On the Relation Between Various Negative Translations

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Several proof translations of classical mathematics into intuitionistic (or even minimal) mathematics have been proposed in the literature over the past century. These are normally referred to as *negative translations* or *double-negation translations*. Amongst those, the most commonly cited are translations due to Kolmogorov, Gödel, Gentzen, Kuroda and Krivine (in chronological order). In this paper we propose a framework for explaining how these different translations are related to each other. More precisely, we define a notion of a (modular) simplification starting from Kolmogorov translation, which leads to a partial order between different negative translations. In this derived ordering, Kuroda, Krivine and Gödel-Gentzen are minimal elements. A new minimal translation is introduced.

1 Introduction

Several proof translations of classical mathematics into intuitionistic (or even minimal) mathematics have been proposed in the literature over the past century. These are normally referred to as *negative translations* or *double-negation translations*. The first such translation is due to Kolmogorov [21] in 1925. He observed that placing a double negation $\neg\neg$ in front of every subformula turns a classically valid formula into an intuitionistically valid one. Formally, defining

$$\begin{array}{rclrcl} (A \wedge B)^{Ko} & :\equiv & \neg\neg(A^{Ko} \wedge B^{Ko}) & P^{Ko} & :\equiv & \neg\neg P, \quad \text{for } P \text{ atomic} \\ (A \vee B)^{Ko} & :\equiv & \neg\neg(A^{Ko} \vee B^{Ko}) & (\forall x A)^{Ko} & :\equiv & \neg\neg\forall x A^{Ko} \\ (A \to B)^{Ko} & :\equiv & \neg\neg(A^{Ko} \to B^{Ko}) & (\exists x A)^{Ko} & :\equiv & \neg\neg\exists x A^{Ko}, \end{array}$$

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one can show that A is provable classically if and only if A^{Ko} is provable intuitionistically. Kolmogorov's translation, however, was apparently not known to Gödel and Gentzen who both came up with similar translations [9, 10, 12] a few years later. Gentzen's translation (nowadays known as Gödel-Gentzen negative translation [4, 18, 29]) simply places a double negation in front of atomic formulas, disjunctions, and existential quantifiers, i.e.

$$\begin{array}{rcl@{\qquad}rcl} (A \wedge B)^{GG} & :\equiv & A^{GG} \wedge B^{GG} & P^{GG} & :\equiv & \neg\neg P, \quad \text{for } P \text{ atomic} \\ (A \vee B)^{GG} & :\equiv & \neg\neg(A^{GG} \vee B^{GG}) & (\forall x A)^{GG} & :\equiv & \forall x A^{GG} \\ (A \to B)^{GG} & :\equiv & A^{GG} \to B^{GG} & (\exists x A)^{GG} & :\equiv & \neg\neg\exists x A^{GG}. \end{array}$$

As with Kolmogorov's translation, we also have that $\mathsf{CL} \vdash A$ if and only if $\mathsf{IL} \vdash A^{GG}$, where CL and IL stand for classical and intuitionistic logic, respectively. Gödel's suggested translation was in fact somewhere in between Kolmogorov's and Gentzen's, as it also placed a double negation in front of the clause for implication, i.e.

$$(A \to B)^{GG} :\equiv \neg(A^{GG} \wedge \neg B^{GG}) \;\Leftrightarrow_{\mathsf{IL}}\; \neg\neg(A^{GG} \to B^{GG}).$$

In the 1950's, Kuroda revisited the issue of negative translations [23], and proposed a different (somewhat simpler) translation:

$$\begin{array}{rcl@{\qquad}rcl} (A \wedge B)_{Ku} & :\equiv & A_{Ku} \wedge B_{Ku} & P_{Ku} & :\equiv & P, \quad \text{for } P \text{ atomic} \\ (A \vee B)_{Ku} & :\equiv & A_{Ku} \vee B_{Ku} & (\forall x A)_{Ku} & :\equiv & \forall x \neg\neg A_{Ku} \\ (A \to B)_{Ku} & :\equiv & A_{Ku} \to B_{Ku} & (\exists x A)_{Ku} & :\equiv & \exists x A_{Ku}. \end{array}$$

Let $A^{Ku} :\equiv \neg\neg A_{Ku}$. Similarly to Kolmogorov, Gödel and Gentzen, Kuroda showed that $\mathsf{CL} \vdash A$ if and only if $\mathsf{IL} \vdash A^{Ku}$. In particular, if A does not contain universal quantifiers then $\mathsf{CL} \vdash A$ if and only if $\mathsf{IL} \vdash \neg\neg A$, since $(\cdot)_{Ku}$ is the identity mapping on formulas not containing universal quantifiers. Finally, relatively recently, following the work of Krivine [22], yet another different translation was developed[1], namely

$$\begin{array}{rcl@{\qquad}rcl} (A \wedge B)_{Kr} & :\equiv & A_{Kr} \vee B_{Kr} & P_{Kr} & :\equiv & \neg P, \quad \text{for } P \text{ atomic} \\ (A \vee B)_{Kr} & :\equiv & A_{Kr} \wedge B_{Kr} & (\forall x A)_{Kr} & :\equiv & \exists x A_{Kr} \\ (A \to B)_{Kr} & :\equiv & \neg A_{Kr} \wedge B_{Kr} & (\exists x A)_{Kr} & :\equiv & \neg\exists x \neg A_{Kr}. \end{array}$$

Letting $A^{Kr} :\equiv \neg A_{Kr}$, we also have that $\mathsf{CL} \vdash A$ if and only if $\mathsf{IL} \vdash A^{Kr}$. This negative translation in fact already appears implicitly in Shoenfield's classical variant [30] of Gödel's dialectica interpretation [13], as recently observed in [3, 31].

[1]Throughout the paper this translation is going to be called "Krivine negative translation" as currently done in the literature (see [20, 31]) even though it should be better called Streicher-Reus translation. Although inspired by the Krivine's work in [22] it is the syntactical translation studied by Streicher and Reus [32] in a version presented in [3, 31] we are using here.

More than translating CL into IL, it is well known that some negative translations produce embeddings of CL into minimal logic ML (i.e. intuitionistic logic without ex-falso-quodlibet). More precisely

$$\mathsf{CL} \vdash A \quad \text{iff} \quad \mathsf{ML} \vdash A^*,$$

where $* \in \{Ko, GG, Kr\}$, for instance. For Kuroda negative translation, however, we only have $\mathsf{CL} \vdash A$ iff $\mathsf{IL} \vdash A^{Ku}$ (see [34]). Nevertheless, we observe that a small change in Kuroda negative translation produces an embedding in ML. More precisely, defining

$$\begin{array}{rclrcl} (A \wedge B)_{mKu} & :\equiv & A_{mKu} \wedge B_{mKu} & P_{mKu} & :\equiv & P, \quad \text{for } P \text{ atomic} \\ (A \vee B)_{mKu} & :\equiv & A_{mKu} \vee B_{mKu} & (\forall x A)_{mKu} & :\equiv & \forall x \neg\neg A_{mKu} \\ (A \to B)_{mKu} & :\equiv & \neg A_{mKu} \vee B_{mKu} & (\exists x A)_{mKu} & :\equiv & \exists x A_{mKu} \end{array}$$

and letting $A^{mKu} :\equiv \neg\neg A_{mKu}$ we have $\mathsf{CL} \vdash A$ if and only if $\mathsf{ML} \vdash A^{mKu}$. We call the translation $(\cdot)^{mKu}$, minimal Kuroda negative translation.

It is also known that all these translations into IL (or ML respectively) lead to intuitionistically (or minimally) equivalent formulas, in the sense that, for instance, A^{Ko}, A^{GG}, A^{Ku} and A^{Kr} are all provably intuitionistically equivalent. As such, one could say that they are all essentially the same. On the other hand, it is obvious that they are intrinsically different, some being much more expensive in terms of negations than others. The goal of the present paper is to explain the precise sense in which Gödel-Gentzen, Kuroda (or minimal Kuroda) and Krivine translations are systematic simplifications of Kolmogorov's original translation, and show that, in a precise sense, they are optimal (modular) translations of classical logic into intuitionistic (or minimal) logic. A new optimal variant is discussed in Section 5 below.

Till Section 5 we develop our study in the more restricted framework of ML. In Section 6, we show how our analysis can easily be adapted to the framework of IL. Finally on Section 7, we discuss non-modular negative translations and some future and related work.

For more comprehensive surveys on the different negative translations, with more historical background, see [19, 20, 24, 33, 34].

Note. This is an extended version of our Classical Logic and Computation (CL&C) workshop 2010 paper, which appeared in [6]. The main differences to the workshop version are that here all proofs are included, and the analysis of the negative translations is first done over the weaker setting of minimal logic (rather than intuitionistic logic). Moreover, following a suggestion of Ulrich Kohlenbach, we judge the optimality of the translations not just by the number of negations it introduces, but rather by the number of implications introduced (counting a negation as a particular form of implication).

1.1 Some useful results

Our considerations on the different negative translations is based on the fact that formulas with various implications (note that negations are a particular kind of implications) can be simplified to equivalent formulas with fewer implications. The cases when this is (or is not) possible are outlined in the following lemma.

Lemma 1. *The following equivalences are provable in* ML*:*

1. $\neg\neg(\neg\neg A \wedge \neg\neg B) \leftrightarrow \neg\neg(A \wedge B)$
2. $\neg\neg(\neg\neg A \vee \neg\neg B) \leftrightarrow \neg\neg(A \vee B)$
3. $\neg\neg(\neg\neg A \to \neg\neg B) \leftrightarrow \neg\neg(\neg A \vee B)$
4. $\neg\neg\exists x\neg\neg A \leftrightarrow \neg\neg\exists xA$
5. $\neg\neg(\neg A \wedge \neg B) \leftrightarrow \neg(A \vee B)$
6. $\neg\neg(\neg A \vee \neg B) \leftrightarrow \neg(A \wedge B)$
7. $\neg\neg(\neg A \to \neg B) \leftrightarrow \neg(\neg A \wedge B)$
8. $\neg\neg\forall x\neg A \leftrightarrow \neg\exists xA$
9. $\neg\neg(\neg\neg A \wedge \neg\neg B) \leftrightarrow (\neg\neg A \wedge \neg\neg B)$
10. $\neg\neg(\neg\neg A \to \neg\neg B) \leftrightarrow (\neg\neg A \to \neg\neg B)$
11. $\neg\neg\forall x\neg\neg A \leftrightarrow \forall x\neg\neg A$
12. $\neg(\neg\neg A \wedge \neg\neg B) \leftrightarrow (\neg\neg A \to \neg B)$
13. $\neg(\neg\neg A \vee \neg\neg B) \leftrightarrow (\neg A \wedge \neg B)$
14. $\neg(\neg\neg A \to \neg\neg B) \leftrightarrow (\neg\neg A \wedge \neg B)$
15. $\neg\exists x\neg\neg A \leftrightarrow \forall x\neg A$.

The following equivalences are provable in CL *but not in* IL *(and hence not in* ML*):*

16. $\neg\neg\forall x\neg\neg A \leftrightarrow \neg\neg\forall xA$
17. $\neg\neg\exists x\neg A \leftrightarrow \neg\forall xA$
18. $\neg\neg(\neg\neg A \vee \neg\neg B) \leftrightarrow (\neg\neg A \vee \neg\neg B)$
19. $\neg\neg(\neg\neg A \to \neg\neg B) \leftrightarrow (\neg\neg\neg A \vee \neg\neg B)$.
20. $\neg\neg\exists x\neg\neg A \leftrightarrow \exists x\neg\neg A$
21. $\neg\forall x\neg\neg A \leftrightarrow \exists x\neg A$
22. $\neg(\neg\neg A \wedge \neg\neg B) \leftrightarrow (\neg A \vee \neg B)$

The following equivalence is provable in IL *but not in* ML*:*

23. $\neg\neg(\neg\neg A \to \neg\neg B) \leftrightarrow \neg\neg(A \to B)$.

