A Note on Frobenius' theorems

1 Some Multilinear Algebra

Let V be a real vector space with finite dimension $n = \dim V$.

Proposition 1. *Given a basis* $\{e_1, \ldots, e_n\}$ *of* V *let* $\{\lambda^1, \ldots, \lambda^n\} \subset V^*$ *be its dual basis, which is characterized by* $\lambda^i(e_j) = \delta_{ij}$ *for* $1 \leq i, j \leq n$. *Given* $\xi \in \bigwedge^k(V)$,

$$\xi = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \xi(e_{i_1}, \ldots, e_{i_k})\, \lambda^{i_1} \wedge \cdots \wedge \lambda^{i_k}.$$

Proof. Exercise. Proved in class. □

Proposition 2. *Given linear forms* $\theta_1, \ldots, \theta_\ell \in V^*$ *and* $v \in V$,

$$i_v(\theta_1 \wedge \cdots \wedge \theta_\ell) = \sum_{j=0}^{\ell} (-1)^{j-1}\, \theta_j(v)\, \theta_1 \wedge \cdots \wedge \widehat{\theta_j} \wedge \cdots \wedge \theta_\ell.$$

Proof. See [3, Proposition 20.7]. □

Proposition 3. *Given* $\xi \in \bigwedge^k(V)$, $\eta \in \bigwedge^\ell(V)$ *and* $v \in V$,

$$i_v(\xi \wedge \eta) = i_v(\xi) \wedge \eta + (-1)^k \xi \wedge i_v \eta.$$

Proof. See [3, Proposition 20.8(ii)]. See also Exercise 5.13. □

Definition 1. *Given a* k*-form* $\xi \in \bigwedge^k(V)$ *we define its kernel to be*

$$\operatorname{Ker}(\xi) := \{v \in V : i_v \xi = 0\}.$$

This kernel is a linear subspace because the map $V \ni v \mapsto i_v \xi \in \bigwedge^{k-1}(V)$ *is linear. This concept (of kernel of* ξ*) is not a standard definition in the bibliography.*

Definition 2. *Given* $k \geq 1$, *a* k*-form* $\xi \in \bigwedge^k(V)$ *is said to be* decomposable *if there exist* 1*-forms* $\theta^1, \ldots, \theta^k \in V^* = \bigwedge^1(V)$ *such that* $\xi = \theta^1 \wedge \cdots \wedge \theta^k$.

It follows from this definition that every 1-form is decomposable.

Proposition 4. *Given $\theta^1, \ldots, \theta^k \in V^*$ such that $\xi = \theta^1 \wedge \cdots \wedge \theta^k \neq 0$,*

$$\mathrm{Ker}(\xi) = \bigcap_{j=1}^{k} \mathrm{Ker}(\theta^j) \ \textit{ has dimension } \ n-k.$$

Proof. Since $\xi = \theta^1, \ldots, \theta^k \neq 0$, the 1-forms $\{\theta^j \colon 0 \leq j \leq k\}$ are linearly independent. It follows that the $(k-1)$-forms $\{\theta \wedge \cdots \wedge \widehat{\theta^j} \wedge \cdots \wedge \theta^k \colon 1 \leq j \leq k\}$ are also linearly independent. By Proposition 2,

$$v \in \mathrm{Ker}(\xi) \ \Leftrightarrow \ i_v \xi = 0 \ \Leftrightarrow \ \theta^j(v) = 0 \, \forall j \ \Leftrightarrow \ v \in \bigcap_{j=1}^{k} \mathrm{Ker}(\theta^j)$$

which implies that $\mathrm{Ker}(\xi) = \cap_{j=1}^{k} \mathrm{Ker}(\theta^j)$. To compute the dimension of this kernel consider the linear map $\theta : V \to \mathbb{R}^k$, $\theta(v) := (\theta^1(v), \cdots, \theta^k(v))$, whose kernel is equal to $\mathrm{Ker}(\xi)$. Since

$$\sum_{j=1}^{k} c_j \, \theta^j = 0 \ \Leftrightarrow \ (c_1, \ldots, c_k) \cdot \theta(v) \, \forall \, v \in V \ \Leftrightarrow \ (c_1, \ldots, c_k) \in \theta(V)^{\perp}$$

