. *For any polytope P and any contracting reflexion law f , $\Phi_{f,P}$ is uniformly partially hyperbolic.*

Proof. Given $x = (p, v)$, $x' = (p', v') \in M$ such that $x' = \Phi(x)$ we denote by $L(x, x')$ the map from $v^\perp \times v^\perp$ to $v'^\perp \times v'^\perp$ that represents the derivative $D\Phi_x$ in the Jacobi

coordinates (see Proposition 3). This linear map is represented by a block upper triangular matrix of the form

$$L(x, x') = \begin{pmatrix} A(x, x') & B(x, x') \\ 0 & C(x, x') \end{pmatrix}$$

where $\|A(x, x')^{-1}\| \leq 1$ and $\|C(x, x')\| \leq \lambda < 1$, whose inverse is

$$L(x, x')^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$$

where $A = A(x, x')$, etc. Given a linear map $H' : v^\perp \rightarrow v^\perp$ the pre-image of its graph by $L(x, x')$ is the graph of another linear function $H : v^\perp \rightarrow v^\perp$ called the *backward graph transform* of H' and denoted by $H =: \Gamma(x, x')H'$. The operator $\Gamma(x, x')$ is hence defined by the relation

$$L(x, x')^{-1}\text{Graph}(H') = \text{Graph}(\Gamma(x, x')H').$$

A simple computation shows that

$$\Gamma(x, x')H' = A(x, x')^{-1}B(x, x') - A(x, x')^{-1}H'C(x, x').$$

We claim that writing $x_n = (p_n, v_n) = \Phi^n x$ and denoting by Z_n the zero endomorphism on v_n^\perp , the following limit exists

$$H^s(x) := \lim_{n \rightarrow +\infty} \Gamma(x, \Phi x) \dots \Gamma(\Phi^{n-1}x, \Phi^n x)Z_n.$$

A recursive computation allows to explicit the right hand side composition in the previous limit, which is a partial sum of the following series

$$H^s(x) = \sum_{j=0}^{\infty} (-1)^j A_0^{-1} \dots A_j^{-1} B_j C_{j-1} \dots C_0$$

where $A_j = A(\Phi^j x, \Phi^{j+1}x)$, etc. This series converges because $\|A_j^{-1}\| \leq 1$ and $\|C_j\| \leq \lambda(f) < 1$ for all $j \geq 0$.

By construction, the subspaces $E^s(x) := \Psi_x^{-1}\text{Graph}(H^s(x))$ determine a $D\Phi$ -invariant subbundle of TM satisfying $\|D\Phi|_{E^s(x)}\| \leq \lambda(f)$ for all $x \in D$. Given $x = (p, v) \in D$, define $E^{cu}(x) := \Psi_x^{-1}\{(J, 0) : J \in v^\perp\}$. Clearly, E^{cu} is invariant. Moreover, $\|D\Phi^{-1}|_{E^{cu}(x)}\| \leq 1$ for all $x \in D$.

Finally, since $T_x M = E^s(x) \oplus E^{cu}(x)$ the previous facts show that Φ is uniformly partially hyperbolic. \square

2.6. Main results.

Definition 2.8. Given $k \in \mathbb{N}$, we say that $x \in M^+$ is *k-generating* if the face normals along any orbit segment of length k of the orbit of x generate the Euclidean space \mathbb{R}^d .

Definition 2.9. Given $\varepsilon > 0$, the polytope P is called *ε -spanning* if for any d distinct faces F_{i_1}, \dots, F_{i_d} of P with interior normals $\eta_{i_1}, \dots, \eta_{i_d}$, the angle between η_{i_1} and $E := \langle \eta_{i_2}, \dots, \eta_{i_d} \rangle$ is at least ε , i.e.

$$\angle(\eta_{i_1}, E) \geq \varepsilon.$$

We also say that P is a *spanning polytope* if it is ε -spanning for some $\varepsilon > 0$.

The following theorem is the main result of this paper. It shows that the contracting billiard map uniformly expands the unstable direction along the orbit of any k -generating point. Moreover, the expanding rate only depends on the polytope and contracting reflection law.

Theorem 2.10. *Suppose P is a spanning polytope and f a contracting reflexion law. There exists $\sigma = \sigma(f, P) > 1$, depending only on f and P , such that for every $k \geq d$ and every k -generating $x \in D$,*

$$\|D\Phi_{f,P}^{-2k}|_{E^u(x)}\| \leq 1/\sigma.$$

We prove this theorem and the following results in section 6.

Definition 2.11. Given $x \in M^+$, the *escaping time* of x , denoted by $T(x)$, is the least positive integer $k \in \mathbb{N}$ such that x is k -generating. If x is not k -generating for any $k \in \mathbb{N}$, then we set $T(x) = \infty$. We also call the function $T : M^+ \rightarrow \mathbb{N} \cup \{\infty\}$ the *escaping time of P with respect to f* .

Theorem 2.12. *Suppose P is a spanning polytope and μ is an ergodic $\Phi_{f,P}$ -invariant Borel probability measure. If T is μ -integrable, then μ is hyperbolic.*

Theorem 2.13. *Suppose P is a spanning polytope and Λ an invariant set of $\Phi_{f,P}$. If T is bounded on Λ , then $\Phi_{f,P}$ is uniformly hyperbolic on Λ .*

The concept of polytope in general position, mentioned in the following corollaries, is defined below (see definition 3.1).

Corollary 3. *Suppose P is a polytope in general position. There exists $\lambda_0 = \lambda_0(P) > 0$ such that for every contracting reflection law f satisfying $\lambda(f) > \lambda_0$ the billiard map $\Phi_{f,P}$ is uniformly hyperbolic.*

A polytope P in general position is called *obtuse* if the barycentric angle at every vertex of P is greater than $\pi/4$ (see section 4 for a precise definition).

Corollary 4. *Suppose P is a polytope in general position and f any contracting reflection law. If P is obtuse, the $\Phi_{f,P}$ is uniformly hyperbolic.*

3. Generic Polytopes.

Definition 3.1. A d -dimensional polytope P is said to be in general position if

1. for any set of d faces of P , $(d-1)$ -dimensional faces, their normals are linearly independent,
2. the normals to the $(d-1)$ -faces of P incident with any given vertex are linearly independent.

Proposition 5. *Given some d -dimensional polytope $P \subset \mathbb{R}^d$ in general position, each vertex has exactly d faces and d edges incident with it.*

Proof. Follows from condition (2) of the Definition 3.1. □

Consider the class \mathcal{P}_N of d -dimensional polyhedra $P \subset \mathbb{R}^d$ that contain the origin, i.e., $0 \in \text{int}(P)$, with exactly N faces. Given N points $(p_1, \dots, p_N) \in (\mathbb{R}^d \setminus \{0\})^N$, define the polytope $Q(p_1, \dots, p_N) \subset \mathbb{R}^d$,

$$Q(p_1, \dots, p_N) := \bigcap_{j=1}^N \{x \in \mathbb{R}^d : \langle x, p_j \rangle \leq \langle p_j, p_j \rangle\}.$$

The set

$$\mathcal{U} := \{(p_1, \dots, p_N) \in (\mathbb{R}^d \setminus \{0\})^N : Q(p_1, \dots, p_N) \text{ has exactly } N\text{-faces}\}$$

is open in $(\mathbb{R}^d \setminus \{0\})^N$, and the range of $Q : \mathcal{U} \rightarrow \cdot$ coincides with \mathcal{P}_N . Locally the map $Q : \mathcal{U} \rightarrow \mathcal{P}_N$ is one-to-one, and determines an atlas for a smooth structure on \mathcal{P}_N . We will consider on this manifold the Lebesgue measure obtained as push-forward of the Lebesgue measure on $(\mathbb{R}^d \setminus \{0\})^N$ by the map Q .

Let \mathcal{P}_N denote the subset of polytopes in \mathcal{P}_N .

In Algebraic Geometry, the following result is a standard consequence of the notion of ‘general position’. We include its proof here for the reader’s convenience, also because we could not find any reference for this precise statement.

Proposition 6. *The subset of polytopes in general position is open and dense, and has full Lebesgue measure in \mathcal{P}_N .*

Proof. Consider the subsets $\mathcal{N}_1 \subset \mathcal{P}_N$, resp. $\mathcal{N}_2 \subset \mathcal{P}_N$, of polytopes where condition (1), resp. (2), of definition 3.1 is violated. It is enough to observe that the sets \mathcal{N}_1 and \mathcal{N}_2 are finite unions of algebraic varieties of co-dimension one.

For any vector $v = (v_1, \dots, v_d) \in \mathbb{R}^d$, let $\hat{v} := (v_1, \dots, v_d, \langle v, v \rangle) \in \mathbb{R}^{d+1}$. Then \mathcal{N}_2 is covered by the union over all $1 \leq i_1 < i_2 < \dots < i_{d+1} \leq N$ of the hypersurfaces defined by the algebraic equation

$$\det[\hat{p}_{i_1}, \hat{p}_{i_2}, \dots, \hat{p}_{i_{d+1}}] = 0. \tag{1}$$

In fact, if there is a point $x_0 \in \mathbb{R}^d$ in the intersection of $d + 1$ distinct hyperplanes

$$\langle p_{i_k}, x \rangle = \langle p_{i_k}, p_{i_k} \rangle \quad k = 1, \dots, d + 1$$

then the matrix with rows $\hat{p}_{i_1}, \hat{p}_{i_2}, \dots, \hat{p}_{i_{d+1}}$ contains the vector $(x_0, -1) \in \mathbb{R}^{d+1}$ in its kernel, which implies (1).

Analogously, \mathcal{N}_1 is contained in the union over all $1 \leq i_1 < i_2 < \dots < i_d \leq N$ of the hypersurfaces defined by the algebraic equation

$$\det[p_{i_1}, p_{i_2}, \dots, p_{i_d}] = 0.$$

□

4. Escaping Times. In this section we study the escaping times of billiards on polyhedral cones with contracting reflection laws.

