Functional Interpretations of Intuitionistic Linear Logic

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Abstract. We present three functional interpretations of intuitionistic linear logic and show how these correspond to well-known functional interpretations of intuitionistic logic via embeddings of IL^ω into ILL^ω. The main difference from previous work of the second author is that in intuitionistic linear logic the interpretations of $!A$ are simpler (at the cost of an asymmetric interpretation of pure ILL^ω) and simultaneous quantifiers are no longer needed for the characterisation of the interpretations.

1 Introduction

This paper presents a family of functional interpretations of intuitionistic linear logic ILL^ω, starting from a single functional interpretation of pure (exponential-free) ILL^ω, followed by three possible interpretations of $!A$.

The second author [7–10] has recently shown how different functional interpretations of intuitionistic logic can be factored into a uniform family of interpretations of classical linear logic combined with Girard's standard embedding $(\cdot)^*$ of intuitionistic logic into linear logic. In the symmetric context of *classical linear logic* each formula A is associated with a simultaneous one-move two-player game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. Intuitively, the two players, say Eloise and Abelard, must pick their moves $\boldsymbol{x}$ and $\boldsymbol{y}$ simultaneously and Eloise wins if and only if $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ holds. The symmetric nature of the game implies that (proof-theoretically) the formula A was interpreted as the formula

$$\exists\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

where $\exists\!\!\forall^{\boldsymbol{x}}_{\boldsymbol{y}} A$ is a simple form of branching quantifier – termed simultaneous quantifier. Following this game-theoretic reading, the different interpretations of the modality $!A$ are all of the following form: First, it (always) turns a symmetric game into an asymmetric one, where Eloise plays first, giving Abelard the advantage of playing second. In the symmetric context, this asymmetric game can be modelled by allowing Abelard to play a function $\boldsymbol{f}$ which calculates his move from a given Eloise move $\boldsymbol{x}$. But also, the game $!A$ gives a second (non-canonical) advantage to Abelard, by allowing him to play a *set of moves*, rather than a single move. The idea being that he wins the game $!A$ if any move $\boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}$ is winning with respect to Eloise's move $\boldsymbol{x}$, i.e. $\neg|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. Formally

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{f}} \equiv \forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Therefore, the game $!A$ always introduces a break of symmetric, but it leaves open *what kind of sets* Abelard is allowed to play. What the second author has shown is that if only singleton sets are allowed the resulting interpretation corresponds to Gödel's dialectica interpretation [1, 4, 9]; if finite sets are allowed then it corresponds to the Diller-Nahm variant of the dialectica interpretation [2, 10]; and if these sets are actually the whole set of moves then it corresponds to Kreisel's modified realizability interpretation [6, 8].

In this paper we show that in the context of *intuitionistic linear logic* every formula can be interpreted as a game where Eloise plays first and Abelard plays second, being the branching quantifiers no longer needed. In other words, Abelard's advantage of playing second, which was limited to the game $!A$ in classical linear logic, is ubiquitous in intuitionistic linear logic. In this way, the game-theoretic interpretation of the modality $!A$ is simply to lift the moves of Abelard from a single move to a set of moves. Formally,

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} \equiv \forall \boldsymbol{y} \in \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Hence, by working in the context of ILL^ω, we can fully separate the canonical part of the interpretation (pure intuitionistic linear logic), where all interpretations coincide, and the non-canonical part where each choice of "sets of moves" gives rise to a different functional interpretation.

As we shall see, the functional interpretation of *pure intuitionistic linear logic* coincides with Gödel's dialectica interpretation of *intuitionistic logic*, reading $\multimap, \otimes$ and $\oplus$ as $\to$, $\wedge$ and $\vee$, respectively. This is so, because the dialectica interpretation identifies the games A and $!A$. The connection between Gödel's dialectica interpretation and intuitionistic linear logic was first studied by de Paiva [11]. One can view our work here as a proof-theoretic reading of de Paiva's category-theoretic work, together with an extension linking the "dialectica" interpretation of intuitionistic linear logic also with Kreisel's modified realizability.

The main contributions of the paper are as follows: In Section 2 we present the basic interpretation of pure intuitionistic linear logic. In the same section we outline which principles are needed for the characterisation of the interpretation (Subsection 2.1). Section 3 describes three different interpretations of the modality $!A$. This is followed (Section 4) by a description of how each of these choices corresponds to the three best-known functional interpretations of intuitionistic logic.

1.1 Intuitionistic Linear Logic

Intuitionistic linear logic can be viewed as a fragment of Girard's linear logic [3] which is sufficient for embedding intuitionistic logic into the linear context. We will make use of the formulation of intuitionistic linear logic shown in Tables 1 and 2. Our system is denoted by ILL^ω since we work in the language of all finite types.

The finite types are inductively defined in the usual way: i is a finite type and if ρ and σ are finite types then $\rho \to \sigma$ is a finite type. Our language has a constant of type i (to ensure that all types are inhabited by a closed term) and variables x^ρ for each finite type ρ. We assume that the terms of ILL^ω contain all typed λ-terms, i.e. constants and variables are terms and if t^σ and $s^{\rho\to\sigma}$ are terms then $(\lambda x^\rho . t^\sigma)^{\rho\to\sigma}$ and $(s^{\rho\to\sigma} t^\rho)^\sigma$ are also terms.

The atomic formulas of ILL^ω are denoted by A_{at} (the linear logic constant 0 is an atomic formula) and if A and B are formulas, then $A \otimes B$, $A \,\&\, B$, $A \oplus B$, $A \multimap B$, $!A$,

$$\frac{}{P \vdash P}\,(\text{id}) \qquad \frac{}{\Gamma, 0 \vdash A}$$

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\,(\text{cut}) \qquad \frac{\Gamma \vdash A}{\pi\{\Gamma\} \vdash A}\,(\text{per})$$

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes\text{R}) \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,(\otimes\text{L})$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,(\multimap \text{R}) \qquad \frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,(\multimap \text{L})$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \,\&\, B}\,(\&\text{R}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, A \,\&\, C \vdash B}\,(\&\text{L}) \qquad \frac{\Gamma, B \vdash C}{\Gamma, A \,\&\, B \vdash C}\,(\&\text{L})$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B}\,(\oplus\text{R}) \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}\,(\oplus\text{R}) \qquad \frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}\,(\oplus\text{L})$$

Table 1. Intuitionistic Linear Logic (connectives)

$\forall x A(x)$ and $\exists x A(x)$ are also formulas. In this paper we will also work with a subsystem of ILL^ω, dubbed ILL^ω_r, where a restriction is assumed on the &R-rule: it is applied just with contexts of the form $!\Gamma$. In subsequent chapters we will see the necessity of this technical restriction. Note, however, that both systems ILL^ω and ILL^ω_r are strong enough to capture intuitionistic logic IL^ω into the linear context, as precised in the following proposition.