Proof. The fact that 1 – 15 are valid in ML are easy to show directly. Equivalences 4, 8, 11, 15, which involve quantifiers, are in fact discussed in [16]. It is also easy to see that 16 – 22 are classically valid. That 16 – 22 are not valid intuitionistically can be shown by constructing different appropriate Kripke models or using statements already known not to be provable in IL (see [33] pages 324 – 328 and [35] pages 12, 75 – 86). 23 is shown to be provable in IL in [33] (page 9). See also [34, 35]. Finally, we claim that $\neg\neg(\neg\neg A \to \neg\neg B) \to \neg\neg(A \to B)$ is not provable in ML. If it was we could replace falsity $\perp$ by the formula A and the premise would be ML-provable, whereas the conclusion would not as it becomes an instance of Peirce's law. □

1.2 Logical framework

In the language of classical logic and minimal logic ML, we consider as primitive the constants $\bot$, $\top$, the connectives $\wedge$, $\vee$, $\to$ and the quantifiers $\forall$ and $\exists$. We write $\neg A$ as an abbreviation for $A \to \bot$. Note that CL can be formulated using a proper subset of the symbols we consider as primitive. It would be sufficient, for instance, to consider the fragment $\{\bot, \to, \vee, \exists\}$ or $\{\bot, \to, \wedge, \forall\}$ (as adopted by Schwichtenberg in [28]). Our choice of dealing directly with the full set $\{\bot, \top, \to, \wedge, \vee, \forall, \exists\}$ in the classical framework has two main reasons: First, it emphasises which symbols are treated in a similar or different manner in classical and minimal logic. Second, in some embeddings of CL into ML we are going to analyse, the translations of certain formulas are syntactically different to the derived translations we would obtain considering just a subset of primitive symbols. In fact, usually when we choose to work with a subset of the logical connectives in classical logic, we are implicitly committing ourselves to one of the particular negative translations.

2 Modular Translations

Let us first observe that all negative translations mentioned above are in general not optimal – in the sense of introducing the least number of implications (counting negations as implications) in order to turn a classically valid formula into a minimally valid one. For instance, minimal Kuroda translation of a purely universal formula $\forall x P(x)$ is $\neg\neg\forall x \neg\neg P(x)$, whereas Gödel-Gentzen would give the optimal translation $\forall x \neg\neg P(x)$. On the other hand, for purely existential formulas $\exists x P(x)$ we have that Kuroda gives the optimal translation, whereas Gödel-Gentzen introduces unnecessary negations. The important property of all these translations, however, is that they are *modular*, i.e. except for a single non-modular step applied to the whole formula, the translation of a formula is based on the translation of its immediate subformulas. The following definition makes this precise.

Definition 1 (Modular negative translations)**.** *We say that a translation* $(\cdot)^{Tr}$ *from* CL *to* ML *is* modular *if there are formula constructors* $I^{Tr}_{\Box}(\cdot,\cdot)$ *for* $\Box \in \{\wedge, \vee, \to\}$, $I^{Tr}_{Q}(\cdot,\cdot)$ *for* $Q \in \{\forall, \exists\}$, $I^{Tr}_{at}(\cdot)$ *and* $I^{Tr}_{\vdash}(\cdot)$ *called translation of connectives, quantifiers, atomic formulas and the provability sign, respectively, such that for each formula A of* CL*:*

$$A^{Tr} \equiv I^{Tr}_{\vdash}(A_{Tr})$$

where $(\cdot)_{Tr}$ is defined inductively as:

$$\begin{array}{lllllll} (A \wedge B)_{Tr} & :\equiv & I^{Tr}_{\wedge}(A_{Tr}, B_{Tr}) & \quad & P_{Tr} & :\equiv & I^{Tr}_{at}(P), \text{ for } P \text{ atomic} \\ (A \vee B)_{Tr} & :\equiv & I^{Tr}_{\vee}(A_{Tr}, B_{Tr}) & & (\forall x A)_{Tr} & :\equiv & I^{Tr}_{\forall}(x, A_{Tr}) \\ (A \to B)_{Tr} & :\equiv & I^{Tr}_{\to}(A_{Tr}, B_{Tr}) & & (\exists x A)_{Tr} & :\equiv & I^{Tr}_{\exists}(x, A_{Tr}). \end{array}$$

A modular translation is called a negative translation *if (i) $A \leftrightarrow_{\mathsf{CL}} I^{Tr}_{\vdash}(A_{Tr})$ and (ii) $\mathsf{ML} \vdash I^{Tr}_{\vdash}(A_{Tr})$ whenever $\mathsf{CL} \vdash A$.*[2]

For instance, Krivine negative translation is a modular translation with

$$\begin{array}{lllllll} I^{Kr}_{\wedge}(A, B) & :\equiv & A \vee B & \quad & I^{Kr}_{at}(P) & :\equiv & \neg P, \text{ for } P \text{ atomic} \\ I^{Kr}_{\vee}(A, B) & :\equiv & A \wedge B & & I^{Kr}_{\forall}(x, A) & :\equiv & \exists x A \\ I^{Kr}_{\to}(A, B) & :\equiv & \neg A \wedge B & & I^{Kr}_{\exists}(x, A) & :\equiv & \neg \forall x \neg A \end{array}$$

and $I^{Kr}_{\vdash}(A) :\equiv \neg A$. Similarly, one can easily see how Kolmogorov, Gödel-Gentzen, and minimal Kuroda translations are also modular translations.

Definition 2 (Relating modular translations)**.** *We define a relation $\sim$ between modular translations as follows: Given translations T_1 and T_2 we define $T_1 \sim T_2$ if the following equivalences are valid in minimal logic:*

$$\begin{array}{lllllll} I^{T_1}_{\Box}(A, B) & \leftrightarrow_{\mathsf{ML}} & I^{T_2}_{\Box}(A, B) & \quad & I^{T_1}_{at}(P) & \leftrightarrow_{\mathsf{ML}} & I^{T_2}_{at}(P) \\ I^{T_1}_{Q}(x, A) & \leftrightarrow_{\mathsf{ML}} & I^{T_2}_{Q}(x, A) & & I^{T_1}_{\vdash}(A) & \leftrightarrow_{\mathsf{ML}} & I^{T_2}_{\vdash}(A), \end{array}$$

for all formulas A, B, and atomic formulas P, $\Box \in \{\wedge, \vee, \to\}$ and $Q \in \{\forall, \exists\}$.

In other words, two modular translations are related via $\sim$ if the corresponding translations of connectives, quantifiers, atoms and provability are equivalent formulas in ML. It is immediate that $\sim$ is an *equivalence relation*. In what follows we say that two modular translations are the *same* if they are in the same equivalence class for the relation $\sim$ (i.e. they are the same mod $\sim$). When two translations are not the same (in the previous sense), we say they are *different*. Two different translations T_1 and T_2 from CL to ML are said to be *equivalent* if for each formula A, the two translations of A, namely A^{T_1} and A^{T_2}, are equivalent formulas in ML. For instance, changing the clause for $\exists x A$ in the Gödel-Gentzen translation to $(\exists x A)^{GG} :\equiv \neg \forall x \neg A^{GG}$ does not change the interpretation, since we have that $\neg \forall x \neg A$ is equivalent (in ML) to $\neg\neg \exists x A$. So, these would be just two ways of writing the same translation. On the other hand, minimal Kuroda translation is different from Gödel-Gentzen's since, for instance, we do not normally have that $I^{GG}_{\forall}(x, A) \equiv \forall x A$ is equivalent, in minimal logic, to $I^{mKu}_{\forall}(x, A) \equiv \forall x \neg\neg A$.

[2]A negative translation is usually assumed to satisfy a third condition (*iii*) $I^{Tr}_{\vdash}(A_{Tr}) \leftrightarrow_{\mathsf{ML}} B$ for some B constructed from doubly negated atomic formulas by means of $\forall, \wedge, \to, \bot$; ensuring that all negative translations are equivalent (see [33] for negative translations into IL).

3 Simplifications

Noticing that the Gödel-Gentzen negative translation could be reached (in a modular way) from Kolmogorov translation via equivalences in ML, arose the idea of looking for a general strategy covering the standard negative translations.

Thus, our goal is to show that the different negative translations are obtained via a systematic simplification of Kolmogorov translation. For that, we need the concept of "simplification" we define below. Intuitively, the idea of a simplification is to transform formulas into equivalent formulas in minimal logic with fewer implications (counting negations also as implications) *preserving the modularity of the translation*. The reason why our "metric of simplification" counts implications instead of just negations is because the logical complexity of a formula increases with the introduction of implications, as we view a negation as a particular form of implication.

Definition 3 (Simplification from inside/outside)**.** *A* simplification r from inside *is a set of transformations (at most one for each connective and quantifier) of the following form:*

$$\neg\neg(NA \,\square\, NB) \quad \overset{r}{\Rightarrow} \quad N(N_1A \,\square^r N_2B)$$

$$\neg\neg QxNA \quad \overset{r}{\Rightarrow} \quad N(Q^r xN_1A),$$

where $\square, \square^r \in \{\wedge, \vee, \rightarrow\}$*, and* $Q, Q^r \in \{\forall, \exists\}$*,* N *stands for a single or a double negation (same choice in all the set of transformations), and* N_1 *and* N_2 *are negations (possible none and not necessarily the same in all transformations) such that*

(i) both sides are equivalent formulas in ML *and*

(ii) the number of implications (counting negations as implications) on the right side is strictly smaller than on the left side.

A simplification r from outside *is defined in a similar way replacing the shape of the transformation before by*

$$N(\neg\neg A \,\square\, \neg\neg B) \quad \overset{r}{\Rightarrow} \quad N_1NA \,\square^r N_2NB$$

$$NQx\neg\neg A \quad \overset{r}{\Rightarrow} \quad Q^r xN_1NA.$$

Intuitively, in the first case we are moving negations N *outwards* over the outer double negation $\neg\neg$, whereas in the second case we are moving N *inwards* over the inner $\neg\neg$. The moving of negations is done so that we reduce the number of negations and implications on total while keeping the modularity of the translation.

Definition 4 (Maximal simplification)**.** *A simplification is* maximal *if* *(i) it is not properly included in any other simplification, i.e. including new transformations for other connectives prevents the new set of being a simplification, and*

(ii) it is not possible to replace $\Box^r$, Q^r, N_1 and N_2 so as to reduce the number of implications (counting negations as implications) on the right side of any transformation.

Intuitively, a simplification being maximal means that we can not get ride of more negations/implications.

Proposition 1. *Let r_1 and r_2 be the set of transformations:*

$$\begin{array}{lcllcl}
\neg\neg(\neg\neg A \wedge \neg\neg B) & \overset{r_1}{\Rightarrow} & \neg\neg(A \wedge B) & \neg\neg(\neg A \wedge \neg B) & \overset{r_2}{\Rightarrow} & \neg(A \vee B) \\
\neg\neg(\neg\neg A \vee \neg\neg B) & \overset{r_1}{\Rightarrow} & \neg\neg(A \vee B) & \neg\neg(\neg A \vee \neg B) & \overset{r_2}{\Rightarrow} & \neg(A \wedge B) \\
\neg\neg(\neg\neg A \to \neg\neg B) & \overset{r_1}{\Rightarrow} & \neg\neg(\neg A \vee B) & \neg\neg(\neg A \to \neg B) & \overset{r_2}{\Rightarrow} & \neg(\neg A \wedge B) \\
\neg\neg\exists x\neg\neg A & \overset{r_1}{\Rightarrow} & \neg\neg\exists x A, & \neg\neg\forall x\neg A & \overset{r_2}{\Rightarrow} & \neg\exists x A,
\end{array}$$

respectively. The sets r_1 and r_2 are maximal simplifications from inside.

