On the reverse mathematics of Lipschitz and Wadge determinacy

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joint work with
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Goal and Plan

Goal: to calibrate the strength of Lipschitz and Wadge determinacy, as well as the Semi-Linear Ordering principle (SLO), in terms of subsystems of second order arithmetic ($\mathbb{Z}_2$).

Plan

- Introduction
- L/W determinacy. Topological results
- Formalization of L/W determinacy and SLO in $\mathbb{Z}_2$
- Results in $\mathbb{Z}_2$
- L/W determinacy and SLO in Cantor space
- L/W determinacy and SLO in Baire space
- Concluding remarks
Introduction. Gale-Stewart and Lipschitz Games

<table>
<thead>
<tr>
<th>Player I</th>
<th>$f(0)$</th>
<th>$f(1)$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player II</td>
<td>$g(0)$</td>
<td>$g(1)$</td>
<td>...</td>
</tr>
</tbody>
</table>

- (Gale-Stewart games) $A \subseteq X^\omega$. \(\textbf{I}\) wins a play of $G(A)$ if \[
\langle f(0), g(0), f(1), g(1), \ldots \rangle \in A
\] Otherwise \(\textbf{II}\) wins.

- (Lipschitz games) $A, B \subseteq X^\omega$. \(\textbf{II}\) wins a play of $G_L(A, B)$ if \[
\langle f(0), f(1), \ldots \rangle \in A \text{ iff } \langle g(0), g(1), \ldots \rangle \in B
\] Otherwise \(\textbf{I}\) wins.
Introduction. Wadge Games

- Variant of Lipschitz games where Player II is allowed to pass, but she must play infinitely often otherwise she loses.

<table>
<thead>
<tr>
<th>Player I</th>
<th>$f(0)$</th>
<th>...</th>
<th>$f(k)$</th>
<th>$f(k+1)$</th>
<th>$f(k+2)$</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Player II</td>
<td>$p \cdots p$</td>
<td>$g(0)$</td>
<td>$p$</td>
<td>$g(1)$</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

(Wadge games) $A, B \subseteq X^\omega$. Player II wins a play of $G_W(A, B)$ if
$$\langle f(0), f(1), \ldots \rangle \in A \text{ iff } \langle g(0), g(1), \ldots \rangle \in B$$
Otherwise Player I wins.

Reduction to Gale-Stewart games:
- Player I wins $G_L(A, B)$ iff Player I wins $G(\neg(A \leftrightarrow B))$.
- Similar for $G_W(A, B)$. 

Introduction. Determinacy axioms in set theory

- (Mycielski/Steinhaus, 1962)

\[ \text{AD} = \forall A \subseteq \omega^\omega, \text{ the Gale-Stewart game } G(A) \text{ is determined,} \]
- i.e. either I or II has a winning strategy.

- (Wadge, 1972)

\[ \text{AD}_{L/W} = \forall A, B \subseteq \omega^\omega, \text{ the game } G_{L/W}(A, B) \text{ is determined.} \]

- Similarly for pointclasses: open determinacy, Borel determinacy, projective determinacy, etc.

- \( \text{AD} \implies \text{AD}_{L/W}, \ BP, \ LM, \ PSP, \ \neg\text{AC}, \ AC_\omega(\omega^\omega) \)

- (Solovay’s conjecture) \( \text{ZF} + V = L(\mathbb{R}) \) proves \( \text{AD} \iff \text{AD}_W \)
Introduction. Lipschitz and Wadge reducibility

- (L/W reducibility) Given $A, B \subseteq X^\omega$,

  $A \leq_{L/W} B$ iff there is a Lipschitz/continuous $F : X^\omega \to X^\omega$
  such that $A = F^{-1}(B)$.

- (L/W degrees) Given $A \subseteq X^\omega$, we define

  $[A]_{L/W} = \{ B \subseteq X^\omega : B \equiv_{L/W} A \}$

- (Wadge’s lemma) $A, B \subseteq 2^\omega$ or $\omega^\omega$. Then:
  1. $\mathbf{II}$ wins the game $G_L(A, B)$ iff $A \leq_L B$.
  2. $\mathbf{II}$ wins the game $G_W(A, B)$ iff $A \leq_W B$.
  3. If $\mathbf{I}$ wins $G_{L/W}(A, B)$ then $B^c \leq_{L/W} A$. 
Semi-Linear Ordering Principle

(Wadge, 1972)

\[ \text{SLO}_{L/W} = \text{For all } A, B \subseteq \omega^\omega, A \leq_{L/W} B \lor B^c \leq_{L/W} A. \]

Wadge’s lemma implies

\[
\begin{align*}
\text{AD}_L & \implies \text{SLO}_L \\
\downarrow \\
\text{AD}_W & \implies \text{SLO}_W
\end{align*}
\]

Classical consequences of AD have been proved under SLO\(_W\):

\[ \text{SLO}_W \implies \text{PSP}, \neg \text{AC}, \text{AC}_\omega(\omega^\omega). \]

(Andretta, 2003/4) ZF + BP + DC proves

\[ \text{SLO}_W \iff \text{AD}_W \iff \text{AD}_L \iff \text{SLO}_L. \]
Introduction. Classical results

- (Martin, 1975) $\mathbf{ZF} + \mathbf{DC}$ proves determinacy for all Borel sets.