the linear independence of $\{\theta^j \colon 0 \leq j \leq k\}$ implies that $\theta(V)^{\perp} = \{0\}$ and hence that $\theta(V) = \mathbb{R}^k$. Therefore

$$\dim Ker(\xi) = \dim \mathrm{Ker}(\theta) = \dim V - \dim \theta(V) = n - k.$$

□

Proposition 5. *Given two decomposable k-forms $\xi, \xi' \in \bigwedge^k(V) \setminus \{0\}$,*

$$\mathrm{Ker}(\xi) = \mathrm{Ker}(\xi') \quad \Leftrightarrow \quad \exists \, \kappa \in \mathbb{R} \setminus \{0\} \ \textit{such that} \ \xi' = \kappa \, \xi.$$

Proof. Assume that $\mathrm{Ker}(\xi) = \mathrm{Ker}(\xi')$ and let $\{v_1, \ldots, v_n\}$ be a basis of V such that $\{v_{k+1}, \ldots, v_n\}$ is a basis of $\mathrm{Ker}(\xi)$. Let $\{\lambda^1, \ldots, \lambda^n\}$ be the dual basis of V^*, which is characterized by the relations $\lambda^j(v_i) = \delta_{ij}$ for all $1 \leq i, j \leq n$. By Proposition 1 we have

$$\xi = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \xi(v_{i_1}, \ldots, v_{i_k}) \, \lambda^{i_1} \wedge \cdots \wedge \lambda^{i_k} = \xi(v_1, \ldots, v_k) \, \lambda^1 \wedge \cdots \wedge \lambda^k.$$

The second equality holds because $\xi(v_{i_1}, \ldots, v_{i_k}) = 0$ for every $(i_1, \ldots, i_k) \neq (1, \ldots, k)$. In fact, if $(i_1, \ldots, i_k) \neq (1, \ldots, k)$ then $i_\alpha > k$ for some index α which implies that $v_{i_\alpha} \in \mathrm{Ker}(\xi)$ and hence that $\xi(v_{i_1}, \ldots, v_{i_k}) = \pm \, i_{v_{i_\alpha}} \xi(\cdots) = 0$. In the same way

$$\xi = \xi'(v_1, \ldots, v_k) \, \lambda^1 \wedge \cdots \wedge \lambda^k.$$

Because ξ and ξ', are non-zero k-forms we have $\xi(v_1, \ldots, v_k) \neq 0$ and $\xi'(v_1, \ldots, v_k) \neq 0$. Thus $\xi' = \kappa \, \xi$ where κ is the ratio between these two non-zero numbers.

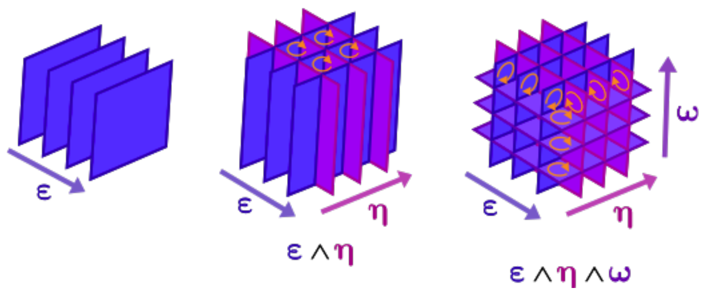
The converse implication is obvious. □

Exercise 1. *Given $\xi \in \bigwedge^k(V)$, show that ξ is decomposable if and only if* $\mathrm{Ker}(\xi)$ *has dimension $\geq n-k$.*
Hint: *Use the argument in the proof of Proposition 5.*

Exercise 2. *Prove that every 2-form $\xi \in \bigwedge^2(\mathbb{R}^3)$ is decomposable.*
Hint: *There exists a skew-symmetric matrix $A \in \mathbb{R}^{3\times 3}$ such that $\xi(u,v) = u^T A\, v$. Conclude that* $\dim \mathrm{Ker}(\xi) = 1$ *when $\xi \neq 0$.*

Exercise 3. *Prove that the 2-form $\xi = dx\, dy + dz\, dw \in \bigwedge^2(\mathbb{R}^4)$ is not decomposable.*
Hint: *Prove that* $\mathrm{Ker}(\xi) = \{(0,0,0,0)\}$.