Proposition 1 ([3]). *Define two translations of IL^ω into ILL^ω inductively as follows:*

$$\begin{array}{llll}
A^*_{\mathsf{at}} & :\equiv A_{\mathsf{at}} & A^\circ_{\mathsf{at}} & :\equiv\ !A_{\mathsf{at}}, \quad \textit{if } A_{\mathsf{at}} \not\equiv \bot \\
\bot^* & :\equiv 0 & \bot^\circ & :\equiv 0 \\
(A \wedge B)^* & :\equiv A^* \,\&\, B^* & (A \wedge B)^\circ & :\equiv A^\circ \otimes B^\circ \\
(A \vee B)^* & :\equiv\ !A^* \oplus\ !B^* & (A \vee B)^\circ & :\equiv A^\circ \oplus B^\circ \\
(A \to B)^* & :\equiv\ !A^* \multimap B^* & (A \to B)^\circ & :\equiv\ !(A^\circ \multimap B^\circ) \\
(\forall x A)^* & :\equiv \forall x A^* & (\forall x A)^\circ & :\equiv\ !\forall x A^\circ \\
(\exists x A)^* & :\equiv \exists x !A^* & (\exists x A)^\circ & :\equiv \exists x A^\circ
\end{array}$$

If A is provable in IL^ω then A^* and A° are provable in ILL^ω_r (and hence also in ILL^ω). Moreover, it is easy to check that $A^\circ \multimapboth\ !A^*$.

Proof. It is already known that if $\Gamma \vdash_{\mathsf{IL}^\omega} A$ then $!\Gamma^* \vdash_{\mathsf{ILL}^\omega} A^*$. The result with ILL^ω replaced by ILL^ω_r just require our attention in the rule &R. The result for A° follows immediately from the fact that in ILL^ω_r we can prove $A^\circ \multimapboth\ !A^*$. □

$$\frac{\Gamma \vdash A}{\Gamma \vdash \forall x A}\ (\forall R) \qquad \frac{\Gamma, A[t/x] \vdash B}{\Gamma, \forall x A \vdash B}\ (\forall L)$$

$$\frac{\Gamma \vdash A[t/x]}{\Gamma \vdash \exists x A}\ (\exists R) \qquad \frac{\Gamma, A \vdash B}{\Gamma, \exists x A \vdash B}\ (\exists L)$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\text{con}) \qquad \frac{\Gamma \vdash B}{\Gamma, !A \vdash B}\ (\text{wkn}) \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (!R) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (!L)$$

Table 2. Intuitionistic Linear Logic (quantifiers and modality)

1.2 Verifying system

As we will show in the next sections, the three functional interpretations we present interpret the formula $A \oplus B$ via a sort of flagged disjoint union, i.e. a boolean and a witness for either A or B. Therefore, in the verifying system, which we shall denote by $\mathsf{ILL}^{\omega}_{b}$, we consider that the language also contains the booleans b as base type, with two boolean constants true and false (T, F), boolean variables, an equality relation $=^b$ between two terms of boolean type and a constant of type $b \to \rho \to \rho \to \rho$ that should be seen as a conditional λ-term $z(t,q)$ that reduces to t or q depending on whether z^b reduces to true or false. $\mathsf{ILL}^{\omega}_{b}$ is assumed to contain the following axioms for equality:

1. $!(x =^b x)$
2. $!(x =^b y) \multimap\ !(y =^b x)$
3. $!(x =^b y) \otimes\ !(y =^b z) \multimap\ !(x =^b z)$
4. $!(x =^b y) \otimes A[x/w] \multimap A[y/w]$.

We would also like to ensure that true and false are distinct and that there are no other elements of boolean type

5. $!(\mathrm{T} =^b \mathrm{F}) \multimap 0$
6. $!(z =^b \mathrm{T}) \oplus !(z =^b \mathrm{F})$.

The axioms for the conditional λ-term are as follows

7. $A[\mathrm{T}(t,q)/w] \multimapboth A[t/w]$ and $A[\mathrm{F}(t,q)/w] \multimapboth A[q/w]$.

For simplicity, we use the following abbreviation:

$$A \diamond_z B \ :\equiv\ (!(z =^b \mathrm{T}) \multimap A) \,\&\, (!(z =^b \mathrm{F}) \multimap B).$$

Lemma 1. *The following are derivable in* $\mathsf{ILL}^{\omega}_{b}$

(i) $\dfrac{\vdash A[\mathrm{T}] \quad \vdash A[\mathrm{F}]}{\vdash A[z]}$

(ii) $!(\mathrm{T} =^b \mathrm{T}) \multimap A \vdash A$ *and* $!(\mathrm{F} =^b \mathrm{F}) \multimap A \vdash A$

(iii) $A \vdash \; !(\mathrm{T} =^b \mathrm{F}) \multimap B$

(iv) $A \diamond_{\mathrm{T}} B \multimapboth A$ *and* $A \diamond_{\mathrm{F}} B \multimapboth B$

(v) $!A \diamond_z !B \multimapboth !(!A \diamond_z !B)$.

Proof. Assertion (*i*) can be derived from axioms 4. and 6.; (*ii*) follows easily from axiom 1.; (*iii*) can be deduced from axiom 5. and the forward implications in (*iv*) follow immediately from item (*ii*) and the inverse implications can easily be deduced using (*iii*). The direct implication in assertion (*v*) can be derived using assertions (*i*) and (*iv*), being the other implication trivial. □

2 A Basic Interpretation of Pure ILL^ω

In this section we present a basic functional interpretation of pure (without the exponential $!A$) intuitionistic linear logic, and prove its soundness. In the next section we then consider different extensions of this interpretation to full intuitionistic linear logic, ILL^ω.

Definition 1 (Basic functional interpretation of pure ILL^ω). *For each formula A of pure ILL^ω, let us associate a new formula $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$, with two fresh sets of free-variables $\boldsymbol{x}$ and $\boldsymbol{y}$, inductively as follows: For atomic formula A_{at} we let $|A_{\mathsf{at}}| :\equiv A_{\mathsf{at}}$. Assume the interpretations of A and B have already been defined as $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ and $|B|^{\boldsymbol{v}}_{\boldsymbol{w}}$, we then define*

$$
\begin{aligned}
|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}} \multimap |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}} \\
|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \,\&\, B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},z} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|A \oplus B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} &:\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
|\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{y}} &:\equiv |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}} \\
|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} &:\equiv |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}}.
\end{aligned}
$$

Intuitively, the meaning of A is reduced to the existence of an object $\boldsymbol{x}$ such that $\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. The $\boldsymbol{x}$'s are called *witnesses* and the $\boldsymbol{y}$'s *challenges*. Note that, contrary to the interpretation of classical linear logic [7, 10], the functional interpretation of intuitionistic linear logic is no longer symmetric. In terms of games, the interpretation above can be seen as associating to each formula A a one-move two-player *sequential* game $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. In this game, Eloise starts by playing a move $\boldsymbol{x}$ followed by Abelard playing a move $\boldsymbol{y}$. Eloise wins if $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ holds, otherwise Abelard wins.