Let Π_1, \dots, Π_s be s hyperplanes in \mathbb{R}^d passing through the origin. For each hyperplane Π_i we take a unit normal vector η_i and we suppose that the set of hyperplanes are in general position, i.e. the normal vectors η_1, \dots, η_s are linearly independent. A set of s hyperplanes in general position define a *convex polyhedral cone*

$$Q = \{x \in \mathbb{R}^d : \langle x, \eta_i \rangle \geq 0, \quad i = 1, \dots, s\}.$$

For polyhedral billiard with the specular reflection law, Sinai proved that there exists a constant $K > 0$, depending only on Q , such that every billiard trajectory in Q has at most K reflections [9]. In this case we say that Q has *finite escaping time*.

By projecting the billiard dynamics to the orthogonal complement of $\bigcap_{i=1}^s \Pi_i$, we may assume that the normal vectors η_1, \dots, η_s defining the polyhedral cone Q span \mathbb{R}^d . Thus, from now on we set $s = d$. Associated with a convex polyhedral cone Q there is a constant measuring the aperture of Q . It is defined as follows.

Definition 4.1. The normal vectors η_1, \dots, η_d regarded as points determine an affine hyperplane H and a unit normal vector e such that

$$\langle \eta_i, e \rangle = \ell, \quad i = 1, \dots, d,$$

where ℓ is the distance of H to the origin. The *barycentric angle* ϕ of Q is defined by $\sin \phi = \ell$ (see Figure 1). Note that $0 < \phi < \pi/2$. We say that a convex polyhedral cone Q is *obtuse* if $\phi > \pi/4$.

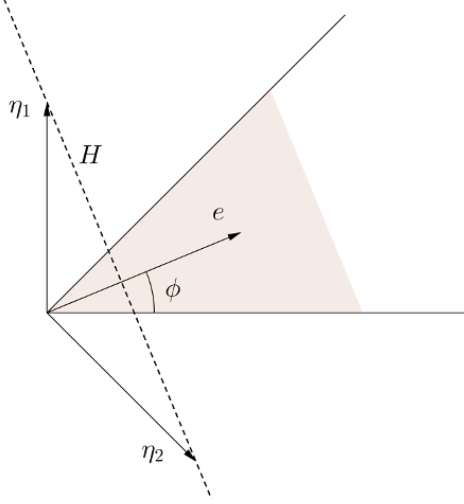


FIGURE 1. Barycentric angle ϕ .

4.1. Zigzag reflections. According to Proposition 2, given any billiard orbit $\{(p_k, v_k)\}_{k \geq 0}$, the sequence of *reflection velocities* satisfies

$$v_{k+1} = \frac{\cos f(\theta_k)}{\cos \theta_k} P_{\eta_{i_k}}(u_k) + \frac{\sin f(\theta_k)}{\sin \theta_k} P_{\eta_{i_k}^\perp}(u_k), \quad k \geq 0, \quad (2)$$

where $u_k = R_{\eta_{i_k}}(v_k)$, $\theta_k = \arccos \langle u_k, \eta_{i_k} \rangle$ and η_{i_k} is the inward normal of P where the $k+1$ -th collision took place.

Lemma 4.2. $\|v_{k+1} - v_k\| = 2 \cos\left(\frac{f(\theta_k) + \theta_k}{2}\right)$ for every $k \geq 0$.

Proof. Simple computation using (2). \square

Given a sequence of consecutive reflection velocities v_0, \dots, v_n we denote by L the length of the zigzag path formed by the reflections, i.e.

$$L(v_0, \dots, v_n) = \sum_{k=0}^{n-1} \|v_{k+1} - v_k\|.$$

We say that Q has *bounded zigzag reflections* if there exists a constant $C > 0$ such that $L(v_0, \dots, v_n) \leq C$ for every sequence of consecutive reflection velocities v_0, \dots, v_n and any $n \geq 0$.

Lemma 4.3. *A convex polyhedral cone has finite escaping time if and only if it has bounded zigzag reflections.*

Proof. If Q has finite escaping time, then there exists an integer $K > 0$ such that every billiard trajectory has at most K reflections. Since the zigzag length $L : \prod_{i=1}^K \mathbb{S}^m \rightarrow \mathbb{R}$ is a continuous function with compact domain, it has a maximum. Thus, Q has bounded zigzag reflections.

Now suppose that Q has not finite escaping time. This means that for every $K > 0$ there exists a billiard trajectory in Q that has at least K reflections with the faces of Q . By Lemma 4.2 we have $\|v_{k+1} - v_k\| \geq \delta > 0$ where $\delta := 2 \cos\left(\frac{f(\pi/2) + \pi/2}{2}\right) > 0$. This means that for every $K > 0$ there exists a sequence of consecutive reflection velocities v_0, \dots, v_n such that $L(v_0, \dots, v_n) \geq \delta K$. So Q cannot have bounded zigzag reflections. \square

Next we provide a sufficient condition on the contracting reflection law that guarantees boundedness of zigzag reflections. Thus finite escaping time.

Lemma 4.4. *For every sequence of consecutive reflection velocities v_0, \dots, v_n we have*

$$\langle v_{k+1} - v_k, e \rangle = \|v_{k+1} - v_k\| \gamma_k, \quad k = 0, \dots, n$$

where

$$\gamma_k = \cos\left(\frac{f(\theta_k) - \theta_k}{2}\right) \sin \phi + \sin\left(\frac{f(\theta_k) - \theta_k}{2}\right) h_k$$

and $h_k = \left\langle P_{\eta_{i_k}^\perp}(u_k) / \sin \theta_k, e \right\rangle$.

Proof. Follows from (2) that

$$v_{k+1} - v_k = \frac{\cos f(\theta_k) + \cos \theta_k}{\cos \theta_k} P_{\eta_{i_k}}(u_k) + \frac{\sin f(\theta_k) - \sin \theta_k}{\sin \theta_k} P_{\eta_{i_k}^\perp}(u_k).$$

Taking into account that $P_{\eta_{i_k}}(u_k) / \cos \theta_k = \eta_{i_k}$ and $\langle \eta_{i_k}, e \rangle = \sin \phi$ we get

$$\langle v_{k+1} - v_k, e \rangle = (\cos f(\theta_k) + \cos \theta_k) \sin \phi + (\sin f(\theta_k) - \sin \theta_k) h_k,$$

where $h_k = \left\langle P_{\eta_{i_k}^\perp}(u_k) / \sin \theta_k, e \right\rangle$. Using classical trigonometric identities we can write

$$\langle v_{k+1} - v_k, e \rangle = 2 \cos\left(\frac{f(\theta_k) + \theta_k}{2}\right) \gamma_k,$$

where

$$\gamma_k = \cos\left(\frac{f(\theta_k) - \theta_k}{2}\right) \sin \phi + \sin\left(\frac{f(\theta_k) - \theta_k}{2}\right) h_k.$$

To conclude the proof apply Lemma 4.2. \square

Theorem 4.5. *If $2\phi > \pi/2 - f(\pi/2)$ then Q has finite escaping time.*

Proof. Let v_0, \dots, v_n be any sequence of consecutive reflection velocities. By Lemma 4.4,

$$2 \geq \langle v_n - v_0, e \rangle = \sum_{k=0}^{n-1} \|v_{k+1} - v_k\| \gamma_k, \quad (3)$$

where

$$\gamma_k = \cos\left(\frac{f(\theta_k) - \theta_k}{2}\right) \sin \phi + \sin\left(\frac{f(\theta_k) - \theta_k}{2}\right) h_k$$

and $h_k = \left\langle P_{\eta_{i_k}^\perp}(u_k) / \sin \theta_k, e \right\rangle$. To estimate γ_k from below note that $h_k \leq \cos \phi$.

Thus

$$\gamma_k \geq \sin\left(\phi + \frac{f(\theta_k) - \theta_k}{2}\right) \geq \sin\left(\phi + \frac{f(\pi/2) - \pi/2}{2}\right).$$

By assumption $\mu := \phi + \frac{f(\pi/2) - \pi/2}{2} > 0$. Then, it follows from (3) that

$$L(v_0, \dots, v_n) < \frac{2}{\sin \mu},$$

for every sequence of consecutive reflection velocities v_0, \dots, v_n . This proves that Q has bounded zigzag reflections. Thus, by Lemma 4.3, Q has finite escaping time. \square

This theorem yields the following corollaries.

Corollary 5. *Any polyhedral cone Q with contracting reflection law f sufficiently close to the specular one has finite escaping time.*

Proof. It is clear that $2\phi > \pi/2 - f(\pi/2)$ for every contraction f sufficiently close to the identity. Thus, Q has finite escaping time, by Theorem 4.5. \square

Recall that a convex polyhedral cone Q is obtuse if $\phi > \pi/4$.

Corollary 6. *Any obtuse polyhedral cone Q has finite escaping time for every contracting reflection law f .*

Proof. If the polyhedral cone is obtuse then $\phi > \pi/4$. Thus, $2\phi > \pi/2 > \pi/2 - f(\pi/2)$ for every contraction f . Thus, Q has finite escaping time, by Theorem 4.5. \square

5. Uniform Expansion. By Proposition 3, the first component of the derivative $D\Phi_{f,P}(p, v)$ of the billiard map is represented in Jacobi coordinates by the map

$$L_{v, \eta, v'} := P_{v'\perp} \circ P_{v, \eta\perp} : \mathbb{R}^d \rightarrow \mathbb{R}^d \quad (4)$$

where $v', v, \eta \in \mathbb{R}^d$ are three coplanar unit vectors with $v' = C_\eta(R_\eta(v))$.

The main result of this section is Theorem 5.3, which gives conditions that ensure the uniform expansion of compositions of such maps. Since the second component of the billiard map is contracting (see Proposition 4), these conditions will imply the uniform hyperbolicity of the billiard map.

5.1. Trajectories. Let P be a d -dimensional polytope in \mathbb{R}^d , and \mathcal{N}_P be the set of its unit inward normals. Denote by \mathbb{N}_0 the set of natural numbers \mathbb{N} including 0.

In the sequel we introduce a space of trajectories containing true orbits of the billiard map of P . The reason is to exploit the compactness of this space which does not hold for the billiard map's phase space, since one has to exclude from the phase space all orbits which eventually hit the skeleton of P .