The following picture from wikipedia (`https://en.wikipedia.org/wiki/Exterior_algebra`) helps to visualize the kernels of decomposable forms in $\mathbb{R}^3$. The picture depicts several level sets of the 1-forms ε, η and ω.



2 A couple of formulas

Let M be some n-dimensional manifold and denote by $\mathfrak{X}(M)$ the space of all smooth vector fields on M. Let $\Omega^k(M)$ be the space of k-differential forms on M.

Proposition 6. *Given $\omega \in \Omega^1(M)$, $X, Y \in \mathfrak{X}(M)$,*

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

Proof. See [3, Proposition 20.13]. □

Proposition 7 (Cartan's Formula)**.** *Given $\omega \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$,*

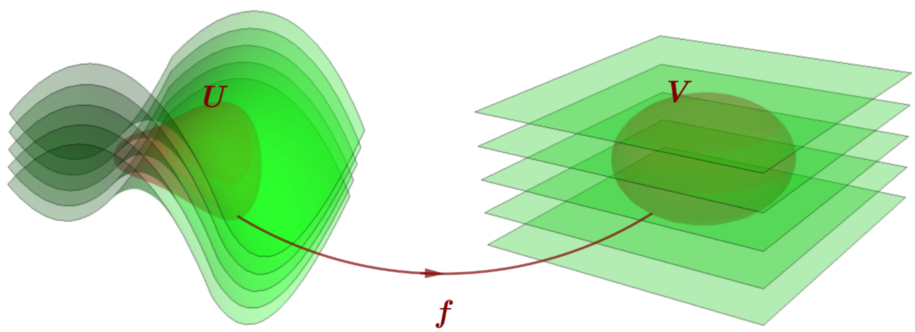
$$\mathcal{L}_X \omega = d\, i_X \omega + i_X\, d\omega.$$

Proof. See [3, Proposition 20.10 (iii)]. □

3 Frobenius Theorems, statements and proofs

A partitioned manifold is a pair $(M, \mathcal{P})$ where M is a manifold and $\mathcal{P}$ is a partition of M. We say that two partitioned manifolds $(M, \mathcal{P})$ and $(N, \mathcal{Q})$ are diffeomorphic if there exists a diffeomorphism $f : M \to N$ that induces a bijective map $F \mapsto f(F)$ between the partitions $\mathcal{P}$ and $\mathcal{Q}$. Given an open set $U \subset M$, we denote by $\mathcal{P}|_U$ the restriction of the partition $\mathcal{P}$ to U.

Definition 3. *A k-dimensional foliation of M is any partition $\mathcal{F}$ of M such that $(M, \mathcal{F})$ is locally diffeomorphic to $(\mathbb{R}^n, \mathcal{E}_n^k)$, where $\mathcal{E}_n^k := \{\mathbb{R}^k \times \{c\} : c \in \mathbb{R}^{n-k}\}$. This means that for every $p \in M$ there are open sets $U \subset M$, $V \subset \mathbb{R}^n$ with $p \in U$ and a diffeomorphism $f : (U, \mathcal{F}|_U) \to (V, \mathcal{E}_n^k|_V)$ between the partitioned manifolds $(U, \mathcal{F}|_U)$ and $(V, \mathcal{E}_n^k|_V)$.*



Given a foliation $\mathcal{F}$ of M, the elements of the partition $\mathcal{F}$ are called the *leaves* of $\mathcal{F}$. The *leaf* of $\mathcal{F}$ that contains a point $x \in M$ is denoted by $\mathcal{F}(x)$. The tangent space $T_x\mathcal{F}(x)$ is abbreviated by $T_x\mathcal{F}$.

Proposition 8. *If $f : M \to N$ is a submersion then $\mathcal{F} = \{f^{-1}(c) : c \in N\}$ is a foliation of M with dimension $k = \dim(M) - \dim(N)$.*

Proof. Exercise. □

Definition 4. *A k-*dimensional distribution on *M is a smooth function $M \ni x \mapsto D_x$ that to each point $x \in M$ associates a k-dimensional subspace $D_x \subset T_xM$. It can be defined as a smooth section of the Grassmannian bundle $\mathrm{Gr}_k(TM)$. Alternatively, the map $x \mapsto D_x$ is smooth if for every $p \in M$ there exists an open set $U \subset M$ with $p \in U$ and there are smooth vector fields $X_1, \ldots, X_k \in \mathfrak{X}(U)$ such that $\{X_1(x), \ldots, X_k(x)\}$ form a basis for D_x, for all $x \in U$.*