Those familiar with the dialectica interpretation might find it puzzling that linear implication $A \multimap B$ is interpreted above precisely as the intuitionistic implication in

Gödel's dialectica interpretation, even though we claim this is the canonical part of the interpretation, which, depending on the interpretation of $!A$, can correspond to modified realizability as well. Again, in terms of games, this is explained by the fact that in the game $A \multimap B$ Eloise plays first in the game B, but she plays second in the game A. In order to circumvent this discrepancy with our general rule that Eloise always plays first, we allow Eloise's move $\boldsymbol{f}$ in game A to depend on Abelard's move $\boldsymbol{x}$. In this way, although she plays first in both games, it is as if she is playing second in game A, since her move is a function which might depend on Abelard's move.

Theorem 1 (Soundness). *Let $A_0, \ldots, A_n, B$ be formulas of pure ILL^ω, with $\boldsymbol{z}$ as the only free-variables. If*

$$A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z}) \vdash B(\boldsymbol{z})$$

is provable in pure ILL^ω then terms $\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n, \boldsymbol{b}$ can be extracted from this proof such that

$$|A_0(\boldsymbol{z})|^{\boldsymbol{x}_0}_{\boldsymbol{a}_0}, \ldots, |A_n(\boldsymbol{z})|^{\boldsymbol{x}_n}_{\boldsymbol{a}_n} \vdash |B(\boldsymbol{z})|^{\boldsymbol{b}}_{\boldsymbol{w}}$$

is provable in ILL^ω_b, where $\mathsf{FV}(\boldsymbol{a}_i) \subseteq \{\boldsymbol{z}, \boldsymbol{x}_0, \ldots, \boldsymbol{x}_n, \boldsymbol{w}\}$ and $\mathsf{FV}(\boldsymbol{b}) \subseteq \{\boldsymbol{z}, \boldsymbol{x}_0, \ldots, \boldsymbol{x}_n\}$.

Proof. By induction on the derivation of $A_0(\boldsymbol{z}), \ldots, A_n(\boldsymbol{z}) \vdash B$. The axioms are trivial since the interpretation does not change atomic formulas and every type is inhabited. The permutation rule is also immediate. Let us consider a few cases:

Cut

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}} \vdash |A|^{\boldsymbol{a}_0}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}'} \vdash |A|^{\boldsymbol{a}_0}_{\boldsymbol{a}_1[\boldsymbol{a}_0]}}\,[\tfrac{\boldsymbol{a}_1[\boldsymbol{a}_0]}{\boldsymbol{y}}] \qquad \dfrac{|\Delta|^{\boldsymbol{v}}_{\boldsymbol{\delta}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}_1[\boldsymbol{x}]} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{|\Delta|^{\boldsymbol{v}}_{\boldsymbol{\delta}'}, |A|^{\boldsymbol{a}_0}_{\boldsymbol{a}_1[\boldsymbol{a}_0]} \vdash |B|^{\boldsymbol{b}'}_{\boldsymbol{w}}}\,[\tfrac{\boldsymbol{a}_0}{\boldsymbol{x}}]}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}'}, |\Delta|^{\boldsymbol{v}}_{\boldsymbol{\delta}'} \vdash |B|^{\boldsymbol{b}'}_{\boldsymbol{w}}}\,(\text{cut})$$

where $\boldsymbol{\gamma}'$ and $\boldsymbol{\delta}', \boldsymbol{b}'$ are obtained from $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}, \boldsymbol{b}$ via the substitutions $[\boldsymbol{a}_1[\boldsymbol{a}_0]/\boldsymbol{y}]$ and $[\boldsymbol{a}_0/\boldsymbol{x}]$, respectively.

Tensor

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}} \quad |\Delta|^{\boldsymbol{v}}_{\boldsymbol{\delta}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |\Delta|^{\boldsymbol{v}}_{\boldsymbol{\delta}} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}} \otimes |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(\otimes\text{R})}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |\Delta|^{\boldsymbol{v}}_{\boldsymbol{\delta}} \vdash |A \otimes B|^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{y},\boldsymbol{w}}}\,(\text{D1}) \qquad \dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}}, |B|^{\boldsymbol{v}}_{\boldsymbol{b}} \vdash |C|^{\boldsymbol{c}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{b}} \vdash |C|^{\boldsymbol{c}}_{\boldsymbol{w}}}\,(\otimes\text{L})}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{a},\boldsymbol{b}} \vdash |C|^{\boldsymbol{c}}_{\boldsymbol{w}}}\,(\text{D1})$$

Linear implication - left introduction

$$\dfrac{\dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{y}]} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{fa}(\boldsymbol{b}[\boldsymbol{ga}])]} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{fa}(\boldsymbol{b}[\boldsymbol{ga}])}}\,[\tfrac{\boldsymbol{fa}(\boldsymbol{b}[\boldsymbol{ga}])}{\boldsymbol{y}}] \qquad \dfrac{|\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{v}]}, |B|^{\boldsymbol{v}}_{\boldsymbol{b}[\boldsymbol{v}]} \vdash |C|^{\boldsymbol{c}[\boldsymbol{v}]}_{\boldsymbol{z}}}{|\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{ga}]}, |B|^{\boldsymbol{ga}}_{\boldsymbol{b}[\boldsymbol{ga}]} \vdash |C|^{\boldsymbol{c}[\boldsymbol{ga}]}_{\boldsymbol{z}}}\,[\tfrac{\boldsymbol{ga}}{\boldsymbol{v}}]}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{fa}(\boldsymbol{b}[\boldsymbol{ga}])]}, |\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{ga}]}, |A|^{\boldsymbol{a}}_{\boldsymbol{fa}(\boldsymbol{b}[\boldsymbol{ga}])} \multimap |B|^{\boldsymbol{ga}}_{\boldsymbol{b}[\boldsymbol{ga}]} \vdash |C|^{\boldsymbol{c}[\boldsymbol{ga}]}_{\boldsymbol{z}}}\,(\multimap\text{L})}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{fa}(\boldsymbol{b}[\boldsymbol{ga}])]}, |\Delta|^{\boldsymbol{w}}_{\boldsymbol{\delta}[\boldsymbol{ga}]}, |A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{a},\boldsymbol{b}[\boldsymbol{ga}]} \vdash |C|^{\boldsymbol{c}[\boldsymbol{ga}]}_{\boldsymbol{z}}}\,(\text{D1})$$

Universal quantifier

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[z]} \vdash |A(z)|^{\boldsymbol{a}[z]}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[z]} \vdash |A(z)|^{(\lambda z.\boldsymbol{a}[z])z}_{\boldsymbol{y}}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[z]} \vdash |\forall z A(z)|^{\lambda z.\boldsymbol{a}[z]}_{\boldsymbol{y},z}}\,(\text{D1}) \qquad \dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{x}]}, |A(t)|^{\boldsymbol{x}}_{\boldsymbol{a}[\boldsymbol{x}]} \vdash |B|^{\boldsymbol{b}[\boldsymbol{x}]}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{f}t]}, |A(t)|^{\boldsymbol{f}t}_{\boldsymbol{a}[\boldsymbol{f}t]} \vdash |B|^{\boldsymbol{b}[\boldsymbol{f}t]}_{\boldsymbol{w}}}\,[\tfrac{\boldsymbol{f}t}{\boldsymbol{x}}]}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{f}t]}, |\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{a}[\boldsymbol{f}t],t} \vdash |B|^{\boldsymbol{b}[\boldsymbol{f}t]}_{\boldsymbol{w}}}\,(\text{D1})$$