Define the map $h : D \rightarrow (\mathbb{S} \times \mathcal{N}_P)^{\mathbb{N}_0}$, $h(p, v) := \{(v_j, \eta_{i_j})\}_{j \in \mathbb{N}_0}$ where for all $j \geq 0$, $\Phi_{f,P}^j(p, v) = (p_j, v_j)$ with $p_j \in F_{i_j}$. Recall that D is maximal invariant set defined in Section 2.5. This map semi-conjugates the billiard map $\Phi_{f,P}$ with the shift on the space of sequences $(\mathbb{S} \times \mathcal{N}_P)^{\mathbb{N}_0}$. Since $h(D)$ is not compact we introduce the following definition extending the notion of billiard trajectory.

Although $\mathcal{N}_P = \{\eta_1, \dots, \eta_N\}$, in order to simplify our notation from now on we will write η_j , $j \in \mathbb{N}_0$, for any normal in \mathcal{N}_P and not necessarily the j -th normal in \mathcal{N}_P .

Definition 5.1. A sequence $\{(v_j, \eta_j)\}_{j \geq 0} \in (\mathbb{S} \times \mathcal{N}_P)^{\mathbb{N}_0}$ is called a *trajectory* if for all $j \in \mathbb{N}$

1. $\langle v_{j-1}, \eta_j \rangle \leq 0$,
2. $v_j = C_{\eta_j} \circ R_{\eta_j}(v_{j-1})$,

where R_η is the reflection introduced in section 2, and C_η is the contracting reflection law defined in subsection 2.3. We denote by $\mathcal{T} = \mathcal{T}_{f,P}$ the space of all trajectories.

Note that

$$h(D) \subset \mathcal{T} \subset (\mathbb{S} \times \mathcal{N}_P)^{\mathbb{N}_0}.$$

Given $i < j$ in \mathbb{N}_0 , we denote by $[i, j] := \{i, i+1, \dots, j\} \subseteq \mathbb{N}_0$ the time interval between the instants i and j . Given a trajectory $\{(v_j, \eta_j)\}_{j \geq 0}$ and a time interval $[i, j]$, the linear span $V_{[i, j]} := \langle\langle v_i, v_{i+1}, \dots, v_j \rangle\rangle$ is called the *velocity front* of the trajectory along the time interval $[i, j]$. The linear span $N_{[i, j]} := \langle\langle \eta_i, \eta_{i+1}, \dots, \eta_j \rangle\rangle$ is called the *normal front* of the trajectory along the time interval $[i, j]$. Given $i \in \mathbb{N}$, let $L_i : v_{i-1}^\perp \rightarrow v_i^\perp$ be the linear map defined by

$$L_i = P_{v_i^\perp} \circ P_{v_{i-1}, \eta_i^\perp}.$$

Finally we define the *velocity tangent flow* along $[i, j]$ to be the linear map $L_{[i, j]} : v_i^\perp \rightarrow v_j^\perp$ defined by

$$L_{[i, j]} = L_j \circ \dots \circ L_{i+1}.$$

When the trajectory is associated to a billiard orbit $\{(p_l, v_l)\}_{l \geq 0}$ of $\Phi_{f, P}$, the linear map $L_{[i, j]}$ represents, in Jacobi coordinates, the first component of the derivative $D\Phi_{f, P}^{j-i}$ at (p_i, v_i) . By definition, given $i < j < k$,

$$L_{[i, k]} = L_{[j, k]} \circ L_{[i, j]}.$$

We now extend Definition 2.8 to trajectories.

Definition 5.2. We say that the trajectory $\{(v_l, \eta_l)\}_{l \geq 0}$ is *generating on* $[i, j]$ if $N_{[i, j]} = \mathbb{R}^d$. Given $k \in \mathbb{N}$, we say that the trajectory is *k-generating* if it is generating on any interval $[i, j]$ with $j - i \geq k$.

We can now state this section's main result.

Theorem 5.3. *Given $\varepsilon > 0$, d -dimensional polytope P and contracting reflection law f , there exists a constant $\sigma = \sigma(\varepsilon, d, f) > 1$ such that for any trajectory $\{(v_j, \eta_j)\}_{j \geq 0}$ in $\mathcal{T}_{f, P}$ the following holds. If*

1. P is ε -spanning,
2. $\{(v_j, \eta_j)\}_{j \geq 0}$ is k -generating, with $k \in \mathbb{N}$,

then the linear map $L_{[0, 2k]} : v_0^\perp \rightarrow v_{2k}^\perp$ satisfies

$$\|L_{[0, 2k]}(v)\| \geq \sigma \|v\| \quad \text{for all } v \in v_0^\perp.$$

The proof of this theorem is done at the end of the section.

Remark 1. From the previous theorem's conclusion, for any $n \geq 0$,

$$\|L_{[0, n]}(v)\| \geq \sigma^{\frac{n}{2k}-1} \|v\| \quad \text{all } v \in v_0^\perp.$$

This means, minimum growth expansion rate larger or equal than $\sigma^{\frac{1}{2k}} > 1$.

5.2. Properties of trajectories. The following result says that the *trajectory space* \mathcal{T} is compact.

Proposition 7. *The space \mathcal{T} is a closed subspace of the product space $(\mathbb{S} \times \mathcal{N}_P)^{\mathbb{N}_0}$. In particular, with the induced topology \mathcal{T} is a compact space.*

Proof. The trajectory space \mathcal{T} is closed in the product space because conditions (1) and (2) in Definition 5.1 are closed conditions. By Thychonoff's theorem $(\mathbb{S} \times \mathcal{N}_P)^{\mathbb{N}_0}$ is compact, and hence \mathcal{T} is compact too. \square

Lemma 5.4. *Given any trajectory $\{(v_j, \eta_j)\}_{j \geq 0}$ there exist scalars $\alpha_j, \beta_j \in \mathbb{R}$ such that for any $j \geq 1$,*

$$v_j = \alpha_j \eta_j + \beta_j v_{j-1}$$

where

$$\cos\left(\frac{\pi}{2}\lambda(f)\right) < \alpha_j < 2 \quad \text{and} \quad 0 \leq \beta_j < 1.$$

Moreover,

$$\left| \frac{\langle v_j, \eta_j \rangle}{\langle v_{j-1}, \eta_j \rangle} \right| = \frac{\cos f(\theta_j)}{\cos \theta_j} \quad \text{where} \quad \theta_j = \arccos |\langle v_{j-1}, \eta_j \rangle|.$$

Proof. According to Proposition 2,

$$v_j = (a_j + b_j) \cos \theta_j \eta_j + b_j v_{j-1}$$

where

$$a_j = \frac{\cos f(\theta_j)}{\cos \theta_j}, \quad b_j = \frac{\sin f(\theta_j)}{\sin \theta_j} \quad \text{and} \quad \theta_j = \arccos |\langle v_{j-1}, \eta_j \rangle|.$$

Since $\lambda(f) < 1$, we have $1 \leq a_j + b_j < 2$ and $0 \leq b_j < 1$. Moreover, $\cos \theta_j > \cos(\frac{\pi}{2}\lambda(f))$. The last claim is a simple computation. \square

Lemma 5.5. *Given a trajectory $\{(v_l, \eta_l)\}_{l \geq 0}$, for all intervals $[i, j]$,*

1. $V_{[i,j]} = \langle\langle v_i \rangle\rangle + N_{[i+1,j]}$ and $V_{[i,j]}^\perp \subseteq v_i^\perp \cap v_j^\perp$.
2. $L_{[i,j]} : v_i^\perp \rightarrow v_j^\perp$ is the identity on $V_{[i,j]}^\perp$.
3. $\mathfrak{m}(L_{[i,j]}) \geq 1$.

Proof. Straightforward computation. \square

5.3. Collinearities. Throughout the rest of this section, we assume that $\varepsilon > 0$ is fixed and that P is ε -spanning.

Consider a trajectory $\{(v_l, \eta_l)\}_{l \geq 0}$ in \mathcal{T} .

Definition 5.6. A time interval $[i, j]$ is called a *collinearity* of the trajectory $\{(v_l, \eta_l)\}_{l \geq 0}$ if its velocity and the normal fronts along the time interval $[i, j]$ coincide, i.e. $V_{[i,j]} = N_{[i,j]}$. The number $j - i$ will be referred as the length of the collinearity $[i, j]$.

Definition 5.7. A collinearity is called *minimal* if it contains no smaller subinterval which is itself a collinearity.

For instance, if $v_i \in \langle\langle \eta_i \rangle\rangle$ then $\{i\}$ is a minimal collinearity of length 0.

Proposition 8. *Given a trajectory $\{(v_l, \eta_l)\}_{l \geq 0}$, assume $v_i \in N_{[i,j]}$ with $i \leq j$. Then there is some $i' \in [i, j]$ such that the time interval $[i', j]$ is a collinearity.*

Proof. The proof goes by induction on the length $r = j - i$. If the length is 0 then $i = j$ and we have necessarily $v_i \in \langle\langle \eta_i \rangle\rangle$, in which case it is obvious that $[i, i] = \{i\}$ is a collinearity. Assume now that the statement holds for all time intervals of length less than r , and let $v_i = \lambda_i \eta_i + \dots + \lambda_j \eta_j$ with $j - i = r$. We consider two cases:

First suppose that $\lambda_i \neq 0$. By item (1) of Lemma 5.5,

$$V_{[i,j]} = \langle\langle v_i \rangle\rangle + N_{[i+1,j]} \subseteq N_{[i,j]}.$$

Conversely, because $\lambda_i \neq 0$ we have $\eta_i \in \langle\langle v_i \rangle\rangle + N_{[i+1,j]}$ which proves that

$$N_{[i,j]} \subseteq \langle\langle v_i \rangle\rangle + N_{[i+1,j]} = V_{[i,j]},$$

where in the last equality we have used again item (1) of Lemma 5.5. Therefore, $[i, j]$ is a collinearity in this case.