Proof. The transformations in r_1 have the shape of transformations in a simplification from inside. Just take $N :\equiv \neg\neg$, $N_1 :\equiv \neg$ in the third transformation, the other N_1 and the N_2 as being the zero negations, and $\wedge^{r_1} :\equiv \wedge$, $\vee^{r_1} :\equiv \vee$, $\to^{r_1} :\equiv \vee$ and $\exists^{r_1} :\equiv \exists$. Moreover they satisfy the conditions of decreasing the number of implications (counting negations as implications) and of equivalence in ML (see Lemma 1). Therefore r_1 is a simplification from inside. To see that r_1 is a maximal simplification note first that no two transformations for the same connective are allowed in a simplification. Hence, any new transformation would have to have $\neg\neg\forall x\neg\neg A$ on the left-hand side. Neither of the possible formulas for the right-hand side (that we know have at most three negations): $\neg\neg\forall x A$, $\neg\neg\forall x\neg A$, $\neg\neg\exists x A$ and $\neg\neg\exists x\neg A$ is equivalent in ML to the left-hand side. For $\neg\neg\forall x A$ see Lemma 1. So, the set r_1 can not be included properly in any simplification. Secondly, in the three transformations where N_1 and N_2 are already zero no other choice of N_1, N_2, $\wedge^{r_1}$, $\vee^{r_1}$ and $\exists^{r_1}$ would lead to fewer implications. In the third transformation the reduction of implications would just be possible if we replace the right-hand side by $\neg\neg(A \wedge B)$ or $\neg\neg(A \vee B)$. Neither of this two possibilities is a valid option because we lose the required equivalence in ML. Therefore r_1 is a maximal simplification. The case of r_2 can be analysed in a similar way, noticing that $N :\equiv \neg$, $\wedge^{r_2} :\equiv \vee$, $\vee^{r_2} :\equiv \wedge$, $\to^{r_2} :\equiv \wedge$ and $\forall^{r_2} :\equiv \exists$. In this case the transformation for implication has again $N_1 :\equiv \neg$, and alternatives introducing fewer implications would be $\neg(A \wedge B)$ and $\neg(A \vee B)$. In each of the two cases we have no simplifications since none of the formulas is equivalent to $\neg\neg(\neg A \to \neg B)$. ◻

Proposition 2. *Let r_3 and r_4 be the set of transformations:*

$$\begin{array}{lcllcl}
\neg\neg(\neg\neg A \wedge \neg\neg B) & \overset{r_3}{\Rightarrow} & \neg\neg A \wedge \neg\neg B & \neg(\neg\neg A \wedge \neg\neg B) & \overset{r_4}{\Rightarrow} & \neg\neg A \to \neg B \\
\neg\neg(\neg\neg A \to \neg\neg B) & \overset{r_3}{\Rightarrow} & \neg\neg A \to \neg\neg B & \neg(\neg\neg A \vee \neg\neg B) & \overset{r_4}{\Rightarrow} & \neg A \wedge \neg B \\
\neg\neg\forall x\neg\neg A & \overset{r_3}{\Rightarrow} & \forall x\neg\neg A, & \neg(\neg\neg A \to \neg\neg B) & \overset{r_4}{\Rightarrow} & \neg\neg A \wedge \neg B \\
 & & & \neg\exists x\neg\neg A & \overset{r_4}{\Rightarrow} & \forall x\neg A,
\end{array}$$

respectively. The sets r_3 and r_4 are maximal simplifications from outside.

Proof. Taking $N :\equiv \neg\neg$, $\wedge^{r_3} :\equiv \wedge$, $\to^{r_3} :\equiv \to$ and $\forall^{r_3} :\equiv \forall$, we see that the shape of the transformations in the set r_3 is compatible with the shape of the transformations in a simplification from outside. Again, by Lemma 1, we have the equivalences needed and the decreasing of implications also happens. Thus, r_3 is a simplification from outside. Possible extensions of this simplification would have to have as left-hand side the formula $\neg\neg(\neg\neg A \vee \neg\neg B)$ or the formula $\neg\neg\exists x\neg\neg A$. But the former formula is not equivalent in ML to any of the formulas $\neg\neg A \wedge \neg\neg B$, $\neg\neg A \vee \neg\neg B$, $\neg\neg A \to \neg\neg B$, $\neg\neg\neg A \wedge \neg\neg B$, $\neg\neg A \wedge \neg\neg\neg B$, $\neg\neg\neg A \vee \neg\neg B$ and $\neg\neg A \vee \neg\neg\neg B$. And the latter formula is not equivalent in ML to $\forall x\neg\neg A$, $\forall x\neg\neg\neg A$, $\exists x\neg\neg A$, neither to $\exists x\neg\neg\neg A$ (see Lemma 1). So the simplification can not be extended. In terms of introducing fewer implications the only transformation to analyse is the one concerning implication. The two possible cases $\neg\neg A \wedge \neg\neg B$ and $\neg\neg A \vee \neg\neg B$ are not equivalent to $\neg\neg(\neg\neg A \to \neg\neg B)$. So r_3 is a maximal simplification. The set r_4 can be analysed in a similar way, this time taking $N :\equiv \neg$, $\wedge^{r_4} :\equiv \to (N_1 :\equiv \neg)$, $\vee^{r_4} :\equiv \wedge$, $\to^{r_4} :\equiv \wedge\ (N_1 :\equiv \neg)$ and $\exists^{r_4} :\equiv \forall$. Concerning maximality, by Lemma 1, we know that $\neg\forall x\neg\neg A$ is not equivalent in minimal logic to $\exists x\neg A$, so we can not extend the simplification. And it is not possible to reduce the number of implications because neither $\neg(\neg\neg A \wedge \neg\neg B)$ nor $\neg(\neg\neg A \to \neg\neg B)$ is equivalent in ML to a formula with two implications, i.e. of the shape $\neg A \square \neg B$ with $\square$ equal to $\wedge$ or $\vee$ and $\neg(\neg\neg A \wedge \neg\neg B)$ is also not equivalent to any formulas with three implications: $\neg\neg A \vee \neg B$, $\neg A \vee \neg\neg B$, $\neg\neg A \wedge \neg B$, $\neg A \wedge \neg\neg B$ and $\neg A \to \neg B$. □

Proposition 3. *The simplifications r_1, r_2, r_3 and r_4 are the only maximal simplifications.*

Proof. Considering the potential simplifications from outside that are maximal, we can have two cases $N :\equiv \neg\neg$ or $N :\equiv \neg$. In the first case, as can be noticed from the proof of Proposition 2, transformations with left-hand side equal to $\neg\neg(\neg\neg A \vee \neg\neg B)$ or $\neg\neg\exists x\neg\neg A$ can never appear in the simplification. Since in r_3 we have $N_1 = N_2 =$ no negations, for $\wedge$, $\to$ and $\forall$, other maximal simplification would have to have other choices for $\square, N_1, N_2$ for those transformations keeping the number of implications. Obviously changing the connectives in $\wedge$ and $\forall$ we lose the equivalences. In the transformation for $\to$, all the other possible formulas with five implications on the right-hand side are not equivalent in ML to the formula $\neg\neg(\neg\neg A \to \neg\neg B)$ (see Lemma 1). So r_3 is the only maximal simplification from outside with $N :\equiv \neg\neg$. In the second case ($N :\equiv \neg$) we already know that a transformation with left-hand side of the form $\neg\forall x\neg\neg A$ never occurs. We can also check that the only possibilities for the transformations of $\wedge$, $\vee$, $\to$ and $\exists$ giving rise to equivalences in ML (with a minimum number of implications) are the ones in r_4. So r_4 is the only maximal simplification from outside when $N :\equiv \neg$.

Similarly, considering the potential simplifications from inside that are maximal, we can also divide them into two cases: $N :\equiv \neg\neg$ or $N :\equiv \neg$. In the case $N :\equiv \neg\neg$, as showed in the proof of Proposition 1, no transformation with left-hand side of the form $\neg\neg\forall x\neg\neg A$ can appear. For $\wedge$, $\vee$, and $\exists$ any transformation in a maximal simplification has to have N_1, N_2's introducing no negations. Note that this happens with r_1. With this restriction on negations, we can check that the transformations for these connectives presented in r_1 are the only possibilities for a maximal simplification. For $\to$, we already know that any transformation in a maximal simplification should have three implications. The alternatives are $\neg\neg(A \to B)$, $\neg\neg(\neg A \vee B)$, $\neg\neg(\neg A \wedge B)$, $\neg\neg(A \wedge \neg B)$ and $\neg\neg(A \vee \neg B)$. Obviously, the last three formulas are not equivalent in ML to $\neg\neg(\neg\neg A \to \neg\neg B)$ and since the first one is equivalent to $\neg\neg(\neg\neg A \to \neg\neg B)$ in IL but not in ML (see Lemma 1), we conclude that the only possible choice is the one in r_1. Thus r_1 is the only maximal simplification from inside with $N :\equiv \neg\neg$.

In the case $N :\equiv \neg$, and since r_2 is a maximal simplification, we know that no transformation with left-hand side of the form $\neg\neg\exists x\neg A$ can appear. For $\wedge$, $\vee$ and $\forall$ since the transformation in r_2 add no negations nor implication (apart from the one corresponding to N), any other transformation in a maximal simplification has to add no negations or implications either. Because the two sides of a transformation have to be equivalent over ML, the only possibilities are, in fact, the ones in r_2. The transformation that has left-hand side of the form $\neg\neg(\neg A \to \neg B)$, as we proved in Proposition 1, has to have a right-hand side with exactly two implications. Easily we can see that the only possible choice for N_1, N_2 and $\Box$ is $\neg(N_1 A \,\Box\, N_2 B) :\equiv \neg(\neg A \wedge B)$, which is exactly the transformation for $\to$ in r_2. Thus r_2 is the only maximal simplification from inside when $N :\equiv \neg$. $\Box$

4 Kolmogorov Simplified

Definition 3 identifies a class of transformations which can be applied to Kolmogorov negative translation without spoiling the modularity property of the translation. We now present standard ways of simplifying Kolmogorov translation via the maximal (or proper subsets of the maximal) simplifications introduced above.

Definition 5 (Simplification path)**.** Applying a simplification to a formula *A consists in changing the formula through successive steps, applying in each step a transformation allowed by the simplification (i.e. transforming a subformula having the shape of the left-hand side of the transformation by the corresponding right-hand side), till no longer be possible to simplify the expression via that simplification. We call the path of formulas starting in A we obtain this way a* simplification path.

Note that every step in a simplification path acts over a particular connective or quantifier and all formulas in a simplification path are equivalent formulas in ML. The process of applying a simplification is not unique and can lead to different formulas. Nevertheless, all simplification paths are obviously finite since in each step the number of implications is decreasing. From now on, we consider that all simplification paths start with formulas in Kolmogorov form (i.e. formulas of the form A^{Ko}).