Existential quantifier

$$\frac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}} \vdash |A(t)|^{\boldsymbol{a}}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}} \vdash |\exists z A(z)|^{\boldsymbol{a},t}_{\boldsymbol{y}}} \text{(D1)} \qquad \frac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[z]}, |A(z)|^{\boldsymbol{x}}_{\boldsymbol{a}[z]} \vdash |B|^{\boldsymbol{b}[z]}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[z]}, |\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{a}[z]} \vdash |B|^{\boldsymbol{b}[z]}_{\boldsymbol{y}}} \text{(D1)}$$

With - right introduction

$$\frac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}_0} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{u}}_{\mathrm{T}(\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_1)} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}} \diamond_{\mathrm{T}} |B|^{\boldsymbol{b}}_{\boldsymbol{w}}} \text{(Ax. 7/ L1}(iv)) \qquad \dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}_1} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\mathrm{F}(\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_1)} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}} \diamond_{\mathrm{F}} |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}}{\dfrac{|\Gamma|^{\boldsymbol{u}}_{z(\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_1)} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{z(\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_1)} \vdash |A \,\&\, B|^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{y},\boldsymbol{w},z}} \text{(D1)}} \text{(L1}(i))$$

With - left introduction and Plus - right introduction

$$\frac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \diamond_{\mathrm{T}} |C|^{\boldsymbol{v}}_{\boldsymbol{c}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}} \text{(L1}(iv))}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A \,\&\, C|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{a},\boldsymbol{c},\mathrm{T}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}} \text{(D1)} \qquad \frac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}} \diamond_{\mathrm{T}} |B|^{\boldsymbol{b}}_{\boldsymbol{w}}} \text{(L1}(iv))}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}} \vdash |A \oplus B|^{\boldsymbol{a},\boldsymbol{b},\mathrm{T}}_{\boldsymbol{y},\boldsymbol{w}}} \text{(D1)}$$

The other &-L and ⊕-R are similar.

Plus - left introduction

$$\frac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}_0}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \vdash |C|^{\boldsymbol{c}_1}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\mathrm{T}(\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_1)}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \diamond_{\mathrm{T}} |B|^{\boldsymbol{v}}_{\boldsymbol{b}} \vdash |C|^{\mathrm{T}(\boldsymbol{c}_1,\boldsymbol{c}_2)}_{\boldsymbol{w}}} \qquad \dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}_1}, |B|^{\boldsymbol{v}}_{\boldsymbol{b}} \vdash |C|^{\boldsymbol{c}_2}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\mathrm{F}(\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_1)}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \diamond_{\mathrm{F}} |B|^{\boldsymbol{v}}_{\boldsymbol{b}} \vdash |C|^{\mathrm{F}(\boldsymbol{c}_1,\boldsymbol{c}_2)}_{\boldsymbol{w}}} \text{(Ax. 7/ L1}(iv))}{\dfrac{|\Gamma|^{\boldsymbol{u}}_{z(\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_1)}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{b}} \vdash |C|^{z(\boldsymbol{c}_1,\boldsymbol{c}_2)}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{z(\boldsymbol{\gamma}_0,\boldsymbol{\gamma}_1)}, |A \oplus B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{a},\boldsymbol{b}} \vdash |C|^{z(\boldsymbol{c}_1,\boldsymbol{c}_2)}_{\boldsymbol{w}}} \text{(D1)}} \text{(L1}(i))$$

The other cases are treated similary. □

2.1 Characterisation

As described in the introduction, one of the main advantages of working in the context of intuitionistic linear logic is that we no longer need (non-standard) branching quantifiers. The asymmetry introduced in ILL^ω turns the symmetric games of classical linear logic into games where Eloise always plays first, so formulas A are interpreted as $\exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$.

Proposition 2. *The following principles characterise the basic interpretation presented above*

$$\begin{aligned}
\mathsf{AC}_l &: \forall x \exists \boldsymbol{y} A_\forall(\boldsymbol{y}) \multimap \exists \boldsymbol{f} \forall x A_\forall(\boldsymbol{f} x) \\
\mathsf{MP}_l &: (\forall x A_{\mathsf{qf}} \multimap B_{\mathsf{qf}}) \multimap \exists x (A_{\mathsf{qf}} \multimap B_{\mathsf{qf}}) \\
\mathsf{IP}_l &: (A_\forall \multimap \exists \boldsymbol{y} B_\forall) \multimap \exists \boldsymbol{y} (A_\forall \multimap B_\forall) \\
\mathsf{EP} &: \forall \boldsymbol{x}, \boldsymbol{v} (A_{\mathsf{qf}} \otimes B_{\mathsf{qf}}) \multimap (\forall \boldsymbol{x} A_{\mathsf{qf}} \otimes \forall \boldsymbol{v} B_{\mathsf{qf}})
\end{aligned}$$

where A_{qf}*,* B_{qf} *and* $A_{\forall}$*,* $B_{\forall}$ *are quantifier-free formulas and purely universal formulas of* ILL^{ω}_b *respectively. Formally,*

$$\mathsf{ILL}^{\omega}_b + \mathsf{AC}_l + \mathsf{MP}_l + \mathsf{IP}_l + \mathsf{EP} \vdash A \multimapboth \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$$

Proof. By induction on the logical structure of A. Let us consider a few cases:

Tensor.

$$\begin{aligned} A \otimes B &\overset{\text{(IH)}}{\multimapboth} \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes \exists \boldsymbol{v} \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\ &\overset{\text{(EP)}}{\multimapboth} \exists \boldsymbol{x}, \boldsymbol{v} \forall \boldsymbol{y}, \boldsymbol{w} (|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ &\equiv \exists \boldsymbol{x}, \boldsymbol{v} \forall \boldsymbol{y}, \boldsymbol{w} |A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}}. \end{aligned}$$

With.

$$\begin{aligned} A \,\&\, B &\overset{\text{(IH)}}{\multimapboth} \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \,\&\, \exists \boldsymbol{v} \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \\ &\multimapboth \forall z (\exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z \exists \boldsymbol{v} \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ &\multimapboth \forall z \exists \boldsymbol{x}, \boldsymbol{v} (\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ &\multimapboth \forall z \exists \boldsymbol{x}, \boldsymbol{v} \forall \boldsymbol{y}, \boldsymbol{w} (|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ &\overset{(\mathsf{AC}_l)}{\multimapboth} \exists \boldsymbol{f}, \boldsymbol{g} \forall z, \boldsymbol{y}, \boldsymbol{w} (|A|^{\boldsymbol{f}z}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{g}z}_{\boldsymbol{w}}) \\ &\multimapboth \exists \boldsymbol{x}, \boldsymbol{v} \forall z, \boldsymbol{y}, \boldsymbol{w} (|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ &\equiv \exists \boldsymbol{x}, \boldsymbol{v} \forall \boldsymbol{y}, \boldsymbol{w}, z |A \,\&\, B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w},z}. \end{aligned}$$