Assume next that $\lambda_i = 0$. By Lemma 5.4, there are scalars α_{i+1} and β_{i+1} such that $v_{i+1} = \alpha_{i+1}\eta_{i+1} + \beta_{i+1}v_i$. We may assume that $\beta_{i+1} \neq 0$. Otherwise $v_{i+1} \in \langle \eta_{i+1} \rangle$ and $[i+1, j]$ is a collinearity. Thus

$$\lambda_{i+1}\eta_{i+1} + \dots + \lambda_j\eta_j = v_i = \frac{1}{\beta_{i+1}}(v_{i+1} - \alpha_{i+1}\eta_{i+1}) .$$

In this case

$$v_{i+1} = \beta_{i+1} \left[\left(\lambda_{i+1} - \frac{\alpha_{i+1}}{\beta_{i+1}} \right) \eta_{i+1} + \lambda_{i+2}\eta_{i+2} + \dots + \lambda_j\eta_j \right] .$$

and the conclusion follows by the induction hypothesis applied to the time interval $[i+1, j]$ of length $p-1$. \square

Proposition 9. *Given a trajectory $\{(v_l, \eta_l)\}_{l \geq 0}$ and $i < j \leq j'$ the following holds:*

1. *If $[i, j]$ is a collinearity then $[i, j']$ is also a collinearity.*
2. *If $v_j \in V_{[i, j-1]}$ and $\eta_j \notin N_{[i, j-1]}$, then there is some $i < i' \leq j$ such that $[i', j]$ is a collinearity.*

Proof. Let $i < j \leq j'$.

1. Assume $V_{[i, j]} = N_{[i, j]}$. Then by Lemma 5.5,

$$V_{[i, j']} = \langle v_i \rangle + N_{[i+1, j]} + N_{[j+1, j']} = N_{[i, j]} + N_{[j+1, j']} = N_{[i, j']} .$$

2. Assume now $v_j \in V_{[i, j-1]}$. By Lemma 5.4,

$$\eta_j = \frac{1}{\alpha_j}(v_j - \beta_j v_{j-1}) ,$$

where $\alpha_j \neq 0$. Thus $\eta_j \in V_{[i, j-1]}$. By Lemma 5.5 we can write $\eta_j = \lambda_i v_i + u$ for some $u \in N_{[i+1, j-1]}$. By assumption, $\lambda_i \neq 0$. Thus $v_i \in N_{[i+1, j]}$. Again by Lemma 5.4, we conclude that $v_{i+1} \in N_{[i+1, j]}$. Now the claim follows by Proposition 8. \square

Corollary 7. *Let $\{(v_l, \eta_l)\}_{l \geq 0}$ be a trajectory and $k \geq i \geq 0$ be integers such that the time segment $[i, k]$ contains no subinterval which is a collinearity. Then for every $j \in [i, k]$ either*

1. $\eta_j \in \{\eta_{i+1}, \dots, \eta_{j-1}\}$, or else
2. $v_j \notin V_{[i, j-1]}$.

Proof. This corollary is a reformulation of item (2) of Proposition 9. \square

5.4. Quantifying collinearities. We are now going to prove quantified versions of Propositions 8, 9 and Corollary 7. The following abstract continuity lemma will be useful.

Lemma 5.8. *Let \mathcal{X} be a compact topological space and $f, g : \mathcal{X} \rightarrow \mathbb{R}$ be continuous functions such that $g(x) = 0$ for all $x \in \mathcal{X}$ with $f(x) = 0$. Given $\delta > 0$ there is $\delta' > 0$ such that for all $x \in \mathcal{X}$, if $f(x) < \delta'$ then $g(x) < \delta$.*

Proof. Assume, to get a contradiction, that the claimed statement does not hold. Then there is $\delta > 0$ such that for all $n \in \mathbb{N}$ there is a point $x_n \in \mathcal{X}$ with $f(x_n) < \frac{1}{n}$ and $g(x_n) \geq \delta$. Since \mathcal{X} is compact, by taking a subsequence we can assume $x_n \rightarrow x$ in \mathcal{X} . By continuity of f and g , $f(x) = 0$ and $g(x) \geq \delta$, which contradicts the lemma hypothesis. \square

Definition 5.9. Given $\delta > 0$, we call δ -collinearity of a trajectory $\{(v_l, \eta_l)\}_l$ to any time interval $[i, j]$ such that $\dim V_{[i,j]} = \dim N_{[i,j]}$ and

$$\angle(V_{[i,j]}, N_{[i,j]}) < \delta.$$

Proposition 10. Given $\delta > 0$ there exists $\delta' > 0$ such that for any trajectory $\{(v_l, \eta_l)\}_l$ the following holds. If

$$\angle(v_i, N_{[i,j]}) < \delta'$$

for some $0 \leq i \leq j$, then there exists $i' \in [i, j]$ for which the time interval $[i', j]$ is a δ -collinearity of the given trajectory.

Proof. Notice that, because the space of trajectories \mathcal{T} is shift invariant, there is no loss of generality in assuming that $[i, j] = [0, p]$. For each $k \geq 0$, define the functions $f_k, g_k : \mathcal{T} \rightarrow \mathbb{R}$ by

$$\begin{aligned} f_k(\{(v_l, \eta_l)\}_l) &= \angle(v_0, N_{[0,k]}), \\ g_k(\{(v_l, \eta_l)\}_l) &= \min_{0 \leq i \leq k} \angle(V_{[i,k]}, N_{[i,k]}). \end{aligned}$$

These functions are clearly continuous.

Proposition 8 shows that for all $x \in \mathcal{T}$ and $0 \leq k \leq p$, $f_k(x) = 0$ implies $g_k(x) = 0$. Thus, given $\delta > 0$, by Lemma 5.8, there exists $\delta' > 0$ such that for any $0 \leq k \leq p$ and $x \in \mathcal{T}$,

$$f_k(x) < \delta' \quad \Rightarrow \quad g_k(x) < \delta.$$

\square

Proposition 11. Given any trajectory $\{(v_l, \eta_l)\}_l$, $i < j \leq j'$ and $\delta > 0$ the following holds.

1. If $[i, j]$ is a δ -collinearity, then $[i, j']$ is a δ' -collinearity, where $\delta' = \arcsin(\frac{\sin \delta}{\sin \varepsilon})$.
2. There exists $\delta' > 0$ such that, if

$$\angle(v_j, V_{[i,j-1]}) < \delta'$$

and $\eta_j \notin N_{[i,j-1]}$, then there is some $i < i' \leq j$ such that $[i', j]$ is a δ -collinearity.

Proof. Denote by H the linear space spanned by the ‘new’ normals η_l in the range $j < l \leq j'$, i.e., normals which are not in $\{\eta_i, \dots, \eta_j\}$. By definition of H we have,

$$\begin{aligned} V_{[i,j']} &= V_{[i,j]} + H, \\ N_{[i,j']} &= N_{[i,j]} + H. \end{aligned}$$

Hence by Lemma 2.2, if $[i, j]$ is a δ -collinearity,

$$\sin \angle(V_{[i,j']}, N_{[i,j']}) \leq \frac{1}{\sin \varepsilon} \sin \angle(V_{[i,j]}, N_{[i,j]}) \leq \frac{\sin \delta}{\sin \varepsilon} = \sin \delta',$$

which proves that $[i, j']$ is a δ' -collinearity. This proves (1).

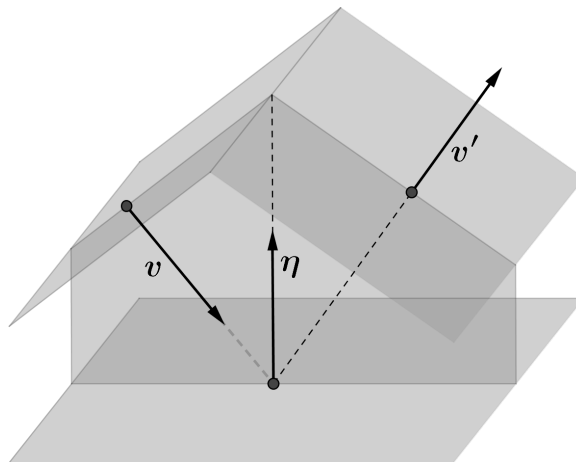


FIGURE 2. Composition of the projections $P_{v'\perp} \circ P_{v,\eta\perp}$

To prove item (2) note that, as in the proof of Proposition 10, there is no loss of generality in assuming that $[i, j] = [0, p]$. Define the functions $f_k, g_k : \mathcal{J} \rightarrow \mathbb{R}$ by

$$f_k(\{(v_l, \eta_l)\}_l) = \angle(v_k, V_{[0, k-1]}),$$

$$g_k(\{(v_l, \eta_l)\}_l) = \min_{1 \leq i \leq k} \angle(V_{[i, k]}, N_{[i, k]}).$$

These functions are clearly continuous.

Item (2) of Proposition 9 shows that for every $x = \{(v_l, \eta_l)\}_l \in \mathcal{J}$ and for every $1 \leq k \leq p$ for which $\eta_k \notin N_{[0, k-1]}$, $f_k(x) = 0$ implies $g_k(x) = 0$. Thus, given $\delta > 0$, by Lemma 5.8, there exists $\delta' > 0$ such that for every $x = \{(v_l, \eta_l)\}_l \in \mathcal{J}$ and for every $1 \leq k \leq p$ for which $\eta_k \notin N_{[0, k-1]}$,

$$f_k(x) < \delta' \quad \Rightarrow \quad g_k(x) < \delta.$$

This proves (2). □

Corollary 8. *Given $\delta > 0$ there is $\delta' > 0$ such that the following dichotomy holds. Let $[i + 1, j]$ be a time segment of a trajectory that contains no subinterval which is a δ -collinearity of that trajectory. Then for every $l \in [i + 1, j]$ either*

1. $\eta_l \in \{\eta_{i+1}, \dots, \eta_{l-1}\}$, or else
2. $\angle(v_l, V_{[i, l-1]}) \geq \delta'$.

Proof. This corollary is a reformulation of Proposition 11 (2). □

5.5. Expansivity lemmas. Recall the map $L_{v,\eta,v'}$ defined in (4). The first lemma says that this map has two singular values: $\lambda = 1$ with multiplicity $d - 1$, and $\lambda = |\langle v', \eta \rangle / \langle v, \eta \rangle|$ with multiplicity 1. See Figure 2.

Lemma 5.10. *Given coplanar unit vectors $v', v, \eta \in \mathbb{R}^d$, the composition $L_{v,\eta,v'} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies:*

- (a) $L_{v,\eta,v'}(v) = 0$,
- (b) $L_{v,\eta,v'}(x) = x$, for every $x \in \eta^\perp \cap v^\perp$,
- (c) $L_{v,\eta,v'}$ maps the line $v^\perp \cap W$ onto the line $v'^\perp \cap W$, where $W = \langle\langle v, \eta \rangle\rangle$, multiplying the vector's norms by the factor $|\langle v', \eta \rangle / \langle v, \eta \rangle|$.