Definition 6 (Length of simplification path)**.** *The length of a simplification path P, denoted $s(P)$, is the number of steps in P, or equivalently the number of nodes in P minus one, where by node we refer to each formula in P.*

Clearly, it is not true that two simplification paths with the same length lead to the same formula, i.e. have the same final node. For instance, consider applying simplification r_1 to the formula below in two different ways:

$$\neg\neg(\neg\neg(\neg\neg A \wedge \neg\neg B) \wedge \neg\neg\exists x \neg\neg A)$$

$$\neg\neg(\neg\neg(\neg\neg A \wedge \neg\neg B) \wedge \neg\neg\exists x A) \qquad\qquad \neg\neg(\neg\neg(A \wedge B) \wedge \neg\neg\exists x \neg\neg A)$$

$$\neg\neg((\neg\neg A \wedge \neg\neg B) \wedge \exists x A) \qquad\qquad \neg\neg((A \wedge B) \wedge \exists x \neg\neg A)$$

Nevertheless, we prove that if a simplification is maximal or is a subset of a maximal simplification then the length of the longest paths is determined by the initial formula and, moreover, all the paths with longest length lead to the same formula. In other words, we have a kind of confluence property for longest paths. First some definitions and auxiliary results.

Notation. In order to simplify the formulation of Lemmas 2 and 3 we use the following abbreviations:

- *Removing the double negation from outside* over $\Box$ or Q, with $\Box \in \{\wedge, \rightarrow\}$ and $Q \in \{\forall, \exists\}$ consists in replacing the formula $\neg\neg(\neg\neg A \Box \neg\neg B)$ by $\neg\neg A \Box \neg\neg B$, or replacing $\neg\neg Qx \neg\neg A$ by $Qx \neg\neg A$.

- *Removing the double negations from inside* over $\Box \in \{\wedge, \vee\}$ or Q, stands for replacing $\neg\neg(\neg\neg A \Box \neg\neg B)$ by $\neg\neg(A \Box B)$, or $\neg\neg Qx \neg\neg A$ by $\neg\neg QxA$.

- *Removing the double negation from inside* over $\rightarrow$ consists in replacing the formula $\neg\neg(\neg\neg A \rightarrow \neg\neg B)$ by $\neg\neg(\neg A \vee B)$.

- *Removing single negations (from inside or outside)* over $\Box \in \{\vee, \rightarrow\}$ in the formula $\neg\neg(\neg\neg A \Box \neg\neg B)$ consists in transforming the double negations in single negations, replacing $\Box$ by $\wedge$ and in the case $\Box \equiv \rightarrow$ adding a negation before A. *Removing a single negation (from inside or outside)* over a quantifier symbol Q in the formula $\neg\neg Qx \neg\neg A$ consists in replacing the double negations by single negations and replacing Q by its dual.

- *Removing a single negation from inside (respectively outside)* over $\wedge$ in the formula $\neg\neg(\neg\neg A \wedge \neg\neg B)$ consists in replacing this formula by $\neg(\neg A \vee \neg B)$ (or replacing this formula by $\neg(\neg\neg A \rightarrow \neg B)$ respectively).

We denote by $\#^A_\Box$ and $\#^A_Q$ the number of symbols $\Box$ and Q respectively, occurring in the formula A. For the sake of counting symbols, the negation symbols $\neg$ introduced by the translations are considered as primitive, and hence do not change the value of $\#^A_\rightarrow$. For example $(\#^A_\rightarrow) = (\#^{A^{Ko}}_\rightarrow)$.

Lemma 2. *For the simplification r_1 and for any formula A^{Ko} there is a simplification path P_{r_1} from A^{Ko} such that*

$$s(P_{r_1}) = (\#^{A^{Ko}}_\wedge) + (\#^{A^{Ko}}_\vee) + (\#^{A^{Ko}}_\rightarrow) + (\#^{A^{Ko}}_\exists)$$

and the formula in the last node can be obtained from A^{Ko} locating in this formula all the occurrences of conjunctions, disjunctions, implications and existential quantifications and removing at once all the double negations from inside these connectives and quantifiers.

Any simplification r_1' obtained from r_1 by removing one or more transformations admits a similar result discounting and disregarding the logical symbols in the left-hand side of the transformations removed.

Proof. The proof is by induction on the complexity of the formula A, simplifying first the subformulas and later the more external connectives and quantifiers whenever possible. If A is an atomic formula then $A^{Ko} :\equiv \neg\neg A$ and we can apply no steps. So, the only simplification path is the path with a single node $\neg\neg A$ which satisfies the lemma.

For $A :\equiv B \wedge C$, we know that by induction hypothesis there is a simplification path P_1 from $B^{Ko} :\equiv \neg\neg B'$ such that

$$s(P_1) = (\#_{\wedge}^{B^{Ko}}) + (\#_{\vee}^{B^{Ko}}) + (\#_{\to}^{B^{Ko}}) + (\#_{\exists}^{B^{Ko}})$$

and the last node of P_1 can be obtained from B^{Ko} removing the double negations from inside all the conjunctions, disjunctions, implications and existential quantifications. We denote that formula by $\neg\neg B'_{-}$. Also, by induction hypothesis, there is a path P_2 from $C^{Ko} :\equiv \neg\neg C'$ such that

$$s(P_2) = (\#_{\wedge}^{C^{Ko}}) + (\#_{\vee}^{C^{Ko}}) + (\#_{\to}^{C^{Ko}}) + (\#_{\exists}^{C^{Ko}})$$

and the last node of P_2 can be obtained from C^{Ko} removing the double negations from inside all the conjunctions, disjunctions, implications and existential quantifications. We denote that formula by $\neg\neg C'_{-}$. Consider the following simplification path from $A^{Ko} \equiv (B \wedge C)^{Ko} \equiv \neg\neg(B^{Ko} \wedge C^{Ko}) \equiv \neg\neg(\neg\neg B' \wedge \neg\neg C')$, which incorporates the two paths P_1 and P_2:

$$\begin{array}{c}
\neg\neg(\neg\neg B' \wedge \neg\neg C') \\
\Big| \; P_1 \\
\neg\neg(\neg\neg B'_{-} \wedge \neg\neg C') \\
\Big| \; P_2 \\
\neg\neg(\neg\neg B'_{-} \wedge \neg\neg C'_{-}) \\
\Big| \\
\neg\neg(B'_{-} \wedge C'_{-})
\end{array}$$

This path has length

$$s(P_1) + s(P_2) + 1 = \#_{\wedge}^{B^{Ko}} + \#_{\wedge}^{C^{Ko}} + 1 + \#_{\vee}^{B^{Ko}} + \#_{\vee}^{C^{Ko}} + \#_{\to}^{B^{Ko}} + \#_{\to}^{C^{Ko}} + \#_{\exists}^{B^{Ko}} + \#_{\exists}^{C^{Ko}} = (\#_{\wedge}^{A^{Ko}}) + (\#_{\vee}^{A^{Ko}}) + (\#_{\to}^{A^{Ko}}) + (\#_{\exists}^{A^{Ko}}).$$

And, by induction hypothesis, easily we can see that the formula in the last node coincide with the formula A^{Ko} after removing the double negations from inside the conjunctions, disjunctions, implications and existential quantifications.

The case $A :\equiv B \vee C$, is done in the same way replacing $\wedge$ by $\vee$.

For $A :\equiv B \to C$ the simplification path becomes:

$$
\begin{array}{c}
\neg\neg(\neg\neg B' \to \neg\neg C') \\
\Big| \; P_1 \\
\neg\neg(\neg\neg B'_- \to \neg\neg C') \\
\Big| \; P_2 \\
\neg\neg(\neg\neg B'_- \to \neg\neg C'_-) \\
\Big| \\
\neg\neg(\neg B'_- \vee C'_-)
\end{array}
$$

This path has length

$$s(P_1) + s(P_2) + 1 = \#_\wedge^{B^{Ko}} + \#_\wedge^{C^{Ko}} + \#_\vee^{B^{Ko}} + \#_\vee^{C^{Ko}} + \#_\to^{B^{Ko}} + \#_\to^{C^{Ko}} + 1 + \#_\exists^{B^{Ko}} + \#_\exists^{C^{Ko}} = (\#_\wedge^{A^{Ko}}) + (\#_\vee^{A^{Ko}}) + (\#_\to^{A^{Ko}}) + (\#_\exists^{A^{Ko}}).$$

For $A :\equiv \exists x B$, the strategy is similar considering, by induction hypothesis, that we have the path P_1 from $B^{Ko} :\equiv \neg\neg B'$ in the conditions of the lemma and constructing the simplification path:

$$
\begin{array}{c}
\neg\neg\exists x \neg\neg B' \\
\Big| \; P_1 \\
\neg\neg\exists x \neg\neg B'_- \\
\Big| \\
\neg\neg\exists x B'_-
\end{array}
$$

For $A :\equiv \forall x B$ we just need to take the path P_1 that exists by induction hypothesis:

$$
\begin{array}{c}
\neg\neg\forall x \neg\neg B' \\
\Big| \; P_1 \\
\neg\neg\forall x \neg\neg B'_-
\end{array}
$$

That concludes the proof. □

The proof above in fact provides an algorithm to construct a simplification path for the simplification r with $r \equiv r_1$ or $r \equiv r'_1$. The simplification path from A^{Ko} constructed this way is called *standard path for r*.

Lemma 3. *For the simplifications r_2, r_3, r_4 and for any formula A^{Ko}, there are simplification paths P_{r_2}, P_{r_3}, P_{r_4} such that*

$$s(P_{r_2}) = (\#_{\wedge}^{A^{Ko}}) + (\#_{\vee}^{A^{Ko}}) + (\#_{\rightarrow}^{A^{Ko}}) + (\#_{\forall}^{A^{Ko}}),$$

$$s(P_{r_3}) = (\#_{\wedge}^{A^{Ko}}) + (\#_{\rightarrow}^{A^{Ko}}) + (\#_{\forall}^{A^{Ko}}) \text{ and}$$

$$s(P_{r_4}) = (\#_{\wedge}^{A^{Ko}}) + (\#_{\vee}^{A^{Ko}}) + (\#_{\rightarrow}^{A^{Ko}}) + (\#_{\exists}^{A^{Ko}}).$$

Moreover, in P_{r_2} the last node can be obtained from A^{Ko} removing at once the single negations from inside all the conjunctions, disjunctions, implications and universal quantifications; the formula in the last node in P_{r_3} can be obtained from A^{Ko} by removing at once the double negations from outside the conjunctions, implications and universal quantifications; and the formula in the last node of P_{r_4} can be obtained from A^{Ko} by removing at once the single negations from outside the conjunctions, disjunctions, implications and existential quantifications.

The result can be adapted in the expected way to simplifications obtained from r_2, r_3 or r_4 by removing one or more transformations.