- (Friedman, 1971) $\mathbf{Z}_2$ cannot prove that all Borel Gale-Stewart games are determined.

- (Montalbán/Shore, 2012) For each $k$, $\mathbf{Z}_2$ proves determinacy for Boolean combinations of $k$ many $\Pi^0_3$ sets, but not for all finite Boolean combinations of $\Pi^0_3$ sets.

- (Steel, ’77; Tanaka, ’90; Nemoto/MedSalem/Tanaka, ’07) A detailed picture of the reverse mathematics of Gale-Stewart determinacy is known.

- (Louveau/Saint-Raymond, 1987) $\mathbf{Z}_2$ does prove that all Borel Lipschitz games are determined.
Introduction. Reverse mathematics of $L/W$ games

- In contrast, the situation for $L/W$ games is completely different:
  - There is no detailed analysis of the strength of $L/W$ determinacy in terms of subsystems of $\mathbb{Z}_2$.
  - Reverse Mathematics of $SLO$ hasn’t been investigated either.

- Known results on Gale-Stewart determinacy give us upper bounds on the strength of $L/W$ determinacy. But these bounds needn’t be optimal.

- This is certainly a notable gap in our understanding of the reverse mathematics of infinite games.
L/W determinacy. Topological results

By reinterpreting Wadge's topological analysis of games we obtain direct proofs of L/W determinacy in $\mathbf{ZF} + \mathbf{DC}$:

<table>
<thead>
<tr>
<th>$(A, B)$</th>
<th>Cantor</th>
<th>Baire</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta^0_1$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\Pi^0_1$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$\Sigma^0_1 \cup \Pi^0_1$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
<tr>
<td>$Df_2 \cap \overline{Df_2}$</td>
<td>$\checkmark$</td>
<td>$\checkmark$</td>
</tr>
<tr>
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<td>$\checkmark$</td>
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We formalize these proofs in $\mathbf{Z}_2$ and obtain a number of results on the strength of L/W determinacy and SLO.
Formalizing L/W determinacy and SLO in $\mathbb{Z}_2$

- $\Gamma$-$\text{Det}_{L/W}$ : L/W determinacy for $\Gamma$ formulas in the Baire space.

  \[ \exists \sigma_I \forall \sigma_{II} [\text{Inf} \rightarrow \neg (A(f) \leftrightarrow B(g))] \lor \exists \sigma_{II} \forall \sigma_I [\text{Inf} \land (A(f) \leftrightarrow B(g))] \]

  $\sigma_I, \sigma_{II}$ range over strategies for I and II, resp.

- $\Gamma$-$\text{SLO}_{L/W}$ : L/W semilinear ordering principle for $\Gamma$ formulas in the Baire space.

  \[ \exists \sigma_{II} \forall \sigma_I [\text{Inf} \land (A(f) \leftrightarrow B(g))] \lor \exists \sigma_{II} \forall \sigma_I [\text{Inf} \land (\neg B(f) \leftrightarrow A(g))] \]

  $\sigma_I, \sigma_{II}$ range over strategies for I and II, resp.

- $\Gamma$-$\text{Det}^*_{L/W}$, $\Gamma$-$\text{SLO}^*_{L/W}$ : similar for the Cantor space.
Results in $\mathbb{Z}_2$

<table>
<thead>
<tr>
<th>Subsystem</th>
<th>Cantor</th>
<th>Baire</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{RCA}_0$</td>
<td>$\Delta^0_1\text{-Det}^*_W$</td>
<td></td>
</tr>
<tr>
<td>$\text{WKL}_0$</td>
<td>$\Delta^0_1\text{-Det}^*_L$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(\Delta^0_1, \Sigma^0_1)\text{-Det}^*_L/W$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(\Sigma^0_1, \Delta^0_1)\text{-Det}^*_L$</td>
<td></td>
</tr>
<tr>
<td>$\text{ACA}_0$</td>
<td>$(\Sigma^0_1 \cup \Pi^0_1)\text{-Det}^*_L/W$,</td>
<td>$\Delta^0_1\text{-Det}_W$,</td>
</tr>
<tr>
<td></td>
<td>$(\Sigma^0_1)^2\text{-Det}^*_L/W$</td>
<td>$(\Sigma^0_1 \cup \Pi^0_1)\text{-Det}_W$</td>
</tr>
<tr>
<td>$\text{ATR}_0$</td>
<td>$\Delta^0_1\text{-Det}_L$,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$(\Sigma^0_1 \cup \Pi^0_1)\text{-Det}_L$</td>
<td></td>
</tr>
<tr>
<td>$\Pi^1_1\text{-CA}_0$</td>
<td>$(\Sigma^0_1)^2 \cup \neg (\Sigma^0_1)^2\text{-Det}_L/W$</td>
<td></td>
</tr>
</tbody>
</table>

