Chapter 6
Verhulst and the logistic equation (1838)

Pierre-François Verhulst was born in 1804 in Brussels. He obtained a PhD in mathematics from the University of Ghent in 1825. He was also interested in politics. While in Italy to contain his tuberculosis, he pleaded without success in favor of a constitution for the Papal States. After the revolution of 1830 and the independence of Belgium, he published a historical essay on an eighteenth century patriot. In 1835 he became professor of mathematics at the newly created Free University in Brussels.

Fig. 6.1 Verhulst (1804–1849)

That same year 1835, his compatriot Adolphe Quetelet, a statistician and director of the observatory in Brussels, published *A Treatise on Man and the Development of his Faculties*. Quetelet suggested that populations could not grow geometrically over a long period of time because the obstacles mentioned by Malthus formed a kind of "resistance", which he thought (by analogy with mechanics) was proportional to the square of the speed of population growth. This analogy had no real basis, but it inspired Verhulst.

N. Bacaër, *A Short History of Mathematical Population Dynamics*,

DOI 10.1007/978-0-85729-115-8_6,

Indeed, Verhulst published in 1838 a *Note on the law of population growth*. Here are some extracts:

> We know that the famous Malthus showed the principle that the human population tends to grow in a geometric progression so as to double after a certain period of time, for example every twenty five years. This proposition is beyond dispute if abstraction is made of the increasing difficulty to find food [...]
>
> The virtual increase of the population is therefore limited by the size and the fertility of the country. As a result the population gets closer and closer to a steady state.

Verhulst probably realized that Quetelet's mechanical analogy was not reasonable and proposed instead the following (still somewhat arbitrary) differential equation for the population $P(t)$ at time t:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right). \tag{6.1}$$

When the population $P(t)$ is small compared to the parameter K, we get the approximate equation

$$\frac{dP}{dt} \simeq rP\,,$$

whose solution is $P(t) \simeq P(0)\,e^{rt}$, i.e. exponential growth[1]. The growth rate decreases as $P(t)$ gets closer to K. It would even become negative if $P(t)$ could exceed K. To get the exact expression of the solution of equation (6.1), we can proceed like Daniel Bernoulli for equation (4.5).

Dividing equation (6.1) by P^2 and setting $p = 1/P$, we get $dp/dt = -rp + r/K$. With $q = p - 1/K$, we get $dq/dt = -rq$ and $q(t) = q(0)\,e^{-rt} = (1/P(0) - 1/K)\,e^{-rt}$. So we can deduce $p(t)$ and $P(t)$.

Finally we get after rearrangement

$$P(t) = \frac{P(0)\,e^{rt}}{1 + P(0)\,(e^{rt} - 1)/K}\,. \tag{6.2}$$

The total population increases progressively from $P(0)$ at time $t = 0$ to the limit K, which is reached only when $t \to +\infty$ (Fig. 6.2). Without giving the values he used for the unknown parameters r and K, Verhulst compared his result with data concerning the population of France between 1817 and 1831, of Belgium between 1815 and 1833, of the county of Essex in England between 1811 and 1831, and of Russia between 1796 and 1827. The fit turned out to be pretty good.

In 1840 Verhulst became professor at the Royal Military School in Brussels. The following year he published an *Elementary Treatise of Elliptic Functions* and was elected to the Royal Academy of Belgium. In 1845 he continued his population

[1] One usually speaks of geometric growth in discrete-time models and of exponential growth in continuous-time models but is is essentially the same thing.

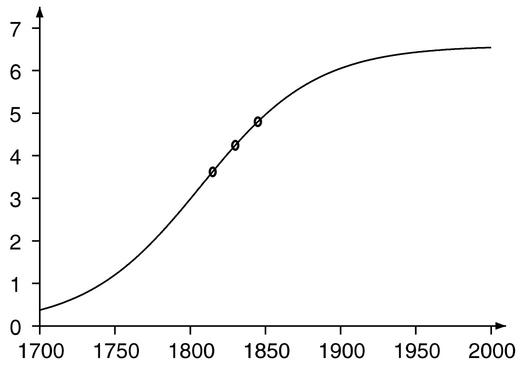

Fig. 6.2 The population of Belgium (in millions) and the logistic curve. The data points correspond to the years 1815, 1830 and 1845. The parameter values are those of the article from 1845.

studies with an article entitled "Mathematical enquiries on the law of population growth". He first turned back to Malthus' remark according to which the population of the USA had doubled every 25 years (Tab. 6.1). If we compute the ratio between

Table 6.1 Official censuses of the population of the USA.