Proof. Straightforward computation. \square

Remark 2. If v, η, v' are collinear vectors then $L_{v, \eta, v'} = \text{id}$ on v^\perp .

The remaining lemmas are abstract. Let V, V', V'' be Euclidean spaces of the same dimension, and $L : V \rightarrow V', L' : V' \rightarrow V''$ be linear isomorphisms.

Given $\sigma \geq 1$ and a subspace $E \subset V$, we say that L is a σ -*expansion on E* if $\|Lv\| \geq \sigma \|v\|$ for all $v \in E$, i.e., $\mathbf{m}(L|_E) \geq \sigma$. Given another linear subspace $H \subseteq V$ such that $E \subseteq H$ we say that L is a *relative σ -expansion on H w.r.t. E* if and only if the quotient map $\bar{L} : V/E \rightarrow V'/L(E)$ is a σ -expansion on H/E . Note that the quotient space V/E is an Euclidean space which can naturally be identified with E^\perp . Finally, we say that L is a σ -*expansion* to mean that L is a σ -expansion on its domain V .

If we do not need to specify the minimal rate of expansion we shall simply say that L is a uniform expansion on E , or that L is a relative uniform expansion on H w.r.t. E .

Lemma 5.11. *Given a linear subspace $H \subseteq V$, if*

1. *L is a σ -expansion on H , and*
2. *L is a relative σ -expansion on V w.r.t. H*

then L is a σ -expansion on V .

Proof. Follows immediately from the definition of σ -expansion and relative σ -expansion. \square

We will now derive some explicit formulas for the minimum expansion of compositions of linear expanding maps. For that purpose we introduce an exotic operation on the set $[0, 1]$ that plays a key role in these formulas.

$$a \oplus b := a + b - ab.$$

With it, $([0, 1], \oplus)$ becomes a commutative semigroup isomorphic to $([0, 1], \cdot)$. In fact, the map $\Psi : ([0, 1], \oplus) \rightarrow ([0, 1], \cdot)$, $\Psi(x) := 1 - x$, is a semigroup isomorphism. The numbers 0 and 1 are respectively the neutral and the absorbing elements of the semigroup $([0, 1], \oplus)$. An important property of this exotic operation is that for all $a, b \in [0, 1]$,

$$a \oplus b < 1 \Leftrightarrow a < 1 \text{ and } b < 1.$$

We will write $\oplus_n x := x \oplus \dots \oplus x$ for any \oplus -sum of n equal terms $x \in [0, 1]$.

The following lemmas use the notation introduced in subsection 2.1.

Lemma 5.12. *Let $L, L' : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be linear maps such that $\mathbf{m}(L) \geq 1$, $\mathbf{m}(L') \geq 1$, $\lambda = \|L\| > 1$ and $\lambda' = \|L'\| > 1$. If the sine of the angle between $\mathbf{v}(L^*)$ and $\mathbf{v}(L')$ is at least ε , then $L' \circ L$ has minimum expansion*

$$\mathbf{m}(L' \circ L) \geq \frac{1}{\sqrt{(1 - \varepsilon^2) \oplus \lambda^{-2} \oplus \lambda'^{-2}}} > 1.$$

Proof. Without loss of generality we can assume that the automorphisms $L, L' \in \text{GL}(\mathbb{R}^2)$ have two singular values, respectively $\{1, \lambda\}$ with $1 < \lambda$, and $\{1, \lambda'\}$ with $1 < \lambda'$. Otherwise simply normalize L and L' dividing them by the minimum expansion. Hence $\|L^{-1}\| = 1 = \|(L')^{-1}\|$. These maps have gap ratios $\|L^{-1}\|/\mathbf{m}(L^{-1}) = \lambda$ and $\|(L')^{-1}\|/\mathbf{m}((L')^{-1}) = \lambda'$. The conclusion of this lemma will follow from [3, Proposition 2.23] applied to the composition of linear maps $L^{-1} \circ (L')^{-1}$. The quantity $\alpha((L')^{-1}, L^{-1})$ in that proposition is the cosine of the

angle between the most expanding directions of the linear maps $((L')^{-1})^*$ and L^{-1} which coincide with the least expanding directions of L' and L^* , respectively. Since these directions are orthogonal to $\mathbf{v}(L')$ and $\mathbf{v}(L^*)$ we have

$$\alpha((L')^{-1}, L^{-1})^2 = \cos^2 \angle(\mathbf{v}(L'), \mathbf{v}(L^*)) \leq 1 - \varepsilon^2.$$

Thus by [3, Proposition 2.23]

$$\begin{aligned} \|(L' \circ L)^{-1}\| &= \|L^{-1} \circ (L')^{-1}\| = \frac{\|L^{-1} \circ (L')^{-1}\|}{\|L^{-1}\| \|(L')^{-1}\|} \\ &\leq \beta((L')^{-1}, L^{-1}) \leq \sqrt{(1 - \varepsilon^2) \oplus \lambda^{-2} \oplus \lambda'^{-2}}. \end{aligned}$$

Since $\mathbf{m}(L' \circ L) = 1/\|(L' \circ L)^{-1}\|$, the claim follows. \square

Lemma 5.13. *Consider linear maps $L: V \rightarrow V'$ and $L': V' \rightarrow V''$ between Euclidean spaces of dimension d . Given $1 \leq k < d$ assume that*

1. $\mathbf{m}(L) \geq 1$ and $\lambda = s_k(L) > 1$,
2. $\lambda' = \|L'\| > 1 = s_2(L')$.

If $\sin \angle(\mathbf{v}(L'), \mathbf{v}_\lambda^\geq(L^*)) \geq \varepsilon$ then

$$s_{k+1}(L' \circ L) \geq \frac{1}{\sqrt{(1 - \varepsilon^2) \oplus \lambda^{-2} \oplus \lambda'^{-2}}} > 1.$$

Proof. We can assume that $V = V' = V'' = \mathbb{R}^d$. Consider the singular value decomposition $L = U D V$, where U and V are orthogonal matrices, and $D = (D_{ij})$ is the diagonal matrix with diagonal entries $D_{ii} = s_i(L)$ for $i = 1, \dots, d$. We can factor D as a product $D = \hat{D} D_0$ of two diagonal matrices: $\hat{D} = \begin{bmatrix} \lambda I_k & 0 \\ 0 & I_{d-k} \end{bmatrix}$ and D_0 with diagonal entries $D_{ii}^{(0)} = D_{ii}/\hat{D}_{ii} \geq 1$. Set $\hat{L} = U \hat{D}$ and $L_0 = D_0 V$, so that $L = \hat{L} \circ L_0$. The linear mapping \hat{L} has singular values

$$s_1(\hat{L}) = \dots = s_k(\hat{L}) = \lambda > 1 = s_{k+1}(\hat{L}) = \dots = s_d(\hat{L}),$$

while $\mathbf{m}(L_0) \geq 1$. Hence $s_{k+1}(L' \circ L) \geq s_{k+1}(L' \circ \hat{L})$. To simplify the geometry we assume from now on that $L = \hat{L}$.

Take a unit vector $v' \in \mathbf{v}(L')$. By assumption $v' \notin \mathbf{v}_\lambda^\geq(L^*)$. Let v'_0 denote the orthogonal projection of v' onto $\mathbf{v}_\lambda^\geq(L^*)$. These two vectors span a plane $P_0 := \langle v', v'_0 \rangle$. Define also the subspaces $E_0 := \mathbf{v}_\lambda^\geq(L^*) \cap (v'_0)^\perp$ and $G_0 := \mathbf{v}_\lambda^\geq(L^*)^\perp \cap (v')^\perp$. These three subspaces determine an orthogonal decomposition $\mathbb{R}^d = P_0 \oplus E_0 \oplus G_0$. Because the mappings $L: \mathbf{v}_\lambda^\geq(L) \rightarrow \mathbf{v}_\lambda^\geq(L^*)$ and $L: \mathbf{v}_\lambda^\geq(L)^\perp \rightarrow \mathbf{v}_\lambda^\geq(L^*)^\perp$ are both conformal, it follows that $\mathbb{R}^d = P_- \oplus E_- \oplus G_-$ is an orthogonal decomposition, where $P_- := L^{-1}P_0$, $E_- := L^{-1}E_0$ and $G_- := L^{-1}G_0$. In fact, since $P_0 \cap \mathbf{v}_\lambda^\geq(L^*) \perp E_0$ and $L: \mathbf{v}_\lambda^\geq(L) \rightarrow \mathbf{v}_\lambda^\geq(L^*)$ is conformal their pre-images are also orthogonal, $P_- \cap \mathbf{v}_\lambda^\geq(L) \perp E_-$. Similarly, since $\mathbf{v}_\lambda^\geq(L^*)^\perp \cap P_0 \perp G_0$ and $L: \mathbf{v}_\lambda^\geq(L)^\perp \rightarrow \mathbf{v}_\lambda^\geq(L^*)^\perp$ is conformal their pre-images are also orthogonal, $\mathbf{v}_\lambda^\geq(L)^\perp \cap P_- \perp G_-$. Define now $P_+ := L'P_0$, $E_+ := L'E_0$ and $G_+ := L'G_0$. Because $\mathbf{m}(L') \geq 1$ with $\mathbf{v}(L') \subset P_0$ and $\mathbf{v}((L')^*) \subset P_+$, it follows that $\mathbb{R}^d = P_+ \oplus E_+ \oplus G_+$ is also an orthogonal decomposition. Therefore the singular values of $L' \circ L$ are the singular values of the restricted compositions $L'|_{P_0} \circ L|_{P_-}$, $L'|_{E_0} \circ L|_{E_-}$ and $L'|_{G_0} \circ L|_{G_-}$. Applying Lemma 5.12 to the linear maps $L'|_{P_0}$ and $L|_{P_-}$ we see that $L'|_{P_0} \circ L|_{P_-}$ has minimum expansion $\beta := ((1 - \varepsilon^2) \oplus \lambda^{-2} \oplus \lambda'^{-2})^{-1/2}$. The map $L'|_{E_0}: E_0 \rightarrow E_+$ is an isometry while $L|_{E_-}: E_- \rightarrow E_0$ is λ -conformal. Therefore the second composition

has a unique singular value λ with multiplicity $k-1 = \dim E_0$. Note that $(1-\varepsilon^2) \oplus \lambda^{-2} \oplus \lambda'^{-2} \geq \lambda^{-2}$ which implies that $\lambda \geq \beta$. Finally notice that $L|_{G_-} : G_- \rightarrow G_0$ and $L'|_{G_0} : G_0 \rightarrow G_+$ are isometries. Hence 1 is the only the singular value of the third composition. Since $\dim(P_- \oplus E_-) = \dim(P_0 \oplus E_0) = k+1$, this proves that $s_{k+1}(L' \circ L) \geq \min\{\beta, \lambda\} = \beta$. \square

The next lemma is designed to be applied to a sequence of linear maps $L_{v_{i-1}, \eta_i, v_i} : v_{i-1}^\perp \rightarrow v_i^\perp$ associated with an orbit segment of the billiard map $\Phi_{f,P}$. Compare assumptions (1)-(2) of Lemma 5.14 with the conclusions of Lemma 5.10 and Remark 2.