Proof. The proof is similar to the proof of the preceding lemma but, contrarily to the simplifications from inside, the paths for r_3, r_4 and its subsets are obtained transforming first the more external logical symbols and just then the symbols in the proper subformulas. For r_2, we have the following inductive path constructions:

$$\begin{array}{c}
(B \wedge C)^{Ko} \equiv \neg\neg(\neg\neg B' \wedge \neg\neg C') \\
\Big| \; P_1 \\
\neg\neg(\neg B'_- \wedge \neg\neg C') \\
\Big| \; P_2 \\
\neg\neg(\neg B'_- \wedge \neg C'_-) \\
\Big| \\
\neg(B'_- \vee C'_-)
\end{array}
\qquad
\begin{array}{c}
(B \rightarrow C)^{Ko} \equiv \neg\neg(\neg\neg B' \rightarrow \neg\neg C') \\
\Big| \; P_1 \\
\neg\neg(\neg B'_- \rightarrow \neg\neg C') \\
\Big| \; P_2 \\
\neg\neg(\neg B'_- \rightarrow \neg C'_-) \\
\Big| \\
\neg(\neg B'_- \wedge C'_-)
\end{array}$$

The case of $(B \vee C)^{Ko}$ is exactly as $(B \wedge C)^{Ko}$, replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

$$
\begin{array}{c}
(\exists x B)^{Ko} \equiv \neg\neg\exists x \neg\neg B' \\
\Big| \; P_1 \\
\neg\neg\exists x \neg B'_{-}
\end{array}
\qquad\qquad
\begin{array}{c}
(\forall x B)^{Ko} \equiv \neg\neg\forall x \neg\neg B' \\
\Big| \; P_1 \\
\neg\neg\forall x \neg B'_{-} \\
\Big| \\
\neg\exists x B'_{-}
\end{array}
$$

For r_3 we have these other path constructions:

$$
\begin{array}{c}
(B \wedge C)^{Ko} \equiv \neg\neg(\neg\neg B' \wedge \neg\neg C') \\
\Big| \\
\neg\neg B' \wedge \neg\neg C' \\
\Big| \; P_1 \\
B'_{-} \wedge \neg\neg C' \\
\Big| \; P_2 \\
B'_{-} \wedge C'_{-}
\end{array}
\qquad
\begin{array}{c}
(B \vee C)^{Ko} \equiv \neg\neg(\neg\neg B' \vee \neg\neg C') \\
\Big| \; P_1 \\
\neg\neg(B'_{-} \vee \neg\neg C') \\
\Big| \; P_2 \\
\neg\neg(B'_{-} \vee C'_{-})
\end{array}
$$

The case $(B \to C)^{Ko}$ is like the case $(B \wedge C)^{Ko}$, replacing $\wedge$ by $\to$.

$$
\begin{array}{c}
(\exists x B)^{Ko} \equiv \neg\neg\exists x \neg\neg B' \\
\Big| \; P_1 \\
\neg\neg\exists x B'_{-}
\end{array}
\qquad\qquad
\begin{array}{c}
(\forall x B)^{Ko} \equiv \neg\neg\forall x \neg\neg B' \\
\Big| \\
\forall x \neg\neg B' \\
\Big| \; P_1 \\
\forall x B'_{-}
\end{array}
$$

For r_4 we have:

$$
\begin{array}{c}
(B \wedge C)^{Ko} \equiv \neg\neg(\neg\neg B' \wedge \neg\neg C') \\
\Big| \\
\neg(\neg\neg B' \to \neg C') \\
\Big| \; P_1 \\
\neg(\neg B'_{-} \to \neg C') \\
\Big| \; P_2^* \\
\neg(\neg B'_{-} \to C'_{-})
\end{array}
\qquad
\begin{array}{c}
(B \vee C)^{Ko} \equiv \neg\neg(\neg\neg B' \vee \neg\neg C') \\
\Big| \\
\neg(\neg B' \wedge \neg C') \\
\Big| \; P_1^* \\
\neg(B'_{-} \wedge \neg C') \\
\Big| \; P_2^* \\
\neg(B'_{-} \wedge C'_{-})
\end{array}
$$

The case $(B \to C)^{Ko}$ is like the case $(B \wedge C)^{Ko}$, swapping $\wedge$ with $\to$.

$$
\begin{array}{c}
(\forall x B)^{Ko} \equiv \neg\neg\forall x \neg\neg B' \\
\Big| \; P_1 \\
\neg\neg\forall x \neg B'_{-}
\end{array}
\qquad
\begin{array}{c}
(\exists x B)^{Ko} \equiv \neg\neg\exists x \neg\neg B' \\
\Big| \\
\neg\forall x \neg B' \\
\Big| \; P_1^* \\
\neg\forall x B'_{-}
\end{array}
$$

The notation P^* is used in the following sense. We know, by induction hypothesis, that there is a path P from $B^{Ko} :\equiv \neg\neg B'$. The last node of this path results from $\neg\neg B'$ removing the single negations from outside all conjunctions, disjunctions, implications and existential quantifications occurring in B^{Ko}. We can show that the last node has the shape $\neg B'_{-}$, where B'_{-} can possibly start with a negation. Moreover, it is possible to prove that every node in the path P starts with a negation and that removing the starting negation in each step along all path we get a sequence of formulas that can be part of a path, i.e. we get a sequence of valid steps in our simplification. We call this sequence of steps P^*. In the above we are using the fact that, after applying a simplification to a symbol $\square$ or Q, we can no longer apply a simplification to the symbol $\square^{r_4}$ or Q^{r_4}, since at least one of the negations inside is not a double negation and never will became (note that in r_4 the number of negations in each position remains the same or decrease). $\square$

Again, the proof above provides algorithms to construct simplification paths for the simplifications r_2, r_3, r_4 and its subsets. The simplification paths from A^{Ko} constructed via these algorithms are called *standard paths*.

Lemma 4. *If the simplification is a subset of a maximal one, in each step of a simplification path we act over a connective or a quantifier already occurring in the initial formula, and we never act twice over the same connective or quantifier.*

Proof. Let r be a subset of a maximal simplification. It is enough to prove that in each step of a simplification path we never act over $\Box^r$ or Q^r. By Proposition 3, we know that the transformations in r are between the ones in r_1, or between the ones in r_2, or the ones in r_3, or the ones in r_4. In the case of r_1, the formulas $\neg\neg(\neg\neg A \,\Box \neg\neg B)$, with $\Box \in \{\wedge, \vee\}$, $\neg\neg(\neg\neg A \;\to\; \neg\neg B)$ and $\neg\neg\exists x \neg\neg A$ are transformed into $\neg\neg(A \,\Box^r B)$, $\neg\neg(\neg A \;\to^r\; B)$ and $\neg\neg\exists^r x A$ respectively. In all the cases we can no longer apply any transformations over $\Box^r$, $\to^r$ or $\exists^r$ since they do not have (and since the negations in every position are kept or reduced they will never became till the last node with) double negations inside them. The cases of r_2, r_3 and r_4 can be checked in a completely similar way. ◻

Note that, in the previous lemma, the hypothesis of considering just subsets of maximal simplifications is essential. In the example below we present a (non maximal) simplification from inside that contradicts the lemma. Consider the simplification:

$$\neg\neg(\neg A \wedge \neg B) \quad \Rightarrow \quad \neg(A \vee \neg\neg B)$$

$$\neg\neg(\neg A \vee \neg B) \quad \Rightarrow \quad \neg(A \wedge B).$$

From $\neg\neg(\neg\neg A \wedge \neg\neg(\neg\neg B \wedge \neg\neg C))$ we can construct the following two paths:

$$\neg\neg(\neg\neg A \wedge \neg\neg(\neg\neg B \wedge \neg\neg C))$$

$$\neg(\neg A \vee \neg\neg\neg(\neg\neg B \wedge \neg\neg C)) \qquad\qquad \neg\neg(\neg\neg A \wedge \neg(\neg B \vee \neg\neg\neg C))$$

$$\neg(\neg A \vee \neg\neg(\neg B \vee \neg\neg\neg C))$$

$$\neg(\neg A \vee \neg(B \wedge \neg\neg C))$$

The two corollaries below are now immediate:

Corollary 1. *For each formula A^{Ko} and each simplification that is a subset of r_1, r_2, r_3 or r_4, any simplification path from A^{Ko} has length smaller or equal to the length of the corresponding standard path.*

Corollary 2. *If the simplification is a subset of a maximal one, two simplification paths with the longest length lead to the same formula.*

The result above justifies the next definition:

Definition 7. *Let r be a subset of a maximal simplification and A^{Ko} a formula in Kolmogorov form. We denote by $r(A^{Ko})$ the formula in the last node of a simplification path with longest length.*

5 Standard Translations

Simplifying the Kolmogorov negative translation via the maximal simplifications r_1, r_2 and r_3 we obtain exactly minimal Kuroda, Krivine and Gödel-Gentzen negative translations.

Proposition 4. $r_1(A^{Ko}) \equiv A^{mKu}$, $r_2(A^{Ko}) \equiv A^{Kr}$ *and* $r_3(A^{Ko}) \equiv A^{GG}$.

Proof. The proof is done by induction on the complexity of the formula A and in order to reach the formula $r_1(A^{Ko})$ we always assume we are going through the standard path (s.p.).
If A is an atomic formula, then $r_1(A^{Ko}) :\equiv r_1(\neg\neg A) \equiv \neg\neg A :\equiv A^{mKu}$.
For $A :\equiv B \wedge C$, writing B^{Ko} in the form $\neg\neg B'$ and C^{Ko} as $\neg\neg C'$, we know that $r_1(\neg\neg B') \equiv r_1(B^{Ko}) \overset{\text{I.H.}}{\equiv} B^{mKu} \equiv \neg\neg B_{mKu}$ and similarly $r_1(\neg\neg C') \equiv r_1(C^{Ko}) \equiv C^{mKu} \equiv \neg\neg C_{mKu}$. Therefore

$$\begin{aligned} r_1((B \wedge C)^{Ko}) &\equiv r_1(\neg\neg(B^{Ko} \wedge C^{Ko})) \\ &\equiv r_1(\neg\neg(\neg\neg B' \wedge \neg\neg C')) \\ &\overset{\text{s.p.}}{\equiv} \neg\neg(B_{mKu} \wedge C_{mKu}) \\ &\equiv (B \wedge C)^{mKu}. \end{aligned}$$

The case $A :\equiv B \vee C$ can be analysed in a similar way.
For $A :\equiv B \to C$ we have:

$$\begin{aligned} r_1((B \to C)^{Ko}) &\equiv r_1(\neg\neg(B^{Ko} \to C^{Ko})) \\ &\equiv r_1(\neg\neg(\neg\neg B' \to \neg\neg C')) \\ &\overset{\text{s.p.}}{\equiv} \neg\neg(\neg B_{mKu} \vee C_{mKu}) \\ &\equiv (B \to C)^{mKu}. \end{aligned}$$

For the quantifiers we have:

$$r_1((\exists x B)^{Ko}) \equiv r_1(\neg\neg\exists x B^{Ko}) \equiv r_1(\neg\neg\exists x \neg\neg B') \overset{\text{s.p.}}{\equiv} \neg\neg\exists x B_{mKu} \equiv (\exists x B)^{mKu}$$

and

$$r_1((\forall x B)^{Ko}) \equiv r_1(\neg\neg\forall x B^{Ko}) \equiv r_1(\neg\neg\forall x \neg\neg B') \overset{\text{s.p.}}{\equiv} \neg\neg\forall x \neg\neg B_{mKu} \equiv (\forall x B)^{mKu}.$$

Therefore $r_1(A^{Ko}) \equiv A^{mKu}$.

In the case of r_2, the proof is done in a similar way, by induction on the complexity of the formula A considering standard paths.

If A is an atomic formula, then $r_2(A^{Ko}) :\equiv r_2(\neg\neg A) \equiv \neg\neg A \equiv \neg A_{Kr} :\equiv A^{Kr}$.