$(\Sigma^0_1)^2$ stands for differences of closed sets ($= \Sigma^0_1 \wedge \Pi^0_1$)
Results in $\mathbb{Z}_2$. Reverse mathematics results

Theorem Over $\text{RCA}_0$ the following are equivalent:

1. $\text{ACA}_0$  
2. $(\Sigma_1^0)_2^{\text{Det}}_L^*$  
3. $(\Sigma_1^0)_2^{\text{SLO}}_L^*$

Theorem Over $\text{ACA}_0$ the following are equivalent:

1. $\text{ATR}_0$  
2. $\Delta_1^0$-$\text{Det}_L$  
3. $\Delta_1^0$-$\text{SLO}_L$  
4. $(\Sigma_1^0 \cup \Pi_1^0)$-$\text{Det}_L$

Theorem Over $\text{RCA}_0$ the following are equivalent:

1. $\text{ATR}_0$  
2. $(\Sigma_1^0 \cup \Pi_1^0)$-$\text{Det}_L$. 
L/W determinacy and SLO in $2^\mathbb{N}$. Methodology

- Closed set $\leftrightarrow \varphi(f) \in \Pi_1^0$. Open set $\leftrightarrow \varphi(f) \in \Sigma_1^0$.

- Basic fact: subset $F$ of $X^\mathbb{N}$ is closed iff it is the body of a tree $T$ on $X$, i.e. $F = [T]$.

- The topological structure of sets $A$, $B$ gives us information for constructing a winning strategy in $G_{L/W}^*(A, B)$.
  - clopen sets $\leftrightarrow$ comparison of associated well-founded trees.
  - $A = [T]$ closed and not open $\leftrightarrow$ existence of a path of $T$ with eventual extensions outside $T$ for any initial sequence.
  - $A = [T_0] - [T_1]$ difference of closed sets $\leftrightarrow$ existence of a path of $T_1$ with eventual extensions in $T_0$ which, in turn, have eventual extensions outside $T_0$. 
**Theorem**

The following assertions are pairwise equivalent over RCA$_0$:

1. ACA$_0$
2. $(\Sigma^0_1)_2$-$\text{Det}_L^*$
3. $(\Sigma^0_1)_2$-$\text{SLO}_L^*$.

**Proof.** (Sketch)

(1 $\to$ 2) We construct winning strategies by analysing several (a lot of!) cases of differences of pruned trees.

(2 $\to$ 3) RCA$_0$ proves that $\Gamma$-$\text{Det}^*_{L/W}$ implies $\Gamma$-$\text{SLO}^*_{L/W}$.

(3 $\to$ 1) Show that if $\mathcal{I}$ wins $G_L^*(A, B) \lor \mathcal{I}$ wins $G_L^*(\neg B, A)$ then 
\[ \{ x : \varphi(x) \in \Sigma^0_1 \} \] exists. Using $\varphi(x)$, we construct $(\Sigma^0_1)_2$ sets $A(f)$ and $B(g)$ s.t. $\mathcal{I}$ cannot have a winning strategy in $G_L^*(\neg B, A)$. Then $\mathcal{I}$ has a winning strategy in $G_L^*(A, B)$, say $\sigma_\mathcal{I}$. We use $\sigma_\mathcal{I}$ to construct a $\Delta^0_1$ formula equivalent to $\varphi(x)$. Finally, 
\[ \{ x : \varphi(x) \} \] exists by $\Delta^0_1$-$\text{CA}_0$. 
Methodology: similar to the Cantor space:

- finite trees $\leftrightarrow$ well-founded (and so ranked) trees
- comparison of maximal lengths $\leftrightarrow$ comparison of tree ranks

(Hirst, 2001) Let $\alpha$ and $\beta$ be countable well orderings. The following are equivalent over $\text{RCA}_0$:

1. $\text{ATR}_0$.
2. $\forall \alpha, \beta (\alpha \leq_w \beta \lor \beta \leq_w \alpha)$.