Lemma 5.14. *Given $\varepsilon > 0$ and $\lambda > 1$ consider unit vectors $\{v_0, v_1, \dots, v_n\}$ in \mathbb{R}^d and a family of linear maps $L_i : v_{i-1}^\perp \rightarrow v_i^\perp$, $1 \leq i \leq n$, such that:*

1. *if $u \in v_{i-1}^\perp \cap v_i^\perp$ then $L_i u = u$,*
2. *if $\{v_{i-1}, v_i\}$ are linearly independent and $u \in v_{i-1}^\perp \cap \langle v_{i-1}, v_i \rangle$ then $L_i u \in v_i^\perp \cap \langle v_{i-1}, v_i \rangle$ and $\|L_i u\| \geq \lambda \|u\|$.*
3. *for each $i = 1, \dots, n$, either $v_i \in \langle v_0, \dots, v_{i-1} \rangle$ or else*

$$\sin \angle(v_i, \langle v_0, \dots, v_{i-1} \rangle) \geq \varepsilon.$$

Then the composition map $L^{(n)} := L_n \circ \dots \circ L_0 : v_0^\perp \rightarrow v_n^\perp$ satisfies $\|L^{(n)} u\| \geq \sigma \|u\|$ for all $u \in v_0^\perp \cap \langle v_0, \dots, v_n \rangle$, where

$$\sigma = 1/\sqrt{(\oplus_{d-1}(1-\varepsilon^2)) \oplus (\oplus_d \lambda^{-2})} > 1.$$

Proof. For each $i = 1, \dots, n$ define, $L^{(i)} := L_i \circ \dots \circ L_0 : v_0^\perp \rightarrow v_i^\perp$ and $W_i := \langle v_0, \dots, v_i \rangle$. Since $W_i^\perp = v_0^\perp \cap \dots \cap v_i^\perp \subset \cap_{j=0}^i v_j^\perp$, and every vector $u \in W_i^\perp$ is fixed by all L_j with $0 \leq j \leq i$, we have $L^{(i)} u = u$ for every $u \in W_i^\perp$.

We can delete from $\{v_0, v_1, \dots, v_n\}$ all vectors v_i such that $\{v_{i-1}, v_i\}$ is linearly dependent, which by item (1) correspond to maps $L_i = \text{id}$, and in this way assume that for all $i = 1, \dots, n$, the vectors $\{v_{i-1}, v_i\}$ are linearly independent and $\|L_i\| \geq \lambda$.

Because W_i^\perp is a singular subspace it follows that $L^{(i)}(v_0^\perp \cap W_i) = v_i^\perp \cap W_i$. We claim that $L^{(i)} : v_0^\perp \cap W_i \rightarrow v_i^\perp \cap W_i$ is a σ_i -expansion, where $k_i := \dim(W_i) - 1$ and

$$\sigma_i := 1/\sqrt{(\oplus_{k_i-1}(1-\varepsilon^2)) \oplus (\oplus_{k_i} \lambda^{-2})}.$$

The proof of this claim goes by induction in i , applying Lemma 5.13.

The claim holds for $i = 1$ with $k_1 = 1$ and $\sigma_1 = \lambda$.

Assume now (induction hypothesis) that $L^{(i-1)}$ is a σ_{i-1} -expansion on $v_0^\perp \cap W_{i-1}$. We know by assumption (3) that either $v_i \in W_{i-1}$ or else $\sin(\angle(v_i, W_{i-1})) \geq \varepsilon$.

- Assume first that $\sin(\angle(v_i, W_{i-1})) \geq \varepsilon$.

We have $\mathfrak{v}_{\sigma_{i-1}}^{\geq}((L^{(i-1)})^*) = v_{i-1}^\perp \cap W_{i-1}$ and $\mathfrak{v}(L_i) = v_{i-1}^\perp \cap \langle v_{i-1}, v_i \rangle$. To apply Proposition 5.13 we need to check that

$$\sin \angle(v_{i-1}^\perp \cap W_{i-1}, v_{i-1}^\perp \cap \langle v_{i-1}, v_i \rangle) \geq \varepsilon. \quad (5)$$

Let v_i^0 denote the unit vector obtained normalizing the orthogonal projection of v_i onto v_{i-1}^\perp , so that

$$v_i = (\cos \alpha) v_i^0 + (\sin \alpha) v_{i-1},$$

with $\|v_i^0\| = 1$, $\langle v_i^0, v_{i-1} \rangle = 0$ and where $\alpha = \angle(v_i^0, v_i)$. Note that $v_{i-1}^\perp \cap \langle v_{i-1}, v_i \rangle$ is the line spanned by v_i^0 . Take any unit vector $v \in v_{i-1}^\perp \cap W_{i-1}$ and let us prove that $\sin(\angle(v_i^0, v)) \geq \varepsilon$. This will establish (5). Define

$$v' := (\cos \alpha) v + (\sin \alpha) v_{i-1}$$

which is a unit vector in W_{i-1} . We can assume that $\langle v_i^0, v \rangle \geq 0$ for otherwise the angle $\angle(v_i^0, v)$ that we want to minimize would be obtuse. Using the previous expressions for v_i and v' we have

$$\langle v_i, v' \rangle = (\cos^2 \alpha) \langle v_i^0, v \rangle + \sin^2 \alpha .$$

Since this expresses $\langle v_i, v' \rangle$ as a convex combination between $\langle v_i^0, v \rangle$ and the number 1, it follows that

$$\cos(\angle(v_i, v')) = \langle v_i, v' \rangle \geq \langle v_i^0, v \rangle = \cos(\angle(v_i^0, v))$$

which implies that

$$\sin(\angle(v_i^0, v)) \geq \sin(\angle(v_i, v')) \geq \varepsilon .$$

This proves (5) and shows the assumptions of Proposition 5.13 are met. From this proposition, we get that on the linear subspace $v_0^\perp \cap W_i$, of dimension $k_i = k_{i-1} + 1$, the linear map $L^{(i)} = L_i \circ L^{(i-1)}$ is a $\hat{\sigma}_i$ -expansion where

$$\hat{\sigma}_i := ((1 - \varepsilon^2) \oplus \sigma_{i-1}^{-2} \oplus \lambda^{-2})^{-1/2} \geq \sigma_i .$$

■ The case $v_i \in W_{i-1}$ is somewhat simpler. We have $W_i = W_{i-1}$, $k_i = k_{i-1}$, and $\sigma_i = \sigma_{i-1}$. Hence, since $\mathfrak{m}(L_i) \geq 1$ by induction hypothesis the map $L^{(i)} = L_i \circ L^{(i-1)}$ is a σ_i -expansion on $v_0^\perp \cap W_i$. \square

5.6. Proof of Theorem 5.3. In this subsection we relate collinearities with expansion of the velocity tangent flow, and then prove Theorem 5.3.

Recall that we are assuming that P is ε -spanning.

Proposition 12. *There exists $\sigma > 1$, depending only on d, f and ε , such that given a collinearity $[i, j_0]$ of some trajectory, for all $j > j_0$, the velocity flow $L_{[i, j]}$ is a relative σ -expansion on $v_i^\perp \cap V_{[i, j]}$ w.r.t. $v_i^\perp \cap V_{[i, j_0]}$.*

Proof. Assume $\{(v_l, \eta_l)\}_l$ is a trajectory with collinearity $[i, j_0]$. Because P is ε -spanning, for all $j > j_0$ such that $\eta_j \notin N_{[i, j-1]}$ we have $\angle(\eta_j, N_{[i, j-1]}) \geq \varepsilon$.

Notice that $V_{[i, j-1]} = N_{[i, j-1]}$, for all $j > j_0$, and by Lemma 5.4, we have $v_j = \alpha_j \eta_j + \beta_j v_{j-1}$ with $\alpha_j \geq \cos(\frac{\pi}{2} \lambda(f)) > 0$. Hence there is some $0 < \varepsilon' < \varepsilon$ depending on ε and on $\lambda(f)$, such that for all $j > j_0$ with $\eta_j \notin N_{[i, j-1]}$,

$$\angle(v_j, V_{[i, j-1]}) \geq \varepsilon' .$$

Consider the set of ‘new normal’ times

$$J := \{j_0 < l \leq j : \eta_l \notin N_{[i, l-1]}\}$$

and the corresponding velocity subspace

$$V_J := \langle\langle v_l : l \in J \rangle\rangle,$$

so that $V_{[i, j]} = V_{[i, j_0]} \oplus V_J$.