For $A :\equiv B \wedge C$, we know that $r_2(\neg\neg B') \equiv r_2(B^{Ko}) \overset{\text{I.H.}}{\equiv} B^{Kr} \equiv \neg B_{Kr}$ and similarly $r_2(\neg\neg C') \equiv r_2(C^{Ko}) \equiv C^{Kr} \equiv \neg C_{Kr}$. But then

$$r_2((B \wedge C)^{Ko}) \equiv r_2(\neg\neg(B^{Ko} \wedge C^{Ko})) \equiv r_2(\neg\neg(\neg\neg B' \wedge \neg\neg C')) \overset{\text{s.p.}}{\equiv} \neg(B_{Kr} \vee C_{Kr}) \equiv (B \wedge C)^{Kr}.$$

Analogously, for $A :\equiv B \vee C$ and $A :\equiv B \to C$ we have the following identities:

$$r_2((B \vee C)^{Ko}) \equiv r_2(\neg\neg(B^{Ko} \vee C^{Ko})) \equiv r_2(\neg\neg(\neg\neg B' \vee \neg\neg C')) \overset{\text{s.p.}}{\equiv} \neg(B_{Kr} \wedge C_{Kr}) \equiv (B \vee C)^{Kr}$$

and

$$r_2((B \to C)^{Ko}) \equiv r_2(\neg\neg(B^{Ko} \to C^{Ko})) \equiv r_2(\neg\neg(\neg\neg B' \to \neg\neg C')) \overset{\text{s.p.}}{\equiv} \neg(\neg B_{Kr} \wedge C_{Kr}) \equiv (B \to C)^{Kr}.$$

For the quantifiers we have:

$$r_2((\exists x B)^{Ko}) \equiv r_2(\neg\neg\exists x B^{Ko}) \equiv r_2(\neg\neg\exists x \neg\neg B') \overset{\text{s.p.}}{\equiv} \neg\neg\exists x \neg B_{Kr} \equiv \neg(\exists x B)_{Kr} \equiv (\exists x B)^{Kr}$$

and

$$r_2((\forall x B)^{Ko}) \equiv r_2(\neg\neg\forall x B^{Ko}) \equiv r_2(\neg\neg\forall x \neg\neg B') \overset{\text{s.p.}}{\equiv} \neg\exists x B_{Kr} \equiv (\forall x B)^{Kr}.$$

Thus $r_2(A^{Ko}) \equiv A^{Kr}$.

We illustrate the case r_3 with the conjunction, since the others are completely similar.

$$\begin{aligned} r_3((B \wedge C)^{Ko}) &\equiv r_3(\neg\neg(B^{Ko} \wedge C^{Ko})) \equiv r_3(\neg\neg(\neg\neg B' \wedge \neg\neg C')) \\ &\overset{\text{s.p.}}{\equiv} r_3(\neg\neg B') \wedge r_3(\neg\neg C') \equiv r_3(B^{Ko}) \wedge r_3(C^{Ko}) \\ &\overset{I.H.}{\equiv} B^{GG} \wedge C^{GG} \equiv (B \wedge C)^{GG}. \end{aligned}$$

□

This study concerning maximal simplifications led us not only to the three standard negative translations above but also to the discovery of a new minimal modular embedding from CL to ML. Consider the translation described below:

$$\begin{array}{rclcrcl} (A \wedge B)_E & :\equiv & \neg A_E \to B_E & \quad & P_E & :\equiv & \neg P, \text{ for } P \text{ atomic} \\ (A \vee B)_E & :\equiv & A_E \wedge B_E & & (\forall x A)_E & :\equiv & \neg \forall x \neg A_E \\ (A \to B)_E & :\equiv & \neg A_E \wedge B_E & & (\exists x A)_E & :\equiv & \forall x A_E \end{array}$$

with $A^E :\equiv \neg A_E$, which is similar to Krivine except that negations are introduced in the $\{\wedge, \forall\}$-clauses whereas Krivine introduces negations on the $\exists$-clause.

Immediately as a corollary of the next proposition, we have that the translation $(\cdot)^E$ is an embedding from CL to ML, different but equivalent to the standard embeddings considered previously.

Proposition 5. $r_4(A^{Ko}) \equiv A^E$.

Proof. We just sketch the case of conjunction $A :\equiv B \wedge C$. The other cases can be done using the same strategy.
Take $B^{Ko} :\equiv \neg\neg B'$ and $C^{Ko} :\equiv \neg\neg C'$. Consider, by induction hypothesis, that $r_4(\neg\neg B') \equiv r_4(B^{Ko}) \overset{\text{I.H.}}{\equiv} B^E \equiv \neg B_E$ and $r_4(\neg\neg C') \equiv r_4(C^{Ko}) \equiv B^E \equiv \neg C_E$. Then

$$\begin{array}{rcl} r_4((B \wedge C)^{Ko}) & \equiv & r_4(\neg\neg(B^{Ko} \wedge C^{Ko})) \equiv r_4(\neg\neg(\neg\neg B' \wedge \neg\neg C')) \\ & \overset{\text{s.p.}}{\equiv} & \neg(\neg B_E \to C_E) \equiv \neg(B \wedge C)_E \equiv (B \wedge C)^E. \end{array}$$

That concludes the proof. □

6 Simplifications over IL

In the previous section we analysed the various negative translation of classical logic CL into *minimal logic* ML. In the present section we shall also consider translations that map into *intuitionistic logic*. In this (less strict) framework, a negative translation is an embedding from CL into IL (not necessarily ML) and simplifications are based on equivalences in IL (not necessarily ML). Working over the stronger system of IL will mean that more equivalences are provable, which in turn will mean that new maximal simplifications are possible.

Obviously, in this context, r_1, r_2, r_3 and r_4 are still simplifications because an equivalence provable in ML is also provable in IL. Using a strategy similar to the one applied on Section 3 in the proofs of Propositions 1, 2, and 3, we can also see that r_1, r_2, r_3 and r_4 are maximal simplifications, but there is a fifth one. The simplification r'_1 (that only differs from r_1 in the transformation for $\to$) defined below

$$\begin{array}{rcl} \neg\neg(\neg\neg A \wedge \neg\neg B) & \overset{r'_1}{\Rightarrow} & \neg\neg(A \wedge B) \\ \neg\neg(\neg\neg A \vee \neg\neg B) & \overset{r'_1}{\Rightarrow} & \neg\neg(A \vee B) \\ \neg\neg(\neg\neg A \to \neg\neg B) & \overset{r'_1}{\Rightarrow} & \neg\neg(A \to B) \\ \neg\neg\exists x \neg\neg A & \overset{r'_1}{\Rightarrow} & \neg\neg\exists x A \end{array}$$

is also a maximal simplification from inside. This appears as no surprise since by Lemma 1, we know that $\neg\neg(\neg\neg A \to \neg\neg B) \leftrightarrow \neg\neg(A \to B)$ in IL but not in ML.

Mimicking Sections 4 and 5, this time in the context of IL, apart from obtaining minimal Kuroda, Krivine, Gödel-Gentzen and the new $(\cdot)^E$ negative translation as maximal simplifications of the Kolmogorov negative translation via r_1, r_2, r_3 and r_4 respectively, we also have the following result:

Proposition 6. $r'_1(A^{Ko}) \equiv A^{Ku}$.

Proof. As in Proposition 4, the proof is done by induction on the complexity of the formula A considering standard paths.

Since the only case that differs is $A :\equiv B \to C$, we sketch it below:

For $A :\equiv B \to C$, writing B^{Ko} in the form $\neg\neg B'$ and C^{Ko} as $\neg\neg C'$, we know that $r'_1(\neg\neg B') \equiv r'_1(B^{Ko}) \overset{\text{I.H.}}{\equiv} B^{Ku} \equiv \neg\neg B_{Ku}$ and similarly $r'_1(\neg\neg C') \equiv r'_1(C^{Ko}) \equiv C^{Ku} \equiv \neg\neg C_{Ku}$. Therefore

$$\begin{array}{rcl} r'_1((B \to C)^{Ko}) & \equiv & r'_1(\neg\neg(B^{Ko} \to C^{Ko})) \\ & \equiv & r'_1(\neg\neg(\neg\neg B' \to \neg\neg C')) \\ & \overset{\text{s.p.}}{\equiv} & \neg\neg(B_{Ku} \to C_{Ku}) \\ & \equiv & (B \to C)^{Ku}. \end{array}$$

□

Hence, simplifying the Kolmogorov negative translation via the (maximal in IL) simplification r'_1 we obtain exactly Kuroda negative translation.

As previously observed, Kuroda negative translation is a translation from CL into IL, not into ML. But a small change in the translation of implications, namely $(A \to B)^{mKu} :\equiv \neg\neg(\neg A_{mKu} \vee B_{mKu})$ produces a negative translation not only in IL

but also in ML. Both translations $(\cdot)^{Ku}$ and $(\cdot)^{mKu}$ are minimal elements in the partial order induced by the simplifications (the former in IL via r_1' and the latter in ML and IL via r_1). In what follows, we present another way of changing Kuroda negative translation so as to obtain an embedding into ML. Consider that we change in r_1 the clause for implication to

$$\neg\neg(\neg\neg A \to \neg\neg B) \overset{\tilde{r}_1}{\Rightarrow} \neg\neg(A \to \neg\neg B).$$

This way, we obtain a non-maximal simplification (in IL) which corresponds to a modular translation $(\cdot)^{\tilde{K}u}$ between Kolmogorov and Kuroda negative translations. Since $\neg\neg(\neg\neg A \to \neg\neg B) \leftrightarrow_{\mathsf{ML}} \neg\neg(A \to \neg\neg B)$, $\tilde{r}_1$ is also a (non-maximal) simplification in ML. Therefore, the modular translation $(\cdot)^{\tilde{K}u}$ that inserts $\neg\neg$ in (*i*) the beginning of the formula, (*ii*) after each universal quantifier, and (*iii*) in front of the conclusion of each implication is such that $\mathsf{CL} \vdash A$ iff $\mathsf{ML} \vdash A^{\tilde{K}u}$. This is the variant of Kuroda considered by Murthy in [25].

7 Final remarks

We conclude with a few remarks on other negative translations, some related work and avenues for further research.

7.1 On non-modular negative translations

Working with *modular* translations brings various benefits. For instance, we can prove properties of the translation by a simple induction on the structure of the formulas, and when applying the translation to concrete proofs this can be done in a modular fashion. On the other hand, if we allow a translation to be non-modular, we can of course construct simpler embeddings, i.e. we can simplify Kolmogorov negative translation even more, getting ride of more implications.

For example, consider the simplification r_3 followed by one more transformation $\neg\neg\exists x \neg\neg A \Rightarrow \neg\forall x \neg A$ to be applied, whenever possible, at the end of the simplification path. As such we could first simplify $\neg\neg(\neg\neg A \wedge \neg\neg\exists x \neg\neg B)$ using r_3 to the formula $\neg\neg A \wedge \neg\neg\exists x \neg\neg B$ and then apply the final simplification to obtain $\neg\neg A \wedge \neg\forall x \neg B$. Although non-modular, these kind of procedures also give rise to translations of classical into minimal or intuitionistic logic (depending on the framework ML or IL of the simplifications).

Avigad [2] presented a more sophisticated non-modular translation of CL into IL that results from a fragment of r_1', avoiding unnecessary negations. More precisely, Avigad's M-translation is defined as:

$$\begin{array}{rcl@{\qquad}rcl} (A \wedge B)^M & :\equiv & \neg(\sim A \vee \sim B)^M & P^M & :\equiv & P, \text{ for } P \text{ atomic} \\ (A \vee B)^M & :\equiv & A^M \vee B^M & \bar{P}^M & :\equiv & \neg P \\ (\forall x A)^M & :\equiv & \neg(\exists x \sim A)^M & (\exists x A)^M & :\equiv & \exists x A^M, \end{array}$$

where in classical logic we consider the negations of atomic formulas $\bar{P}$ as primitive and the formula $\sim A$ is obtained from A replacing $\wedge$, $\forall$, P respectively by $\vee$, $\exists$ and $\bar{P}$ and conversely. Avigad showed that

(1) $\vdash_{\mathsf{IL}} \neg A^M \leftrightarrow \neg A^S$

(2) If $\vdash_{\mathsf{CL}} A$ then $\vdash_{\mathsf{IL}} \neg(\sim A)^M$,

where A^S stands for any of the standard equivalent translations from CL into IL mentioned before such as Gödel-Gentzen, Kolmogorov, Kuroda or Krivine negative translation.