Year	Population
1790	3,929,827
1800	5,305,925
1810	7,239,814
1820	9,638,131
1830	12,866,020
1840	17,062,566

the population in year $n+10$ to that in year n, we find respectively 1.350, 1.364, 1.331, 1.335 and 1.326, which is fairly constant. The population was hence multiplied on average by 1.34 every 10 years and by $1.34^{25/10} \simeq 2.08$ every 25 years. So it had continued to double every 25 years since Malthus' essay, almost half a century earlier. However Verhulst added:

> We shall not insist on the hypothesis of geometric progression, given that it can hold only in very special circumstances; for example, when a fertile territory of almost unlimited size happens to be inhabited by people with an advanced civilization, as was the case for the first American colonies.

In his article Verhulst also returned to equation (6.1), which he called "logistic". He noticed that the curve $P(t)$ increases with a positive curvature (it is convex) as long as $P(t) < K/2$ and then continues to increase towards K but with a negative curvature (it is concave) as soon as $P(t) > K/2$. So the curve has the shape of a distorted letter S (Fig. 6.2).

Indeed, $d^2P/dt^2 = r(1-2P/K)\,dP/dt$. So $d^2P/dt^2 > 0$ if $P < K/2$ and $d^2P/dt^2 < 0$ if $P > K/2$.

Verhulst also explained how the parameters r and K can be estimated from the population $P(t)$ in three different but equally spaced years. If P_0 is the population at time $t=0$, P_1 that at time $t=T$ and P_2 that at time $t=2T$, then a tedious computation starting from equation (6.2) shows that

$$K = P_1\,\frac{P_0P_1+P_1P_2-2P_0P_2}{P_1^2-P_0P_2}\,, \quad r=\frac{1}{T}\log\left[\frac{1/P_0-1/K}{1/P_1-1/K}\right].$$

Using the estimations for the population of Belgium in the years 1815, 1830 and 1845 (respectively 3.627, 4.247 and 4.801 million), he obtained $K = 6.584$ million and $r = 2.62\%$ per year. He could then use equation (6.2) to predict that the population of Belgium would be 4.998 million at the beginning of the year 1851 and 6.064 million at the beginning of the year 1900 (Fig. 6.2). Verhulst did a similar study for France. He obtained $K = 39.685$ million and $r = 3.2\%$ per year. As the populations of Belgium and France have in the mean time largely exceeded these values of K, we see that the logistic equation can be a realistic model only for periods of time of a few decades, as in Verhulst's 1838 article, but not for longer periods.

In 1847 appeared a *Second enquiry on the law of population growth* in which Verhulst gave up the logistic equation and chose instead a differential equation that can be written in the form

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right).$$

He thought that this equation would hold when the population $P(t)$ is above a certain threshold. The solution is

$$P(t) = K + (P(0)-K)\,e^{-rt/K}.$$

Using the same demographic data for Belgium, Verhulst estimated anew the parameters r and K. This time he found $K = 9.4$ million for the maximum population. We see how much the result can depend on the choice of the model!

Verhulst became president of the Royal Academy of Belgium in 1848, but died the following year in Brussels, probably of tuberculosis. Despite Verhulst's hesitation between model equations, the logistic equation was reintroduced independently several decades later by different people. Robertson used it in 1908 to model the individual growth of animals, plants, humans and body organs. McKendrick and Kesava Pai used it in 1911 for the growth of populations of microorganisms. Pearl and Reed used it in 1920 for the growth of the population of the USA, which had started to slow down. In 1922 Pearl finally noticed the work of Verhulst. From then on, the logistic equation inspired many works (see Chapters 13, 20 and 24). The maximum population K eventually became known as the "carrying capacity".

Further reading

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