Linear implication.

$$\begin{aligned} A \multimap B &\overset{\text{(IH)}}{\multimapboth} \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap \exists \boldsymbol{v} \forall \boldsymbol{w} |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \overset{(\mathsf{IP}_l, \mathsf{MP}_l)}{\multimapboth} \forall \boldsymbol{x} \exists \boldsymbol{v} \forall \boldsymbol{w} \exists \boldsymbol{y} (|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimap |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\ &\overset{(\mathsf{AC}_l)}{\multimapboth} \exists \boldsymbol{f}, \boldsymbol{g} \forall \boldsymbol{x}, \boldsymbol{w} (|A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}} \multimap |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}}) \equiv \exists \boldsymbol{f}, \boldsymbol{g} \forall \boldsymbol{x}, \boldsymbol{w} |A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}}. \end{aligned}$$

Universal quantifier.

$$\forall z A \overset{\text{(IH)}}{\multimapboth} \forall z \exists \boldsymbol{x} \forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \overset{(\mathsf{AC}_l)}{\multimapboth} \exists \boldsymbol{f} \forall \boldsymbol{y}, z |A|^{\boldsymbol{f}z}_{\boldsymbol{y}} \equiv \exists \boldsymbol{f} \forall \boldsymbol{y}, z |\forall z A|^{\boldsymbol{f}}_{\boldsymbol{y},z}.$$

The other cases are treated similarly. In fact, for the remaining cases (once the induction hypothesis is assumed) the equivalence can be proved in ILL^{ω}_b alone. □

Remark 1. Note that if we are embedding IL^{ω} via the standard embedding $(\cdot)^*$ then the connective $A \otimes B$ is not needed, and hence the extra principle EP is not needed either.

3 Some Interpretations of ILL^{ω}

In this section we consider a few choices of how the basic interpretation given in Definition 1 can be extended to full intuitionistic linear logic, i.e. we give some alternative interpretations of the modality $!A$. All choices considered will have the form:

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv \, !\forall \boldsymbol{y}' \sqsubset \boldsymbol{y} \, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'} \tag{1}$$

for some notion of bounded quantified formula $\forall \boldsymbol{y}' \sqsubset \boldsymbol{y} A$. In fact, very little structure is required in order to obtain a sound interpretation.

Proposition 3. *Given a formula $A[\boldsymbol{y}]$, assume the formula $\forall \boldsymbol{y} \sqsubset \boldsymbol{a} A$ is such that for some terms*[1] $\eta(\cdot), (\cdot) \otimes (\cdot)$ *and* $(\cdot) \circ (\cdot)$ *the following are provable in* $\mathsf{ILL}^{\omega}_{b}$

(A1) $!\forall \boldsymbol{y} \sqsubset \eta(z)\, A[\boldsymbol{y}] \multimap A[z]$

(A2) $!\forall \boldsymbol{y} \sqsubset \boldsymbol{y}_1 \otimes \boldsymbol{y}_2\, A[\boldsymbol{y}] \multimap !\forall \boldsymbol{y} \sqsubset \boldsymbol{y}_1\, A[\boldsymbol{y}] \otimes !\forall \boldsymbol{y} \sqsubset \boldsymbol{y}_2\, A[\boldsymbol{y}]$

(A3) $!\forall \boldsymbol{y} \sqsubset \boldsymbol{f} \circ \boldsymbol{z}\, A[\boldsymbol{y}] \multimap !\forall \boldsymbol{x} \sqsubset \boldsymbol{z}\, !\forall \boldsymbol{y} \sqsubset \boldsymbol{f}\boldsymbol{x}\, A[\boldsymbol{y}]$.

The interpretation of $!A$ *as above leads to a sound functional interpretation of* ILL^{ω}.

Proof. By Theorem 1 we just have to analyse the rules of contraction, weakening, !-right introduction and !-left introduction.

Contraction

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |!A|^{\boldsymbol{x}_0}_{\boldsymbol{a}_0}, |!A|^{\boldsymbol{x}_1}_{\boldsymbol{a}_1} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |!A|^{\boldsymbol{x}}_{\boldsymbol{a}_0}, |!A|^{\boldsymbol{x}}_{\boldsymbol{a}_1} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,[\tfrac{\boldsymbol{x}}{\boldsymbol{x}_0}, \tfrac{\boldsymbol{x}}{\boldsymbol{x}_1}]}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, !\forall \boldsymbol{y}' \sqsubset \boldsymbol{a}_0\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'}, !\forall \boldsymbol{y}' \sqsubset \boldsymbol{a}_1\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(1)}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, !\forall \boldsymbol{y}' \sqsubset \boldsymbol{a}_0\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'} \otimes\, !\forall \boldsymbol{y}' \sqsubset \boldsymbol{a}_1\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(\otimes \mathrm{L})}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, !\forall \boldsymbol{y}' \sqsubset \boldsymbol{a}_0 \otimes \boldsymbol{a}_1\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(\mathrm{A2})}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |!A|^{\boldsymbol{x}}_{\boldsymbol{a}_0 \otimes \boldsymbol{a}_1} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(1)$$

Weakening

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, !\forall \boldsymbol{y}' \sqsubset \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(\mathrm{wkn})}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |!A|^{\boldsymbol{x}}_{\boldsymbol{a}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(1)$$

Note that every type is inhabited by a closed term.

Bang - right introduction

$$\dfrac{\dfrac{\dfrac{\dfrac{|!\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}[\boldsymbol{y}']} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}'}}{!\forall \boldsymbol{w}' \sqsubset \boldsymbol{\gamma}[\boldsymbol{y}']\, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{w}'} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{y}'}}\,(1)}{!\forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\, !\forall \boldsymbol{w}' \sqsubset (\lambda \boldsymbol{y}'.\boldsymbol{\gamma}[\boldsymbol{y}'])\boldsymbol{y}'\, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{w}'} \vdash !\forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\, |A|^{\boldsymbol{a}}_{\boldsymbol{y}'}}}{!\forall \boldsymbol{w}' \sqsubset (\lambda \boldsymbol{y}'.\boldsymbol{\gamma}[\boldsymbol{y}']) \circ \boldsymbol{y}\, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{w}'} \vdash !\forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\, |A|^{\boldsymbol{a}}_{\boldsymbol{y}'}}\,(\mathrm{A3})}{|!\Gamma|^{\boldsymbol{u}}_{(\lambda \boldsymbol{y}'.\boldsymbol{\gamma}[\boldsymbol{y}']) \circ \boldsymbol{y}} \vdash |!A|^{\boldsymbol{a}}_{\boldsymbol{y}}}\,(1)$$

Bang - left introduction

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, !\forall \boldsymbol{y} \sqsubset \eta(\boldsymbol{a})\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(\mathrm{A1})}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |!A|^{\boldsymbol{x}}_{\eta(\boldsymbol{a})} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{w}}}\,(1)$$

That concludes the proof. □

[1] Note that these terms are allowed to be specific to the formula A, in particular, the free variables of $\eta(\cdot), (\cdot) \otimes (\cdot)$ and $(\cdot) \circ (\cdot)$ are assumed to be contained in the free-variables of $\forall \boldsymbol{y} A[\boldsymbol{y}]$ (i.e. all free-variables of A except $\boldsymbol{y}$).