Definition 5. *A k-dimensional distribution D on M is said to be* completely integrable *if for every $p \in M$ there exists an open set $U \subset M$ with $p \in U$ and there exists a foliation $\mathcal{F}$ on U such that $D_x = T_x\mathcal{F}$ for all $x \in U$.*

Proposition 9. *Every 1-dimensional distribution on M is completely integrable.*

Proof. Given a 1-dimensional distribution D, for every $p \in M$ there is an open set $U \subset M$ containing p and there exists a non-zero smooth vector field $X \in \mathfrak{X}(U)$ such that $X(x) \in D_x$ for all $x \in U$. The integral 1-dimensional foliation $\mathcal{F}$ follows from the existence of solutions of the ordinary differential equation $x'(t) = X(x(t))$ on M. The leaves of $\mathcal{F}$ are the trajectories of the vector field X. □

Given a distribution D, we define now the subspace of vector fields $X \in \mathfrak{X}(M)$ which are tangent to D:

$$\mathcal{V}(D) := \{X \in \mathfrak{X}(M) : \; X(x) \in D_x, \; \forall x \in M\}$$

We will write $X \in D$ to mean that $X \in \mathcal{V}(D)$.

Remark 1. *$\mathcal{V}(D)$ is a linear subspace of $\mathfrak{X}(M)$.*

Proof. Exercise. □

Theorem 1 (Frobenius)**.** *Given a k-dimensional distribution D on M, D is completely integrable if and only if $\mathcal{V}(M)$ is a Lie subalgebra of $\mathfrak{X}(M)$, i.e., $[X,Y] \in \mathcal{V}(D)$ whenever $X, Y \in \mathcal{V}(D)$.*

Proof. See [2, Theorem 5.1]. We present here a geometric sketch of the argument.

If D is completely integrable, let $\mathcal{F}$ be a foliation on some open set $U \subset M$ such that $p \in U$ and $D_x = T_x\mathcal{F}$ for all $x \in U$. Applying Exercise 3.17 to the submanifold $\mathcal{F}(x)$, we see that for any $X, Y \in \mathcal{V}(D)$, $[X,Y](p) \in T_p\mathcal{F} = D_p$. Hence $[X,Y] \in \mathcal{V}(D)$, which proves that $\mathcal{V}(D)$ is a sub-algebra of $\mathfrak{X}(M)$.

The proof of the converse implication goes by induction in the dimension of the distribution. For $k = 1$, Frobenius' Theorem reduces to Proposition 9.

Assume now that this theorem holds for any $(k-1)$-dimensional distribution F such that $\mathcal{V}(F)$ is a Lie algebra and let D be a k-dimensional distribution such that $\mathcal{V}(F)$ is a Lie algebra. Because 'complete integrability' is a local concept, given $p \in M$ we can take an open set $U \subset M$ with $p \in U$ and choose vector fields $X_1, \ldots, X_n \in \mathfrak{X}(U)$ such that $\{X_1(x), \ldots, X_n(x)\}$ is a basis of T_xM, while $\{X_1(x), \ldots, X_k(x)\}$ is a basis of D_x for all $x \in U$. By Exercise 3.13 (flow-box theorem) we can assume that $X_1 = e_1 = (1, 0, \ldots, 0)$. Define $Y_1 = X_1$ and $Y_i = X_i - (X_i \cdot X_1)\, X_1$ for $i = 2, \ldots, n$. The vector Y_i is the orthogonal projection of X_i onto the hyperplane $X_1^{\perp}$. We still have that $\{Y_1(x), \ldots, Y_n(x)\}$ is a basis of T_xM, while $\{Y_1(x), \ldots, Y_k(x)\}$ is a basis of D_x for all $x \in U$, but these new vector fields satisfy for all $2 \le i \le n$ and $x \in U$:

(a) $Y_1(x) \cdot Y_i(x) = 0$,

(b) first component of $Y_i(x) = Y_i(x_1) = 0$,

(c) $[Y_1, Y_i](x) = 0$.