Lemma 5. $\neg(\sim A)^M \leftrightarrow_{\mathsf{IL}} \neg\neg A^M$

Proof. The proof follows from an easy analysis of all the possibilities for the formula A. If A is an atomic formula then $\neg(\sim A)^M :\equiv \neg\bar{A}^M :\equiv \neg\neg A :\equiv \neg\neg A^M$.
For $A :\equiv \bar{P}$, we have $\neg(\sim \bar{P})^M :\equiv \neg P^M :\equiv \neg P \leftrightarrow \neg\neg\neg P \leftrightarrow \neg\neg\bar{P}^M$.
If $A :\equiv B \wedge C$ then $\neg(\sim (B \wedge C)^M :\equiv \neg(\sim B \vee \sim C)^M \leftrightarrow \neg\neg\neg(\sim B \vee \sim C)^M :\equiv \neg\neg(B \wedge C)^M$.
The disjunction case $\neg(\sim (B \vee C))^M :\equiv \neg(\sim B \wedge \sim C)^M :\equiv \neg\neg(\sim\sim B \vee \sim\sim C)^M :\equiv \neg\neg(B \vee C)^M$.
The quantifications are studied below:

$$\neg(\sim \forall x B)^M :\equiv \neg(\exists x \sim B)^M \leftrightarrow \neg\neg\neg(\exists x \sim B)^M :\equiv \neg\neg(\forall x B)^M,$$

and

$$\neg(\sim \exists x B)^M :\equiv \neg(\forall x \sim B)^M :\equiv \neg\neg(\exists x \sim\sim B)^M \neg\neg(\exists x B)^M.$$

That concludes the proof. □

Although the translation $(\cdot)^M$, as presented by Avigad, is not modular, notice that it can be equivalently written in a modular way as

$$\begin{array}{rcl@{\qquad}rcl} (A \wedge B)^{M'} & :\equiv & \neg\neg A^{M'} \wedge \neg\neg B^{M'} & P^{M'} & :\equiv & P, \text{ for } P \text{ atomic} \\ (A \vee B)^{M'} & :\equiv & A^{M'} \vee B^{M'} & \bar{P}^{M'} & :\equiv & \neg P \\ (\forall x A)^{M'} & :\equiv & \forall x \neg\neg A^{M'} & (\exists x A)^{M'} & :\equiv & \exists x A^{M'}, \end{array}$$

since $(\forall x A)^M :\equiv \neg(\exists x \sim A)^M :\equiv \neg\exists x((\sim A)^M) \leftrightarrow_{\mathsf{IL}} \forall x \neg(\sim A)^M \overset{\text{(Lemma 5)}}{\leftrightarrow}_{\mathsf{IL}} \forall x \neg\neg A^M$ and

$$\begin{aligned}(A \wedge B)^M \quad &:\equiv \quad \neg(\sim A \vee \sim B)^M :\equiv \neg((\sim A)^M \vee (\sim B)^M) \\ &\leftrightarrow_{\mathsf{IL}} \quad \neg(\sim A)^M \wedge \neg(\sim B)^M \overset{\text{(Lemma 5)}}{\leftrightarrow}_{\mathsf{IL}} \neg\neg A^M \wedge \neg\neg B^M.\end{aligned}$$

The translation $(\cdot)^{M'}$ can be obtained from Kolmogorov negative translation via a non-maximal simplification, more precisely the simplification r_1' (corresponding to Kuroda translation) without the transformation $\neg\neg(\neg\neg A \wedge \neg\neg B) \overset{r_1'}{\Rightarrow} \neg\neg(A \wedge B)$.

Avigad's translation $(\cdot)^M$ is a *non-modular* simplification of $(\cdot)^{M'}$ since for universal quantifications, for conjunctions and for provability we replace $\neg\neg A^M$ by $\neg(\sim A)^M$ which, although equivalent, has possibly fewer negations, as we see in the proof of Lemma 5. Moreover, as pointed by Avigad in [2], we can simplify the translation $(\cdot)^M$ even further defining $(A \wedge B)^M$ as being $A^M \wedge B^M$. The corresponding modular version in this case is exactly Kuroda negative translation.

7.2 On Gödel-Gentzen negative translation

Although nowadays it is common to name the translation $(\cdot)^{GG}$, presented in Section 1, by *Gödel-Gentzen negative translation*, a few remarks should be made at this point. The translations due to Gödel and Gentzen ([12] and [10], respectively) where introduced in the context of number theory translating an atomic formula P into P itself. Later Kleene [19] considered the translation of the pure logical part, observing that double-negating atomic formulas was necessary, since one does not have stability $\neg\neg P \to P$ in general.

Rigorously, Gentzen's original formulation instead of double negating disjunctions and existential quantifiers used the following equivalent (in IL and ML) definitions $(A \vee B)^{GG} :\equiv \neg(\neg A^{GG} \wedge \neg B^{GG})$ and $\exists x A^{GG} :\equiv \neg\forall x \neg A^{GG}$, since, as such, one can then work in the $\{\exists, \vee\}$-free fragment of minimal or intuitionistic logic.

Moreover, as pointed in Section 1 already, Gödel's original double-negation translation differs from Gentzen's negative translation in the way implication is treated. In the context of IL, Gödel's translation also introduces a double negation before implications. We can easily see, that this translation can be obtained from Kolmogorov negative translation via the non-maximal simplification consisting in r_3 without the transformation $\neg\neg(\neg\neg A \to \neg\neg B) \Rightarrow \neg\neg A \to \neg\neg B$, being, therefore, more expensive in term of implications than Gentzen's negative translation. Another non-maximal simplification, more precisely r_3 without the transformation $\neg\neg(\neg\neg A \wedge \neg\neg B) \Rightarrow \neg\neg A \wedge \neg\neg B$, leads to Aczel's $(\cdot)^N$ variant [1].

Finally, we observe that sometimes in Kolmogorov or Gödel-Gentzen negative translations, $\perp$ is transformed differently from the other atomic formulas, not into

$\neg\neg\bot$ but into $\bot$ itself. This change is easily adapted to our framework, considering in the modular definition of a translation an extra operator $I_{\bot}^{Tr}(\bot)$ and defining $\bot_{Tr} :\equiv I_{\bot}^{Tr}(\bot)$. Note that the translations where $I_{\bot}^{Tr}(\bot) :\equiv \bot$ are the same as the ones with $I_{\bot}^{Tr}(\bot) :\equiv \neg\neg\bot$, since $\bot \leftrightarrow \neg\neg\bot$ in IL and ML.

7.3 Translations from IL into ILL

In the present paper, we saw that the standard translations from CL to IL result from systematic simplifications on Kolmogorov negative translation. Motivated by this idea, we observe that something similar can be said about the embeddings from intuitionistic logic to intuitionistic linear logic (ILL). In the linear framework, and replacing the moves of the double negations from inside or outside by moves of the exponential ! (whenever allowed by linear equivalences), we obtain the standard Girard $(\cdot)^*$ and $(\cdot)^\circ$-translation from a Kolmogorov-like translation from IL into ILL.

We start by reminding the reader of the Girard translations $(\cdot)^*$ and $(\cdot)^\circ$ [11]

$$
\begin{array}{llll}
P^* & :\equiv P & P^\circ & :\equiv\, !P, \quad \text{for } P \text{ atomic}, P \not\equiv \bot \\
\bot^* & :\equiv 0 & \bot^\circ & :\equiv 0 \\
(A \wedge B)^* & :\equiv A^* \,\&\, B^* & (A \wedge B)^\circ & :\equiv A^\circ \otimes B^\circ \\
(A \vee B)^* & :\equiv\, !A^* \oplus\, !B^* & (A \vee B)^\circ & :\equiv A^\circ \oplus B^\circ \\
(A \to B)^* & :\equiv\, !A^* \multimap B^* & (A \to B)^\circ & :\equiv\, !(A^\circ \multimap B^\circ) \\
(\forall x A)^* & :\equiv \forall x A^* & (\forall x A)^\circ & :\equiv\, !\forall x A^\circ \\
(\exists x A)^* & :\equiv \exists x !A^* & (\exists x A)^\circ & :\equiv \exists x A^\circ
\end{array}
$$

which satisfy the following: if $\mathsf{IL} \vdash A$ then $\mathsf{ILL} \vdash A^*$ and $\mathsf{ILL} \vdash A^\circ$. Consider also the following translation, which we denote by $(\cdot)^{lKo}$ since it mimics the Kolmogorov approach:

$$
\begin{array}{rcl}
P^{lKo} & :\equiv & !P, \text{ for } P \text{ atomic} \\
(A \wedge B)^{lKo} & :\equiv & !(A^{lKo} \otimes B^{lKo}) \\
(A \vee B)^{lKo} & :\equiv & !(A^{lKo} \oplus B^{lKo}) \\
(A \to B)^{lKo} & :\equiv & !(A^{lKo} \multimap B^{lKo}) \\
(QA)^{lKo} & :\equiv & !QA^{lKo}, \quad \text{for } Q \in \{\forall x, \exists x\}.
\end{array}
$$

One also has:

Proposition 7. *If* $\mathsf{IL} \vdash A$ *then* $\mathsf{ILL} \vdash A^{lKo}$.

This result, however, follows (as we are going to see throughout this section) from the homologous results for $(\cdot)^*$ and $(\cdot)^\circ$-translations. Not surprisingly, $(\cdot)^{lKo}$ is an unpolished translation, i.e. it is possible to simplify it by removing some *bangs* while still maintaining an equivalent (over intuitionistic linear logic) embedding.

Mimicking the simplifications from outside of Sections 3 and 4, start with A^{lKo} and systematically remove, from outside to inside the formula, the exponential ! from all multiplicative conjunctions, disjunctions, existential quantifications and before 0. Example: if $!(!A \otimes !B)$ appears in the formula we change it to $!A \otimes !B$.

Lemma 6. *The following equivalences are provable in* ILL*:*

$$!0 \multimap\!\!\circ\; 0$$

$$!(!A \otimes !B) \multimap\!\!\circ\; !A \otimes !B$$

$$!(!A \oplus !B) \multimap\!\!\circ\; !A \oplus !B$$

$$!\exists x !A \multimap\!\!\circ\; \exists x !A.$$

It is easy to see that with the previous strategy we obtain exactly Girard's embedding A°. Once we notice that $A^{lKo} \multimap\!\!\circ\; !A^{lKo}$ and $\mathsf{ILL} \vdash !(!A \otimes !B) \multimap\!\!\circ\; !(!A \,\&\, !B)$, we realise that in $(\cdot)^{lKo}$ we can define the translation of $A \wedge B$ by $!(A^{lKo} \,\&\, B^{lKo})$.

On the other hand, mimicking the simplifications from inside, start with A^{lKo} and systematically remove, from inside to outside the formula, the exponential ! from all additive conjunctions, universal quantifications and in the consequent of the implications. Example: if $!(!A \,\&\, !B)$ appears in the formula we change it to $!(A \,\&\, B)$. The next lemma justifies this approach:

Lemma 7. *The following equivalences are provable in* ILL*:*

$$!(!A \,\&\, !B) \multimap\!\!\circ\; !(A \,\&\, B)$$

$$!(!A \multimap !B) \multimap\!\!\circ\; !(!A \multimap B)$$

$$!\forall x !A \multimap\!\!\circ\; !\forall x A.$$

Again, it is easy to see that applying the strategy above to A^{lKo} we obtain exactly $!A^*$.