By Lemma 5.14 there exists $\sigma > 1$, depending only on d, f and ε such that $L_{[j_0, j]}$ is a σ -expansion on $v_{j_0}^\perp \cap V_{[j_0, j]}$. In particular, $L_{[j_0, j]}$ is also a σ -expansion on $v_{j_0}^\perp \cap V_J \subseteq v_{j_0}^\perp \cap V_{[j_0, j]}$. By Lemma 2.5 we have

$$\angle_{\min}(V_{[i, j_0]}, V_J) \geq \arcsin(\sin^d(\varepsilon')) =: \tilde{\varepsilon} .$$

Hence there exists $1 < \tilde{\sigma} < \sigma$ depending only on $\tilde{\varepsilon}$ and σ such that $L_{[j_0, j]}$ is a $\tilde{\sigma}$ -expansion on $(V_{[i, j_0]})^\perp \cap V_{[j_0, j]}$. This implies that $L_{[i, j]}$ is a relative $\tilde{\sigma}$ -expansion on $v_i^\perp \cap V_{[i, j]}$ w.r.t. $v_i^\perp \cap V_{[i, j_0]}$. \square

Corollary 9. *Given the constant $\sigma > 1$ in Proposition 12, and $1 < \sigma' < \sigma$, there is $\delta > 0$ such that for every trajectory $\{(v_l, \eta_l)\}_l$, if $[i, j_0]$ is a δ -collinearity then for all $j > j_0$, the velocity flow $L_{[i, j]}$ is a relative σ' -expansion on $v_i^\perp \cap V_{[i, j]}$ w.r.t. $v_i^\perp \cap V_{[i, j_0]}$.*

Proof. This follows from Proposition 12 with a continuity argument like the one used in the proof of Proposition 10. \square

Proposition 13. *Given $\delta > 0$ there exists $\sigma > 1$, depending on d, f, ε and δ , such that if a time interval $[i + 1, j]$ of some trajectory contains no subinterval which is a δ -collinearity then $L_{[i, j]}$ is a σ -expansion on $v_i^\perp \cap V_{[i, j]}$.*

Proof. Let $[i, j]$ be a time interval such that $[i + 1, j]$ contains no subinterval which is itself a δ -collinearity. By Corollary 8, there is $\delta' > 0$ such that for every $l \in [i + 1, j]$ either $\eta_l \in \{\eta_{i+1}, \dots, \eta_{l-1}\}$, or else

$$\angle(v_l, V_{[i, l-1]}) \geq \delta' .$$

Thus by Lemma 5.14 $L_{[i, j]}$ is a uniform expansion on $v_i^\perp \cap V_{[i, j]}$. \square

Now we can prove Theorem 5.3.

Proof of Theorem 5.3. Take the constant $\sigma > 1$ given in Proposition 12. Set $\sigma' = \frac{1}{2} + \frac{1}{2}\sigma \in (1, \sigma)$, and pick $\delta = \delta(\sigma') > 0$ as provided by Corollary 9. Fix the constant $\sigma'' = \sigma(\delta) > 1$ given by Proposition 13 and set $\sigma_0 = \min\{\sigma', \sigma''\}$.

Fix some integer $k \geq 0$ and let $\{(v_j, \eta_j)\}_{j \in \mathbb{N}_0}$ be a trajectory. We consider three cases:

- If $[0, k]$ contains no δ -collinearity, by Proposition 13 $L_{[0, k]}$ is a σ'' -expansion on $v_0^\perp \cap V_{[0, k]}$. But since any trajectory is generating on $[0, k]$, we have $v_0^\perp = v_0^\perp \cap V_{[0, k]}$, which proves that $L_{[0, k]}$ is a σ'' -expansion. Finally, because $L_{[k, 2k]}$ is non contracting, $L_{[0, 2k]} = L_{[k, 2k]} \circ L_{[0, k]}$ is also a σ'' -expansion.
- If $[0, k]$ contains a δ -collinearity $[i, j] \subseteq [0, k]$, we can assume it is minimal, in the sense that $[i, j]$ contains no proper subinterval which is itself a δ -collinearity. Consider first the case $j \geq i + 1$. By Proposition 13, $L_{[i, j]}$ is a σ'' -expansion on $v_i^\perp \cap V_{[i, j]}$. Because $L_{[i, 2k]} = L_{[j, 2k]} \circ L_{[i, j]}$, and $L_{[j, 2k]}$ is non contracting, the map $L_{[i, 2k]}$ is also a σ'' -expansion on $v_i^\perp \cap V_{[i, j]}$. Remark that since $i \leq k$, the trajectory is generating on $[i, 2k]$, and hence $v_i^\perp = v_i^\perp \cap V_{[i, 2k]}$. Hence by Proposition 9, $L_{[i, 2k]}$ is a relative σ' -expansion on v_i^\perp w.r.t. $v_i^\perp \cap V_{[i, j]}$. Thus by Lemma 5.11, $L_{[i, 2k]}$ is a σ_0 -expansion, which implies so is $L_{[0, 2k]}$.
- Finally we consider the case $[0, k]$ contains δ -collinearities, but the minimal ones have length zero, say $\{i\} \subset [0, k]$ is a δ -collinearity. In this case we have $\angle(v_i, \eta_i) < \delta$, and the proof is somehow simpler. By Lemma 2.2

$$\begin{aligned} \angle(V_{[i, j-1]}, N_{[i, j-1]}) &= \angle(\langle v_i \rangle + N_{[i+1, j-1]}, \langle \eta_i \rangle + N_{[i+1, j-1]}) \\ &\leq \arcsin \left(\frac{\sin \delta}{\sin \varepsilon} \right) =: \hat{\delta} . \end{aligned}$$

On the other hand, because $v_j = \alpha_j \eta_j + \beta_j v_{j-1}$ with $\alpha_j \geq c$ and $c = \cos(\frac{\pi}{2} \lambda(f))$, whenever $\eta_j \notin \{\eta_i, \dots, \eta_{j-1}\}$ we have

$$\begin{aligned} \angle(v_j, V_{[i,j-1]}) &\geq \frac{c}{2} \angle(\eta_j, V_{[i,j-1]}) \\ &\geq \frac{c}{2} \angle(\eta_j, N_{[i,j-1]}) - \frac{c\hat{\delta}}{2} \\ &\geq \frac{c}{2} (\varepsilon - \hat{\delta}) \geq \frac{c\varepsilon}{4}, \end{aligned}$$

provided δ is small enough. Thus, using Lemma 5.10 we get by induction that $L_{[i,i+k]}$ is a uniform expansion, and as before that $L_{[0,2k]}$ is also a uniform expansion.

Therefore, $L_{[0,2k]}$ is a σ_0 -expansion in all cases. \square

6. Proof of the Main Statements. Let P be a spanning polytope and f a contracting reflection law. Denote by $\Phi = \Phi_{f,P}: D \rightarrow D$ the billiard map for P and f .

Proof of Theorem 2.10. Let $x = (p, v) \in D$ be any k -generating point. We can identify the tangent space $T_x M$ with $v^\perp \times v^\perp$ using the Jacobi coordinates (J, J') . From the proof of Proposition 4, the subbundle $E^{cu}(x)$ in the coordinates (J, J') is $\{(J, J') \in v^\perp \times v^\perp: J' = 0\}$. Moreover, by Theorem 5.3, there exists $\sigma > 1$ depending only on P and f such that

$$\|D\Phi^{2k}(x)(J, 0)\| = \|L_{[0,2k]}(J)\| \geq \sigma \|J\|, \quad \forall J \in v^\perp.$$

This uniform minimum growth expansion on E^{cu} proves the theorem. \square

Proof of Theorem 2.12. Assume that (Φ, μ) is ergodic and $\int T d\mu < +\infty$. First note that, by Proposition 4,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \|D\Phi^n(x)|_{E^s}\| = \log \lambda(f) < 0$$

for every $x \in D$. Consider now the partition $\{A_n = T^{-1}\{n\}\}_{n \in \mathbb{N}}$ of D , and define the measurable function $\tilde{T}: D \rightarrow \mathbb{N}$, $\tilde{T} = n$ on $A'_n := \Phi(A_n)$. This function satisfies

$$T(\Phi^{-\tilde{T}(x)}(x)) = \tilde{T}(x) \quad \text{for all } x \in D.$$

Moreover $\int \tilde{T} d\mu = \int T d\mu < +\infty$. From Theorem 2.10 we have

$$\|D\Phi^{-2\tilde{T}(x)}(x)|_{E^{cu}}\| \leq 1/\sigma \quad \text{for all } x \in D.$$

Define recursively the following sequence of backward iterates and stopping times

$$\begin{cases} x_0 &= x \\ t_0 &= 2\tilde{T}(x_0) \end{cases} \quad \begin{cases} x_{j+1} &= \Phi^{-t_j}(x_j) \\ t_{j+1} &= 2\tilde{T}(x_{j+1}) \end{cases}.$$

Let us write $\tau_n = \sum_{j=0}^{n-1} t_j$. Since $t_j \geq 2d$ for all j , this sequence tends to $+\infty$, and we have

$$\begin{aligned} -\frac{1}{\tau_n} \log \|D\Phi^{-\tau_n}(x)|_{E^{cu}}\| &\geq -\frac{1}{\sum_{j=0}^{n-1} t_j} \sum_{j=0}^{n-1} \log \|D\Phi^{-t_j}(x_j)|_{E^{cu}}\| \\ &\geq -\frac{n}{\sum_{j=0}^{n-1} t_j} \log \sigma^{-1} = \frac{\log \sigma}{\frac{1}{n} \sum_{j=0}^{n-1} \tilde{T}(x_{j+1})}. \end{aligned}$$

Thus, by Birkhoff's ergodic theorem, for μ -almost every $x \in D$,

$$\limsup_{n \rightarrow +\infty} -\frac{1}{n} \log \|D\Phi^{-n}(x)|_{E^{cu}}\| \geq \frac{\log \sigma}{\int \tilde{T} d\mu} > 0.$$

By Kingman's ergodic theorem, the above lim sup is actually a limit. Thus,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \|D\Phi^{-n}(x)|_{E^{cu}}\| < 0$$

for μ -almost every $x \in D$. This proves that μ is a hyperbolic measure. \square

Proof of Theorem 2.13. Assume that $\Lambda \subset D$ is Φ -invariant. By Proposition 4, Φ is uniformly partially hyperbolic on Λ . Moreover, it follows from Theorem 2.10 that there exists a constant $C > 0$ depending only on P and f such that

$$\|D\Phi^{-n}(x)|_{E^{cu}(x)}\| \leq C \left(\frac{1}{\sigma}\right)^{\frac{n}{2k}}$$

for every $x \in \Lambda$ that is k -generating. Since the escaping time function T is bounded on Λ , every $x \in \Lambda$ is τ -generating where $\tau := \sup_{x \in \Lambda} T(x)$. So the expansion rate can be made uniform and equal to $\sigma^{1/\tau} > 1$. This shows that Φ is uniformly hyperbolic on Λ . \square

Proof of Corollary 3. Suppose P is in general position, in particular P is a spanning polytope. By Corollary 5 there exists a positive constant $\lambda_0 = \lambda_0(P)$ such that the escaping time function T is bounded on D . The claim follows by Theorem 2.13. \square

Proof of Corollary 4. Suppose P is an obtuse polytope in general position, in particular P is a spanning polytope. By Corollary 6 the escaping time function T is bounded on D . The claim follows by Theorem 2.13. \square

7. Examples. In this section we study in detail the contracting billiard on a family of 3-dimensional simplexes, illustrating the applicability of our main theorems.