Because $\mathcal{V}(D)$ is a Lie algebra, $[Y_i, Y_j] \in \mathcal{V}(D)$ for any $2 \le i, j \le n$. Hence there are smooth functions $c_{ij}^{\ell} \in C^{\infty}(U)$, with $1 \le \ell \le k$, such that

$$[Y_i, Y_j] = \sum_{\ell=1}^{k} c_{ij}^{\ell}\, Y_\ell.$$

From item (b) a simple calculation shows that the first component of $[Y_i, Y_j]$ is also zero, which implies that $c_{ij}^{1} = 0$. Therefore

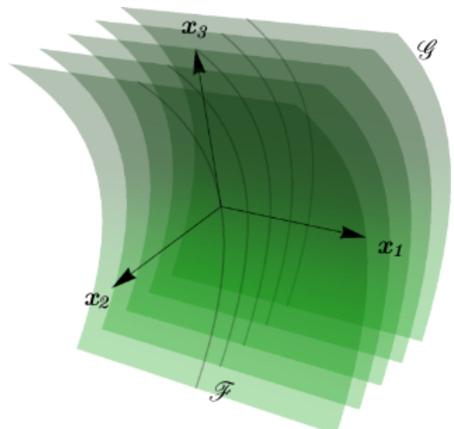
$$[Y_i, Y_j] = \sum_{\ell=2}^{k} c_{ij}^{\ell}\, Y_\ell$$

and the vector fields $Y_2, \ldots, Y_k$ span a $(k-1)$-distribution F such that $\mathcal{V}(F)$ is a Lie algebra. By induction hypothesis this distribution integrates to a foliation $\mathcal{F}$ that we will assume to be defined on the same neighborhood U. By item (c) and Exercise 3.11 the flows of the vector fields Y_i commute with the flow of Y_1. Hence the leaves of $\mathcal{F}$

are invariant under the flow of Y_1. In other words, the leaves of $\mathcal{F}$ are invariant under translations along the direction e_1. Thus we can define a new foliation $\mathcal{G}$, with leaves

$$\mathcal{G}(x) = U \cap (\mathbb{R}\, e_1 + \mathcal{F}(x))$$

which integrates the distribution D. See the figure below. □



Define the linear subspace $\mathcal{I}(D) := \oplus_{k=0}^{n} \mathcal{I}^k(D)$ where

$$\mathcal{I}^k(D) := \{\omega \in \Omega^k(M) \colon \omega_x(v_1, \ldots, v_k) = 0,\ \forall\, x \in M,\ \forall\, v_1, \ldots, v_k \in D_x\,\}.$$

We will say that ω vanishes on D to mean that $\omega \in \mathcal{I}(D)$.

Remark 2. *$\mathcal{I}(D)$ is an ideal of the graded algebra $\Omega^*(M) := \oplus_{k=0}^{n} \Omega^k(M)$.*

Proof. Exercise. □

Theorem 2 (Frobenius)**.** *Given a k-dimensional distribution D on M, D is completely integrable if and only if $d(\mathcal{I}(D)) \subset \mathcal{I}(D)$, i.e., $d\omega \in \mathcal{I}(D)$ whenever $\omega \in \mathcal{I}(D)$.*

Proof. Since the theorem's content is local we can assume that there exist vector fields $X_1, \ldots, X_n \in \mathfrak{X}(M)$ such that $\{X_1(x), \ldots, X_n(x)\}$ is a basis of T_xM for all $x \in M$. Moreover, by Definition 4 we can assume that $\{X_1(x), \ldots, X_k(x)\}$ is a basis of D_x, for all $x \in M$. Consider the dual 1-forms $\omega^1, \ldots, \omega^n \in \Omega^1(M)$ which are characterized by $\omega^i(X_j) = \delta_{ij}$. Then for all $x \in M$,

$$D_x = \bigcap_{j=k+1}^{n} \operatorname{Ker}(\omega^j(x)). \tag{1}$$

Lemma 1. *Any form $\omega \in \mathcal{I}^\ell(D)$ can be represented as*

$$\omega = \sum_{\substack{1 \le i_1 < \ldots < i_\ell \le n \\ i_\ell \ge k+1}} h_{i_1, \cdots, i_\ell}\, \omega^{i_1} \wedge \cdots \wedge \omega^{i_\ell} \tag{2}$$

with $h_{i_1, \cdots, i_\ell} \in C^\infty(M)$.