7.4 Other related work

Strong monads. Part of the present study could have been developed in a more general context, as done in [5]. Let T be a (logical operator having the properties of a) strong monad and consider the translation $(\cdot)^{\mathsf{T}}$ that inserts T in the beginning of each subformula. Assuming that $(\mathsf{T}A)^{\mathsf{T}} \leftrightarrow \mathsf{T}A^{\mathsf{T}}$ what we obtain is a translation of $\mathsf{ML} + (\mathsf{T}A \to A)$ into ML. We name such embedding *Kolmogorov*

T-*translation*. It can be seen that all the transformations in simplifications $\tilde{r}_1$ and r_3 remain valid equivalences in ML when we replace $\neg\neg$ by any strong monad T. Thus, from Kolmogorov T-translation we can obtain, by means of the previous simplifications, the corresponding Kuroda (non-maximal ML variant) and Gödel-Gentzen T-translations. As particular cases we have

- $\mathsf{T}A :\equiv \neg\neg A$ (recovering the standard double-negation translations),
- $\mathsf{T}B :\equiv (B \to A) \to A$ (corresponding to Friedman A-translations [7]),
- $\mathsf{T}A :\equiv \neg A \to A$ or $\mathsf{T}A :\equiv (A \to R) \to A$ (Peirce translations [5]).

As references on these more general embeddings see [1, 5].

Semantical approaches. In this paper we did not discuss semantical approaches to the negative translations. Some considerations concerning conversions between Heyting and Boolean algebras whose valuation of formulas is related via negative translations can be found in [14, 27] and more abstract treatment of negative translations in terms of categorical logic can be found in [17].

Other "metrics". The key ingredient to establish a partial order between negative translations here is the notion of a simplification. The definition of simplification used throughout this study is based on the counting of the total number of implications involved in a formula. More sophisticated "metrics" could be tried in the future, for instance one that instead of just counting implications could be sensible to the "nesting" effect. For instance, an immediate consequence of considering the "nesting" of implications would be that r_4 would no longer be a simplification. Notice that the transformation $\neg(\neg\neg A \wedge \neg\neg B) \overset{r_4}{\Rightarrow} \neg\neg A \to \neg B$, would have depth three on both the left and the right-hand sides.

Translations from CL to CLL. Although not addressed in this paper, we could try to adapt the notion of simplification to translations from classical logic into classical linear logic. As future work we intend to focus in this question not only to capture and motivate standard translations such as Girard ?!-translation [11], but to see which new translations could be revealed with this approach.

CPS **transformations**. There is a close connection between negative translations and *continuation passing style* (CPS) transformations. In the literature [8, 15, 26, 32], we can find various CPS-translations from $\lambda\mu$-calculus into λ-calculus that correspond (at the type level) to the standard negative translations.

Note that the CPS technique captures evaluation ordering for the source language, such as call-by-name, call-by-value or call-by-need. The two schemes below sketch how Kolmogorov negative translation $(\cdot)^{Ko}$ simulates call-by-name in a call-by-value interpreter and Kuroda negative translation $(\cdot)^{\tilde{K}u}$ simulates call-by-value in a call-by-name interpreter. Consider first the proof-tree of the soundness for Kolmogorov translation of the cut rule

$$
\dfrac{\dfrac{\dfrac{\dfrac{[A^{Ko} \to B^{Ko}]\colon \alpha \qquad A^{Ko}\colon N}{B^{Ko}\colon \alpha N} \qquad [\neg B^*]\colon k}{\dfrac{\bot\colon \alpha N k}{\neg(A^{Ko} \to B^{Ko})\colon \lambda\alpha.\alpha N k}} \qquad \neg\neg(A^{Ko} \to B^{Ko})\colon M}{\bot\colon M(\lambda\alpha.\alpha N k)}}{\neg\neg B^*\colon \lambda k.M(\lambda\alpha.\alpha N k)}
$$

where $(A \to B)^{Ko} :\equiv \neg\neg(A^{Ko} \to B^{Ko})$ and B^* is such that $B^{Ko} \equiv \neg\neg B^*$. If the strategy used is call-by-value the term N which proves A would be first evaluated and then passed to the function M which proves $A \to B$. After the Kolmogorov translation, however, N (the proof of A) is encapsulated into a λ-term in head normal form. That then forces the evaluation of M (the proof of $A \to B$) instead, hence, simulating call-by-name.

On the other hand, consider the proof-tree of the soundness of the Kuroda translation of the cut rule

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A_{\tilde{K}u}]\colon a \qquad [A_{\tilde{K}u} \to \neg\neg B_{\tilde{K}u}]\colon \alpha}{\neg\neg B_{\tilde{K}u}\colon \alpha a} \qquad [\neg B_{\tilde{K}u}]\colon k}{\dfrac{\bot\colon \alpha a k}{\neg A_{\tilde{K}u}\colon \lambda a.\alpha a k}} \qquad \neg\neg A_{\tilde{K}u}\colon N}{\bot\colon N(\lambda a.\alpha a k)}}{\neg(A_{\tilde{K}u} \to \neg\neg B_{\tilde{K}u})\colon \lambda\alpha.N(\lambda a.\alpha a k)} \qquad \neg\neg(A_{\tilde{K}u} \to \neg\neg B_{\tilde{K}u}) : M}{\bot : M(\lambda\alpha.N(\lambda a.\alpha a k))}}{\neg\neg B_{\tilde{K}u} : \lambda k.M(\lambda\alpha.N(\lambda a.\alpha a k))}}{}
$$

where $(A \to B)^{\tilde{K}u} :\equiv \neg\neg(A \to B)_{\tilde{K}u} :\equiv \neg\neg(A_{\tilde{K}u} \to \neg\neg B_{\tilde{K}u})$ is the Kuroda negative translation presented on Section 6. In a call-by-name setting the proof of $A \to B$ would be the first to be evaluated. In order to force the argument (the proof of A) to be evaluated first the Kuroda translation puts N (the proof of the translation of A) in the function position with input $\lambda a.\alpha a k$. That forces the evaluation of N (the proof of A) first, hence simulating call-by-value.

For more on this subject see [26] and Chapters 9 and 10 in [25]. It would be interesting to see if our simplifications linking the standard negative translations can be expressed and are meaningful at the calculus reduction strategy level.

References

[1] P. Aczel (2001): *The Russel-Prawitz modality*. Math. Structures Comput. Sci. 11(4), pp. 541–554.

[2] J. Avigad (2000): *A realizability interpretation for Classical Arithmetic*. In: P Häjek S. R. Buss & P. Pudläk, editors: Logic Colloquium'98, Lecture Notes in Logic 13, AK Peters, pp. 57–90.

[3] J. Avigad (2006): *A variant of the double-negation translation*. Technical Report 179, Carnegie Mellon Technical Report CMU-PHIL.

[4] J. Avigad & S. Feferman (1998): *Gödel's functional ("Dialectica") interpretation*. In: S. R. Buss, editor: Handbook of proof theory, Studies in Logic and the Foundations of Mathematics 137, North Holland, Amsterdam, pp. 337–405.

[5] M. H. Escardó & P. Oliva (2010): *The Peirce Translation and the Double Negation Shift*. In: F. Ferreira, B. Löwe, E. Mayordomo & L. M. Gomes, editors: Programs, Proofs, Processes - CiE 2010, LNCS 6158, Springer, pp. 151–161.

[6] G. Ferreira & P. Oliva (2011): *On various negative translations*. In: Proceedings of Classical Logic and Computation 2010, Electronic Proceedings in Theoretical Computer Science 47, EPTCS, pp. 21–33.

[7] H. Friedman (1978): *Classically and intuitionistically provably recursive functions*. In: D. Scott & G. Müller, editors: Higher Set Theory, Lecture Notes in Mathematics 669, Springer, Berlin, pp. 21–28.

[8] K. Fujita (1995): *On embedding of classical substructural logics*. In: Proc. Theory of Rewriting Systems and Its Applications, Kyoto University 918, RIMS, pp. 178–195.

[9] G. Gentzen (1933): *Ueber das verhältnis zwischen intuitionistischer und klassischer arithmetik, Galley proof (received in 1933)*. Mathematische Annalen .

[10] G. Gentzen (1936): *Die Widerspruchsfreiheit der reinen Zahlentheorie*. Mathematische Annalen 112, pp. 493–565.

[11] J.-Y. Girard (1987): *Linear Logic*. Theoretical Computer Science 50(1), pp. 1–102.

[12] K. Gödel (1933): *Zur intuitionistischen Arithmetik und Zahlentheorie*. Ergebnisse eines Mathematischen Kolloquiums 4, pp. 34–38.

[13] K. Gödel (1958): *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*. Dialectica 12, pp. 280–287.

[14] K. Gödel (1986): *Introduction to paper "Zur intuitionistischen Arithmetik und Zahlentheorie"*. In: S. Feferman et al., editor: Collected Works, Vol. I, Oxford University Press, Oxford, pp. 286–295.

[15] P. de Groote (1994): *A CPS-Translation of the λμ-Calculus*. In: S. Tison, editor: Proc. of the Colloquium on Trees in Algebra and Programming, Lecture Notes in Computer Science 787, Springer-Verlag, pp. 85–99.

[16] A. Heyting (1946): *On weakened quantification*. The Journal of Symbolic Logic 11(4), pp. 119–121.

[17] J. M. E. Hyland (2002): *Proof theory in the abstract*. Annals of Pure and Applied Logic 114, pp. 43–78.

[18] H. Ishihara (2000): *A note on the Gödel-Gentzen Translation*. Mathematical Logic Quarterly 46(1), pp. 135–137.

[19] S. C. Kleene (1952): *Introduction to Metamathematics*. D. Van Nostrand Co., Inc., New York, N. Y.

[20] U. Kohlenbach (2008): *Applied Proof Theory: Proof Interpretations and their Use in Mathematics*. Monographs in Mathematics. Springer.

[21] A. N. Kolmogorov (1925): *On the principle of the excluded middle (Russian)*. Mat. Sb. 32, pp. 646–667.

[22] J. Krivine (1990): *Opérateurs de mise en mémoire et traduction de Gödel*. Arch. Math. Logic 30(4), pp. 241–267.

[23] S. Kuroda (1951): *Intuitionistische Untersuchungen der formalistischen Logik*. Nagoya Mathematical Journal 3, pp. 35–47.

[24] H. Luckhardt (1973): *Extensional Gödel Functional Interpretation: A Consistency Proof of Classical Analysis*, Lecture Notes in Mathematics 306. Springer, Berlin.

[25] C. Murthy (1990): *Extracting Constructive Content from Classical Proofs*. Ph.D. thesis, Cornell University.

[26] G. D. Plotkin (1975): *Call-by-name, Call-by-value and the λ-calculus*. *Theoretical Computer Science* 1, pp. 125–159.

[27] H. Rasiowa & R. Sikorski (1963): *The mathematics of metamathematics*. Warsaw. PWN (Polish Scientific Publishers).

[28] H. Schwichtenberg (2006): *Proof Theory-Notes for a lecture course, Sommersemester 2006*. Mathematisches Institut der Ludwig-Maximilians-Iniversität.

[29] M. Shirahata (2006): *The Dialectica interpretation of first-order classical linear logic*. *Theory and Applications of Categories* 17(4), pp. 49–79.

[30] J. R. Shoenfield (1967): *Mathematical Logic*. Addison-Wesley Publishing Company.

[31] T. Streicher & U Kohlenbach (2007): *Shoenfield is Gödel after Krivine*. *Mathematical Logic Quarterly* 53, pp. 176–179.

[32] T. Streicher & B. Reus (1998): *Classical logic, continuation semantics and abstract machines*. *J. Funct. Prog.* 8(6), pp. 543–572.

[33] A. S. Troelstra (1973): *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, *Lecture Notes in Mathematics* 344. Springer, Berlin.

[34] A. S. Troelstra & H. Schwichtenberg (2000): *Basic Proof Theory*. Cambridge University Press, Cambridge (2nd edition).

[35] A. S. Troelstra & D. van Dalen (1988): *Constructivism in Mathematics. An Introduction*, *Studies in Logic and the Foundations of Mathematics* 121. North Holland, Amsterdam.