Remark 2. Assume that the types of $\boldsymbol{y}^{\rho}$ and $\boldsymbol{a}^{T\rho}$ in $\forall \boldsymbol{y} \sqsubset \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ are as shown, for a fixed A. Then, our three families of terms have types

$$\eta \;:\; \rho \to T\rho$$

$$\otimes \;:\; T\rho \times T\rho \to T\rho$$

$$\circ \;:\; (\tau \to T\rho) \times T\tau \to T\rho.$$

In category theory, one could think of $(T, \eta, \circ)$ as forming a Kleisli triple ($\sim$ monad), with $\otimes$ being a commutative monoid on $T\rho$. This in turn extends to a comonad on formulas as

$$T(A[\boldsymbol{y}]) :\equiv\, !(\forall \boldsymbol{y} \sqsubset \boldsymbol{a}\, A)[\boldsymbol{a}].$$

See e.g. the work of Valeria de Paiva [12] and Martin Hyland ([5], section 3.1) on categorical logic for more information about the connection between functional interpretations and comonads.

Proposition 4. *The following are three sound interpretations of* $!A$*:*

$(a)\quad |!A|^{\boldsymbol{x}} :\equiv\, !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

$(b)\quad |!A|^{\boldsymbol{x}}_{\boldsymbol{a}} :\equiv\, !\forall \boldsymbol{y} \in \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$

$(c)\quad |!A|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv\, !|A|^{\boldsymbol{x}}_{\boldsymbol{y}}.$

Proof. (a) This interpretation of $!A$ corresponds to the choice $\forall \boldsymbol{y} \sqsubset \boldsymbol{t}\, A[\boldsymbol{y}] :\equiv \forall \boldsymbol{y} A[\boldsymbol{y}]$. It is easy to check that conditions $(A1)$, $(A2)$ and $(A3)$ become

$$!\forall \boldsymbol{y} A[\boldsymbol{y}] \multimap A[\boldsymbol{z}]$$

$$!\forall \boldsymbol{y} A[\boldsymbol{y}] \multimap\, !\forall \boldsymbol{y} A[\boldsymbol{y}] \otimes\, !\forall \boldsymbol{y} A[\boldsymbol{y}]$$

$$!\forall \boldsymbol{y} A[\boldsymbol{y}] \multimap\, !\forall \boldsymbol{x} !\forall \boldsymbol{y} A[\boldsymbol{y}]$$

respectively, which are trivially derivable in $\mathsf{ILL}^{\omega}_{b}$.

(b) Consider that the language of $\mathsf{ILL}^{\omega}_{b}$ has a new finite type σ^* for each finite type σ. An element of type σ^* is a finite set of elements of type σ. The extended language has a relation symbol $\in$ infixing between a term of type σ and a term of type σ^* with axioms to ensure that $!(x \in y)$ if and only if x is an element in the set y. Consider also the existence of three more constants of types $\sigma \to \sigma^*$, $\sigma^* \to \sigma^* \to \sigma^*$ and $\sigma^* \to (\sigma \to \rho^*) \to \rho^*$ that should be seen as terms such that $\eta(t)$ is the singleton set with t^{σ} as the only element (in particular $!(t \in \eta(t))$), $t \otimes q$ is the union of two finite sets t and q, and $f \circ q$ is the set that results from the union of all sets fx with $x \in q$. The interpretation $|!A|^{\boldsymbol{x}}_{\boldsymbol{a}} :\equiv\; !\forall \boldsymbol{y} \in \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ corresponds to the choice $\forall \boldsymbol{y} \sqsubset \boldsymbol{t}\, A[\boldsymbol{y}] :\equiv \forall \boldsymbol{y} \in \boldsymbol{t}\, A[\boldsymbol{y}]$, which is an abbreviation for $\forall \boldsymbol{y}(!(\boldsymbol{y} \in \boldsymbol{t}) \multimap A[\boldsymbol{y}])$. In this context, the conditions $(A1)$, $(A2)$ and $(A3)$ become

$$!\forall \boldsymbol{y} \in \eta(\boldsymbol{z})\, A[\boldsymbol{y}] \multimap A[\boldsymbol{z}]$$

$$!\forall \boldsymbol{y} \in \boldsymbol{y}_1 \otimes \boldsymbol{y}_2\, A[\boldsymbol{y}] \multimap\, !\forall \boldsymbol{y} \in \boldsymbol{y}_1\, A[\boldsymbol{y}] \otimes\, !\forall \boldsymbol{y} \in \boldsymbol{y}_2\, A[\boldsymbol{y}]$$

$$!\forall \boldsymbol{y} \in \boldsymbol{f} \circ \boldsymbol{z}\, A[\boldsymbol{y}] \multimap\, !\forall \boldsymbol{x} \in \boldsymbol{z}\; !\forall \boldsymbol{y} \in \boldsymbol{f}\boldsymbol{x} A[\boldsymbol{y}],$$

which are provable in the extension of ILL_b^ω outlined above.

(c) This interpretation of $!A$ corresponds to the choice $\forall \boldsymbol{y} \sqsubset \boldsymbol{t}\, A[\boldsymbol{y}] :\equiv A[\boldsymbol{t}/\boldsymbol{y}]$. Given a formula $A[\boldsymbol{y}]$ we define $\eta(\cdot)$, as being the identity, $\circ$ is defined as $f \circ x :\equiv fx$ and $\boldsymbol{y}_1 \otimes \boldsymbol{y}_2$ is

$$\boldsymbol{y}_1 \otimes \boldsymbol{y}_2 := \begin{cases} \boldsymbol{y}_1 & \text{if } !A[\boldsymbol{y}_1] \multimap 0 \\ \boldsymbol{y}_2 & \text{if } !A[\boldsymbol{y}_1]. \end{cases}$$

We are assuming that ILL_b^ω has also an extra axiom (asserting the decidability of A) $\vdash\ !A \oplus (!A \multimap 0)$. Conditions ($A1$), ($A2$) and ($A3$) become

$$!A[\eta(\boldsymbol{z})] \multimap A[\boldsymbol{z}]$$

$$!A[\boldsymbol{y}_1 \otimes \boldsymbol{y}_2] \multimap !A[\boldsymbol{y}_1] \otimes !A[\boldsymbol{y}_2]$$

$$!A[\boldsymbol{f} \circ \boldsymbol{z}] \multimap !!A[\boldsymbol{f}\boldsymbol{z}]$$

respectively. ($A1$) and ($A3$) are trivially derivable. In the derivation of ($A2$) use

$$\vdash\ !A \oplus (!A \multimap 0)$$

$$!A[\boldsymbol{y}_1], !A[\boldsymbol{y}_1 \otimes \boldsymbol{y}_2] \vdash !A[\boldsymbol{y}_1] \otimes !A[\boldsymbol{y}_2], \text{ and}$$

$$!A[\boldsymbol{y}_1] \multimap 0, !A[\boldsymbol{y}_1 \otimes \boldsymbol{y}_2] \vdash 0.$$