(Hirst, 2000) $\text{ATR}_0$ proves: well-founded tree $\leftrightarrow$ ranked tree.

(Increasing functions on trees) $S \preceq T$ iff there is a function $f : S \rightarrow T$ such that $s_1 \subset s_2 \rightarrow f(s_1) \subset f(s_2)$.

$\text{ACA}_0$ proves: if $S \preceq T$ then $\text{rk}(S) \leq_w \text{rk}(T)$. 
Theorem
The following are equivalent over $\textbf{ACA}_0$:

1. $\text{ATR}_0$
2. $\Delta^0_1\text{-Det}_L$
3. $\Delta^0_1\text{-SLO}_L$

Proof. (Sketch $3 \rightarrow 1$ )

$I\!I$ wins $G_L(A, B) \lor I\!I$ wins $G_L(\neg B, A) \rightarrow \alpha \leq_w \beta \lor \beta \leq_w \alpha$

\[
\begin{align*}
\alpha, \beta & \rightarrow \text{ranked trees } S(\alpha), T(\beta) \\
\sigma_{\text{II}} & \rightarrow \text{function } f : S(\alpha) \rightarrow T(\beta)
\end{align*}
\]

\[
\begin{align*}
S(\alpha) \leq T(\beta) \\
\alpha \leq_w \beta
\end{align*}
\]
L/W determinacy and SLO in $\mathbb{N}^\mathbb{N}$. Main result

**Theorem**

The following are equivalent over $\text{RCA}_0$:

1. $\text{ATR}_0$
2. $(\Sigma^0_1 \cup \Pi^0_1)$-Det$_L$
3. $\Pi^0_1$-Det$_L$
4. $(\Delta^0_1, \Pi^0_1)$-Det$_L$

**Proof.** (Sketch $4 \rightarrow 1$)

Since we have

$$\Delta^0_1\text{-SLO} + \text{ACA}_0 \rightarrow \text{ATR}_0,$$

it suffices to show

$$(\Delta^0_1, \Pi^0_1)$-Det$_L \rightarrow \text{ACA}_0$$

over the base subsystem $\text{RCA}_0$. 
Concluding remarks

A first step towards a better understanding of the reverse mathematics of L/W determinacy ...

- Reinterpreting Wadge’s classical work in descriptive set theory, we developed a topological analysis of the L/W games for the first levels of the difference hierarchy.

- We showed that this analysis can be carried out within natural subsystems of $\mathbb{Z}_2$.

- Main results: we obtained new reversals for ACA$_0$ and ATR$_0$ in terms of Lipschitz determinacy and SLO.

- In particular, we investigated the reverse mathematics of the salient Semi-Linear Ordering Principle for the first time.
A reversal for $\textbf{WKL}_0$ in terms of $L/W$ determinacy is left pending.

- Natural candidates: $\Delta^0_1\text{-Det}^*_L$ or $\Sigma^0_1\text{-Det}^*_L$.
- Main obstacle: to determine the exact strength of the Dichotomy Principle:

  $$\text{BinaryTree}(T) \rightarrow (\text{TrueClosed}(T) \lor T \text{ defines a clopen set})$$

We developed our analysis of $L/W$ games up to the difference of closed sets. But it seems plausible to extend this analysis to all finite levels of the difference hierarchy.
(Conjecture) The following are equivalent over $\text{RCA}_0$:

1. $\text{ACA}_0$.
2. For each natural number $k$, $(\Sigma^0_1)_k\text{-Det}_L^*$.
3. For each natural number $k$, $(\Sigma^0_1)_k\text{-SLO}_L^*$.

(Conjecture) The following are equivalent over $\text{RCA}_0$:

1. $\Pi^1_1\text{-CA}_0$.
2. For each natural number $k$, $(\Sigma^0_1)_k\text{-Det}_L$.
3. For each natural number $k$, $(\Sigma^0_1)_k\text{-SLO}_L$. 
Lines of future work

- (Andretta, 2003/4) \( \text{ZF} + \text{BP} + \text{DC} \) proves

\[
\text{SLO}_W \iff \text{AD}_W \iff \text{AD}_L \iff \text{SLO}_L.
\]

(L1) To formalize Andretta’s proof within \( \mathbb{Z}_2 \) in order to obtain equivalences between subsystems of second order arithmetic and Wadge determinacy principles.

- (Louveau/St.-Raymond, ’87) \( \mathbb{Z}_2 \) proves full Borel L/W determinacy.

(L2) To study in detail Louveau/St.-Raymond’s proof in order to isolate a natural subsystem of \( \mathbb{Z}_2 \) which can prove full Borel L/W determinacy, and to investigate whether that subsystem would turn out to be actually equivalent to Borel L/W determinacy.