Let $\{e_1, \dots, e_{d+1}\}$ be the canonical basis of \mathbb{R}^{d+1} . Given $d \geq 2$, we denote by Δ_h^d the d -simplex in \mathbb{R}^{d+1} defined as the convex hull of the vertexes $v_j = e_j$ for $1 \leq j \leq d$ and $v_{d+1} = \frac{1-h}{d} \sum_{j=1}^d e_j + h e_{d+1}$. For any set of d facets of Δ_h^d ($(d-1)$ -dimensional faces), their normals are linearly independent. Therefore, Δ_h^d is in general position according to Definition 3.1 and it is spanning according to Definition 2.9.

7.1. Near conservative billiards. We firstly consider contracting reflection laws close to the specular one. It will be shown that the escaping time is uniformly bounded, by computing the barycentric angles of Δ_h^d .

The simplex Δ_h^d has $d+1$ barycentric angles, one for each vertex. By symmetry all barycentric angles at base vertexes v_j , with $1 \leq j \leq d$, are the same. Denote the barycentric angle at the base vertexes by $\phi_1 = \phi_1(h)$ and the barycentric angle at v_{d+1} by $\phi_2 = \phi_2(h)$. Define

$$\lambda_0(h) := 1 - 4 \min\{\phi_1(h), \phi_2(h)\}/\pi. \quad (6)$$

Proposition 14. *For every $h > 0$ and every contracting reflection law f satisfying $\lambda_0(h) < \lambda(f) < 1$ the billiard map Φ_{f, Δ_h^d} is uniformly hyperbolic.*

Proof. Notice that Δ_h^d is in general position and spanning. Moreover, by Theorem 4.5, if $2\phi_i > \pi/2 - f(\pi/2) > \pi/2 - \lambda(f)\pi/2$ for $i = 1, 2$ then the polyhedral cones have bounded escaping time. This is the case when $\lambda(f) > 1 - 4 \min\{\phi_1, \phi_2\}/\pi$. Thus, by Theorem 2.13, Φ_{f, Δ_h^d} is uniformly hyperbolic. \square

Figure 3 shows the graphs of the $\lambda_0(h)$ defined in (6) for $d = 3, 4$ and 5. The shaded regions bounded between these graphs and $\lambda = 1$ are called admissible regions. This figure shows that the admissible regions decrease as the dimension increases. The bottom tips of these admissible regions correspond to the heights h of the regular d -simplexes.

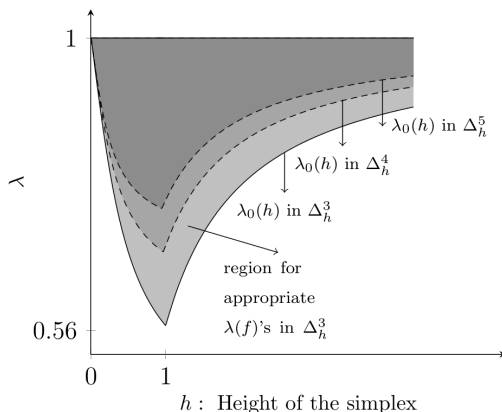


FIGURE 3. Parameter regions with uniform bounded escaping time

7.2. Near slap billiards. Here we consider the situation when $\lambda(f) \approx 0$ for a given contracting reflection law f . These reflection laws are called strongly contracting (see [5]). In this context the dynamics may lose uniformity due to unbounded escaping times. To any strongly contracting billiard we can associate a degenerate billiard map called the ‘slap map’ corresponding to $f = 0$, where reflections are always orthogonal to the faces. When h is small enough the slap map has a trapping region, called a *chamber*, away from acute wedges. Hence the escaping time is bounded on the chamber. This concept generalizes the notion of chamber introduced in [6].

For simplicity we will assume $d = 3$.

Proposition 15. *For any $h \in (0, 1/2)$ there exists $\lambda_0 = \lambda_0(h) > 0$ such that for every contracting reflection law f satisfying $\lambda(f) < \lambda_0(h)$, the billiard map Φ_{f, Δ_h^3} is uniformly hyperbolic.*

Proof. Firstly, let us assume that $\lambda(f) = 0$. This means that the billiard particle always reflects orthogonally to each face of the polytope. Since after the first iterate the angle is zero, we can reduce Φ_{f, Δ_h^3} to a multi-valued map $\Phi_0 : \Delta_h^3 \rightarrow \Delta_h^3$ (skeleton points may have more than one image). Let $A_i, i = 1, \dots, 4$ denote the vertices of the simplex Δ_h^3 with A_4 being the top vertex. The triangle $A_1A_2A_3$ is called the base of the simplex (see Figure 4). We show that there is a set \mathcal{V} on the base of the simplex which is invariant by Φ_0^2 . Let C_0 denote the center of $A_1A_2A_3$, i.e., the point mapped by Φ_0 to the top vertex of the simplex. Then, the base triangle is partitioned into three triangles, namely $A_1A_2C_0, A_1A_3C_0$ and $A_2A_3C_0$. Since $\Phi_0(C_0)$ is the intersection of the three faces, it has three distinct images by Φ_0 . A simple calculation shows that when $h < h_0$ for some $h_0 > 0$,

these images belong to the base of the simplex. Denote them by C_1 , C_2 and C_3 . The image of triangles $A_1A_2C_0$, $A_2A_3C_0$ and $A_3A_1C_0$ under Φ_0^2 are respectively triangles $A_1A_2C_3$, $A_2A_3C_1$ and $A_3A_1C_2$. Therefore Φ_0^2 maps the triangle $A_1A_2A_3$ to itself.

Now we construct an hexagon $\mathcal{H} = M_1M_2M_3M_4M_5M_6$ as follows (see Figure 4): the point M_1 is the intersection of A_1C_2 with the perpendicular to A_1C_0 through C_1 . Likewise, M_2 is the intersection of A_2C_1 with the perpendicular to A_2C_0 through C_2 . The other M_j 's are similarly defined. The hexagon \mathcal{H} is the union of three pentagons whose images under Φ_0^2 are in the hexagon \mathcal{H} . On the right of Figure 4, we can see the image of the pentagon $\mathcal{P} = C_0C_2M_2M_1C_1$, $\Phi_0^2(\mathcal{P}) = C'_0C'_2M'_2M'_1C'_1$.

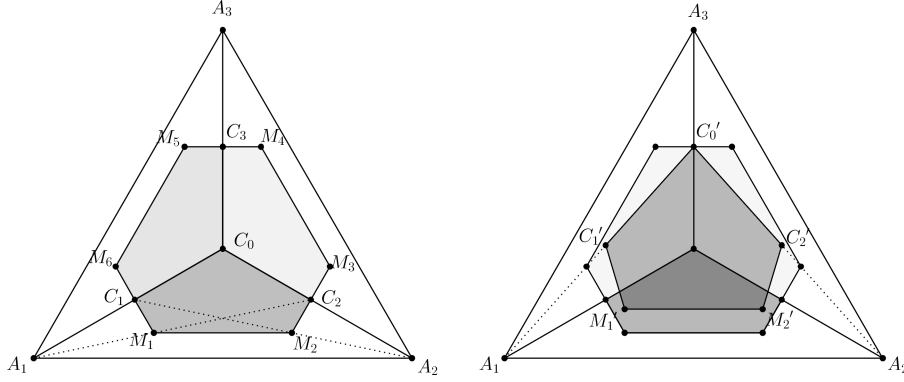


FIGURE 4.

Moreover, the intersection of the pentagon $\Phi_0^2(\mathcal{P})$ with the boundary of \mathcal{H} is just the point $C'_0 = C_3$. Hence, for some small enough neighborhood \mathcal{V} of \mathcal{H} on the base triangle $A_1A_2A_3$ we have $\Phi_0^2(\overline{\mathcal{V}}) \subset \mathcal{V}$.

It is easy to see that every orbit of Φ_0 eventually enters \mathcal{V} . In fact, every orbit starting near the wedges of the simplex will escape by a zig-zag movement and enter $\mathcal{H} \subset \mathcal{V}$. Since \mathcal{V} is away from the wedges, the escaping time $T(x)$ for $x \in \mathcal{V}$ is uniformly bounded.

Denote by η the inward normal to the base of the simplex and given $\lambda_0 > 0$ define

$$\mathbb{S}_{\eta, \lambda_0}^+ = \left\{ v \in \mathbb{S}_\eta^+ : \langle v, \eta \rangle > \cos\left(\lambda_0 \frac{\pi}{2}\right) \right\}.$$

Also define $\Lambda_{\lambda_0} := \left(\mathcal{V} \times \mathbb{S}_{\eta, \lambda_0}^+ \right) \cap M^+$. Then, by continuity, there exists $\lambda_0 = \lambda_0(h) > 0$ such that for every contracting reflection law satisfying $\lambda(f) < \lambda_0$ we have

$$\Phi_{f, \Delta_h^3}^2(\Lambda_{\lambda_0}) \subset \Lambda_{\lambda_0} \quad \text{and} \quad D = \bigcap_{n \geq 0} \Phi_{f, \Delta_h^3}^n(\Lambda_{\lambda_0}).$$

The previous equality follows from the fact that for each $x \in M^+$ there exists $n \geq 0$ such that $\Phi_{f, \Delta_h^3}^n(x) \in \Lambda_{\lambda_0}$. Since the escaping time is bounded on Λ_{λ_0} , it is also bounded on D . Therefore, the proposition follows from Theorem 2.13. \square

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