□

4 Relation to Standard Interpretations of IL^ω

We argued in the introduction (see Proposition 1) that for the purpose of analysing IL^ω via linear logic it sufficies to work with the system ILL_r^ω. As it turns out, in ILL_r^ω, we can simplify our definition of functional interpretation as follows:

Proposition 5. *When interpreting the subsystem ILL_r^ω, the interpretation of $A \,\&\, B$ presented in Definition 1 can be simplified so that the parametrised interpretation*

$$|A \multimap B|^{\boldsymbol{f},\boldsymbol{g}}_{\boldsymbol{x},\boldsymbol{w}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{f}\boldsymbol{x}\boldsymbol{w}} \multimap |B|^{\boldsymbol{g}\boldsymbol{x}}_{\boldsymbol{w}}$$

$$|A \otimes B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \otimes |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$$

$$|A \,\&\, B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \,\&\, |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$$

$$|A \oplus B|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} :\equiv |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B|^{\boldsymbol{v}}_{\boldsymbol{w}}$$

$$|\exists z A(z)|^{\boldsymbol{x},z}_{\boldsymbol{y}} :\equiv |A(z)|^{\boldsymbol{x}}_{\boldsymbol{y}}$$

$$|\forall z A(z)|^{\boldsymbol{f}}_{\boldsymbol{y},z} :\equiv |A(z)|^{\boldsymbol{f}z}_{\boldsymbol{y}}$$

$$|!A|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv !\forall \boldsymbol{y}' \sqsubset \boldsymbol{y}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}'}$$

is sound for ILL_r^ω, assuming (A1), (A2), *and* (A3) *are satisfied.*

Proof. We just have to analyse the rules for & having in mind that, in the case of the system under interpretation, the &-right introduction is restricted of the form $!\Gamma$. The simplified interpretation of $A \,\&\, B$ is shown sound as:

$$\dfrac{\dfrac{\dfrac{|!\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}_0} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}{!\forall \boldsymbol{y}' \sqsubset \boldsymbol{\gamma}_0 \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{y}'} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}\,(\mathrm{P5})}{!\forall \boldsymbol{y}' \sqsubset \boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1 \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{y}'} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}}}\,(\mathrm{A2}) \quad \dfrac{\dfrac{|!\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}_1} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}{!\forall \boldsymbol{y}' \sqsubset \boldsymbol{\gamma}_1 \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{y}'} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}\,(\mathrm{P5})}{!\forall \boldsymbol{y}' \sqsubset \boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1 \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{y}'} \vdash |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}\,(\mathrm{A2})}{\dfrac{!\forall \boldsymbol{y}' \sqsubset \boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1 \, |\Gamma|^{\boldsymbol{u}}_{\boldsymbol{y}'} \vdash |A|^{\boldsymbol{a}}_{\boldsymbol{x}} \,\&\, |B|^{\boldsymbol{b}}_{\boldsymbol{y}}}{|!\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}_0 \otimes \boldsymbol{\gamma}_1} \vdash |A \,\&\, B|^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{x},\boldsymbol{y}}}\,(\mathrm{P5})}\,(\&\mathrm{R})$$

And for the left introduction:

$$\dfrac{\dfrac{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \vdash |C|^{\boldsymbol{c}}_{\boldsymbol{w}}}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A|^{\boldsymbol{x}}_{\boldsymbol{a}} \,\&\, |B|^{\boldsymbol{v}}_{\boldsymbol{b}} \vdash |C|^{\boldsymbol{c}}_{\boldsymbol{w}}}\,(\&\mathrm{L})}{|\Gamma|^{\boldsymbol{u}}_{\boldsymbol{\gamma}}, |A \,\&\, B|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{a},\boldsymbol{b}} \vdash |C|^{\boldsymbol{c}}_{\boldsymbol{w}}}\,(\mathrm{P5})$$

The other &-left introduction is similar. □

Since in the remaining part of this section we work with translations of intuitionistic logic into linear logic, by $|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$ we refer to the (simplified) parametrised interpretation described in Proposition 5. Next we prove that the three different ways of interpreting $!A$ (cf. Proposition 4) give rise to interpretations of $\mathsf{ILL}^{\omega}_{r}$ that correspond (via the translations of intuitionistic logic into intuitionistic linear logic) to Kreisel's modified realizability, the Diller-Nahm interpretation, and Gödel's dialectica interpretation, as:

$\lvert !A\rvert^{\boldsymbol{x}}_{\boldsymbol{a}}$	Interpretation of IL^{ω}
$!\forall \boldsymbol{y} \lvert A\rvert^{\boldsymbol{x}}_{\boldsymbol{y}}$	Kreisel modified realizability
$!\forall \boldsymbol{y} \in \boldsymbol{a}\, \lvert A\rvert^{\boldsymbol{x}}_{\boldsymbol{y}}$	Diller-Nahm interpretation
$!\lvert A\rvert^{\boldsymbol{x}}_{\boldsymbol{a}}$	Gödel's dialectica interpretation.

But first a consideration concerning translation $(\cdot)^*$ of IL^{ω} into $\mathsf{ILL}^{\omega}_{r}$, which we will use in the treatment of the Diller-Nahm and the dialectica interpretations (for modified realizability we use the translation $(\cdot)^{\circ}$).

Proposition 6. *Consider the following simplification of Girard's translations* $(\cdot)^*$ *(cf. Proposition 1)*

$$
\begin{array}{ll}
A_{\mathsf{at}}^{+} & :\equiv A_{\mathsf{at}}, \quad \textit{if } A_{\mathsf{at}} \not\equiv \bot \\
\bot^{+} & :\equiv 0 \\
(A \wedge B)^{+} & :\equiv A^{+} \mathbin{\&} B^{+} \\
(A \vee B)^{+} & :\equiv A^{+} \oplus B^{+} \\
(A \to B)^{+} & :\equiv {!A^{+}} \multimap B^{+} \\
(\forall x A)^{+} & :\equiv \forall x A^{+} \\
(\exists x A)^{+} & :\equiv \exists x A^{+}.
\end{array}
$$

If A is provable in IL^{ω} *then* A^{+} *is provable in* $\mathsf{ILL}_r^{\omega} + \mathrm{P}_{\oplus} + \mathrm{P}_{\exists}$*, where*

$$
\begin{array}{lll}
\mathrm{P}_{\oplus} & : & !(A \oplus B) \multimap {!A} \oplus {!B} \\
\mathrm{P}_{\exists} & : & !\exists x A \multimap \exists x {!A}.
\end{array}
$$