In particular, $\mathcal{I}(D)$ is generated (as an ideal) by the monomials $\omega^{k+1}, \ldots, \omega^n$.

Proof. It is clear that $\omega^{k+1}, \dots, \omega^n \in \mathcal{I}^1(D)$. Hence, since $\mathcal{I}(D)$ is an ideal, it must contain all k- forms (2).

Conversely, any k-form $\omega \in \Omega^k(M)$ can be written as

$$\omega = \sum_{1 \le i_1 < \ldots < i_\ell \le n} h_{i_1, \cdots, i_\ell}\, \omega^{i_1} \wedge \cdots \wedge \omega^{i_\ell}$$

with $h_{i_1,\cdots,i_\ell} \in C^\infty(M)$. In fact we have $h_{i_1,\cdots,i_\ell} = \omega(X_{i_1}, \dots, X_{i_\ell})$ because of Proposition 1. But if $\omega \in \mathcal{I}^\ell(D)$ and $i_\ell \le k$ then $X_{i_1}, \dots, X_{i_\ell} \in D$ and

$$h_{i_1,\cdots,i_\ell} = \omega(X_{i_1}, \dots, X_{i_\ell}) = 0.$$

This proves that ω takes the form (2). □

Remark 3. *In the previous setting* $d(\mathcal{I}(D)) \subset \mathcal{I}(D) \quad \Leftrightarrow \; d\omega^j \in \mathcal{I}^2(D)\; \forall\, k+1 \le j \le n.$

Proof. Exercise. □

By Proposition 6, given $r \ge k+1$ and $i, j \le k$

$$d\omega^r(X_i, X_j) = X_i(\underbrace{\omega^r(X_j)}_{=0}) - X_j(\underbrace{\omega^r(X_i)}_{=0}) - \omega^r([X_i, X_j]) = -\omega^r([X_i, X_j]). \tag{3}$$

Hence each of the following statements is equivalent to the next one:

- D is completely integrable;
- $\mathcal{V}(D)$ is a Lie sub-algebra of $\mathfrak{X}(M)$; (by Theorem 1)
- $[X_i, X_j] \in D$ for all $1 \le i < j \le k$; (because $D = \langle X_1, \dots, X_k \rangle$)
- $\omega^r([X_i, X_j]) = 0$ for all $1 \le i < j \le k$ and $k+1 \le r \le n$; (by (1))
- $d\omega^r(X_i, X_j) = 0$ for all $1 \le i < j \le k$ and $k+1 \le r \le n$; (by (3))
- $d\omega^r \in \mathcal{I}^2(D)$ for all $k+1 \le r \le n$; (by definition of $\mathcal{I}(D)$)
- $d(\mathcal{I}(D)) \subset \mathcal{I}(D)$. (by Remark 3)

□

Theorem 3. *Given* 1*-forms* $\omega^{k+1}, \dots, \omega^n \in \Omega^1(M)$ *and a* k*-distribution* D *such that*

$$D_x = \mathrm{Ker}(\omega^{k+1}(x)) \cap \ldots \cap \mathrm{Ker}(\omega^n(x)) \qquad \forall\, x \in M$$

consider the form $\Omega = \omega^{k+1} \wedge \cdots \wedge \omega^n$. *Then the following statements are equivalent:*

(a) D *is completely integrable;*

(b) $d\omega^r = \sum_{\substack{1 \le \alpha < \beta \\ k < \beta \le n}} h^k_{\alpha,\beta}\, \omega^\alpha \wedge \omega^\beta$ *with* $h^k_{\alpha,\beta} \in C^\infty(M)$*;*

(c) $d\omega^r \wedge \Omega = 0$*, for all* $k < r \le n$.

Proof. The assumption implies that $\omega^{k+1}(x), \ldots, \omega^n(x)$ are linearly independent for all $x \in M$. Hence, working locally we can take 1-forms $\omega^1, \ldots, \omega^k$ such that $\{\omega^1(x), \ldots, \omega^n(x)\}$ is a basis of $(T_xM)^*$ for all x. Consider then the dual vector fields $X_1, \ldots, X_n \in \mathfrak{X}(M)$ which satisfy $\omega^i(X_j) = \delta_{ij}$ for all $1 \leq i, j \leq n$.

Assuming (a) we have by Theorem 2, for all $r > k$, and $i < j \leq k$, $d\omega^r(X_i, X_j) = 0$. On the other hand by Proposition 1

$$d\omega^r = \sum_{1 \leq \alpha < \beta \leq n} h^k_{\alpha,\beta}\, \omega^\alpha \wedge \omega^\beta$$

with $h^k_{\alpha,\beta} = d\omega^r(X_\alpha, X_\beta)$. This implies (b).

Assuming (b), in each summand $\omega^\alpha \wedge \omega^\beta$ of $d\omega^r$ we have $k < \beta$. Hence $\omega^\alpha \wedge \omega^\beta \wedge \Omega = 0$ because the factor ω^β is present in Ω. This implies (c), that is $d\omega^r \wedge \Omega = 0$.

To finish we prove that $(c) \Rightarrow (a)$. By Proposition 4, for all $x \in M$,

$$D_x = \bigcap_{r=k+1}^{n} \mathrm{Ker}(\omega^r(x)) = \mathrm{Ker}(\Omega(x)).$$

Thus, given $1 \leq j \leq k$, $i_{X_j}\Omega = 0$. For any $r > k$, since $d\omega^r \wedge \Omega = 0$, by Proposition 3 we have $(i_{X_j} d\omega^r) \wedge \Omega = 0$. We can write

$$i_{X_j} d\omega^r = \sum_{s=1}^{n} h_s\, \omega^s$$

for some functions $h_s \in C^\infty(M)$. Since

$$0 = (i_{X_j} d\omega^r) \wedge \Omega = \sum_{s=1}^{k} h_s\, \omega^s \wedge \Omega + \sum_{s=k+1}^{n} h_s\, \underbrace{\omega^s \wedge \Omega}_{=0} = \sum_{s=1}^{k} h_s\, \omega^s \wedge \Omega$$

where the forms $\{\omega^s \wedge \Omega\}_{s \leq k}$ are linearly independent we must have $h_s = 0$ for all $s \leq k$. Therefore

$$i_{X_j} d\omega^r = \sum_{s=k+1}^{n} h_s\, \omega^s.$$

Hence for all $1 \leq i, j \leq k$,

$$d\omega^r(X_j, X_i) = i_{X_j} d\omega^r(X_i) = \sum_{s=k+1}^{n} h_s\, \omega^s(X_i) = 0$$

which implies that $d\omega^r \in \mathcal{I}(D)$. The complete integrability (a) follows by Theorem 2 .

We have shown that

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a).$$

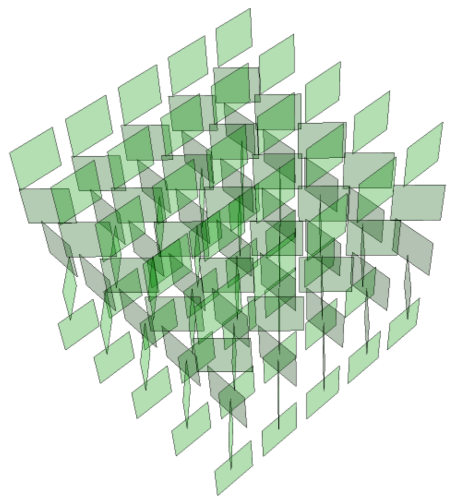
□

Exercise 4. *Under the assumptions of Theorem 3 prove that* $d\Omega = 0$ *is sufficient for the complete integrability of the distribution* $\mathrm{Ker}(\Omega)$.

Exercise 5. *Prove that the kernel of the* 1*-form*

$$\omega = \cos z\, dx + \sin z\, dy \in \Omega^2(\mathbb{R}^3)$$

is not an integrable 2*-distribution in* $\mathbb{R}^3$*. See the figure below.*



Exercise 6. *Let* $U = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ *and consider the vector fields* $X, Y \in \mathfrak{X}(U)$*,* $X(x, y, z) = (0, -z, y)$*,* $Y(x, y, z) = (z, 0, -x)$*. Prove that these two vector fields span a completely integrable* 2*-distribution and determine the corresponding integral foliation.*

Exercise 7. *Given a submersion* $f : M^n \to N^m$*, let* Ω *be volume form on* N^m*. Prove that the kernel of* $f^*\Omega$ *is a completely integrable* $(n - m)$*-distribution and identify the corresponding integral foliation.*

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