Proof. First we show that given the principles $\mathrm{P}_{\oplus}$ and $\mathrm{P}_{\exists}$, we have $!A^{*} \multimapboth {!A^{+}}$. The proof is done by induction on the complexity of the formula A. Conjunction, implication and universal quantification follow easily by induction hypothesis using that ILL_r^{ω} proves:

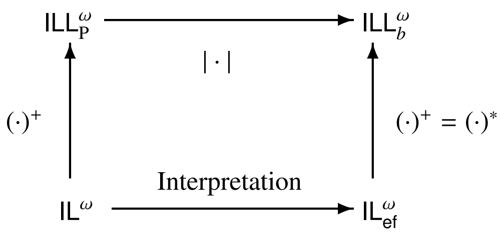
$$
\begin{array}{lll}
!(A \mathbin{\&} B) & \multimapboth & {!A} \otimes {!B} \\
!(!A \multimap B) & \multimapboth & !(!A \multimap {!B}) \\
!\forall x A & \multimapboth & !\forall x {!A}
\end{array}
$$

respectively. Disjunction and existential quantification are studied below:

$$
\begin{array}{l}
!(A \vee B)^{*} \equiv {!(!A^{*} \oplus {!B^{*}})} \multimapboth {!A^{*}} \oplus {!B^{*}} \\
\qquad \overset{(\mathrm{IH})}{\multimapboth} {!A^{+}} \oplus {!B^{+}} \overset{(\mathrm{P}_{\oplus})}{\multimapboth} !(A^{+} \oplus B^{+}) \equiv {!(A \vee B)^{+}}
\end{array}
$$

and $!(\exists x A)^{*} \equiv {!\exists x {!A^{*}}} \multimapboth \exists x {!A^{*}} \overset{(\mathrm{IH})}{\multimapboth} \exists x {!A^{+}} \overset{(\mathrm{P}_{\exists})}{\multimapboth} {!\exists x A^{+}} \equiv {!(\exists x A)^{+}}$. Applying Proposition 1, we know that from $\mathsf{IL}^{\omega} \vdash A$ we have $\mathsf{ILL}_r^{\omega} \vdash A^{*}$. So, $\mathsf{ILL}_r^{\omega} \vdash {!A^{*}}$ and hence $\mathsf{ILL}_r^{\omega} + \mathrm{P}_{\oplus} + \mathrm{P}_{\exists} \vdash {!A^{*}}$. Using the equivalence proved before we have $\mathsf{ILL}_r^{\omega} + \mathrm{P}_{\oplus} + \mathrm{P}_{\exists} \vdash {!A^{+}}$. In particular, we conclude $\mathsf{ILL}_r^{\omega} + \mathrm{P}_{\oplus} + \mathrm{P}_{\exists} \vdash A^{+}$. □

The reason for assuming the principles $\mathrm{P}_{\oplus}$ and $\mathrm{P}_{\exists}$ is that they are validated by the interpretations we shall consider. As such, we can make use of these to simplify the embeddings of intuitionistic logic into (this extension of) linear logic, since the interpretation of linear logic will interpret these principles taking us back to standard linear logic, as suggested by the following diagram:



In the diagram, $\mathsf{ILL}^\omega_{\mathsf{P}}$ abbreviates $\mathsf{ILL}^\omega_r + \mathrm{P}_\oplus + \mathrm{P}_\exists$. Note that all our interpretations transform proofs in IL^ω into existential-free proofs, i.e. proofs in $\mathsf{IL}^\omega_{\mathsf{ef}}$, where the two translations $(\cdot)^*$ and $(\cdot)^+$ coincide.

4.1 Modified realizability

Kreisel's modified realizability associates to each formula A of intuitionistic logic a new formula $\boldsymbol{x}\ \mathsf{mr}\ A$ (see [13] for the formal definition). We are going to prove that this form of realizability once translated to the linear logic context via the $(\cdot)^\circ$ translation corresponds (according to the theorem bellow) to the interpretation of ILL^ω_r with $|!A|^{\boldsymbol{x}} :\equiv\ !\forall \boldsymbol{y} |A|^{\boldsymbol{x}}_{\boldsymbol{y}}$. First an auxiliar result:

Lemma 2. $|A^\circ|^{\boldsymbol{x}} \multimapboth\ !|A^\circ|^{\boldsymbol{x}}$.

Proof. Note that, because of the way we interpret $!A$, it can be checked by induction on A that the interpretation of A° has an empty tuple of challenge variables, i.e. we obtain a formula of the form $|A^\circ|^{\boldsymbol{x}}$. To verify the lemma, it is enough to prove that $|A^\circ|^{\boldsymbol{x}} \multimapboth\ !A'$, for some formula A', since assuming this we have $!|A^\circ|^{\boldsymbol{x}} \multimapboth\ !!A' \multimapboth\ !A' \multimapboth |A^\circ|^{\boldsymbol{x}}$. The proof is done by induction on the complexity of the formula A. We just sketch the cases of conjunction and disjunction, being the other cases immediate.

$$|(A \wedge B)^\circ|^{\boldsymbol{x},\boldsymbol{y}} \equiv |A^\circ \otimes B^\circ|^{\boldsymbol{x},\boldsymbol{y}} \equiv |A^\circ|^{\boldsymbol{x}} \otimes |B^\circ|^{\boldsymbol{y}} \overset{\text{(IH)}}{\multimapboth}\ !A' \otimes\ !B' \multimapboth\ !(A' \,\&\, B').$$

$$|(A \vee B)^\circ|^{\boldsymbol{x},\boldsymbol{y},z} \equiv |A^\circ \oplus B^\circ|^{\boldsymbol{x},\boldsymbol{y},z} \equiv |A^\circ|^{\boldsymbol{x}} \diamond_z |B^\circ|^{\boldsymbol{y}} \overset{\text{(IH)}}{\multimapboth}\ !A' \diamond_z !B' \overset{\text{(L1}(v))}{\multimapboth}\ !(!A' \diamond_z !B'). \quad \square$$

Theorem 2. $|A^\circ|^{\boldsymbol{x}} \multimapboth (\boldsymbol{x}\ \mathsf{mr}\ A)^\circ$.

Proof. The proof is done by induction on the complexity of the formula A. If A is an atomic formula, the result is trivial. Consider the case of conjunction:

$$\begin{aligned} |(A \wedge B)^\circ|^{\boldsymbol{x},\boldsymbol{y}} &\equiv |A^\circ \otimes B^\circ|^{\boldsymbol{x},\boldsymbol{y}} \equiv |A^\circ|^{\boldsymbol{x}} \otimes |B^\circ|^{\boldsymbol{y}} \\ &\overset{\text{(IH)}}{\multimapboth} (\boldsymbol{x}\ \mathsf{mr}\ A)^\circ \otimes (\boldsymbol{y}\ \mathsf{mr}\ B)^\circ \equiv (\boldsymbol{x}\ \mathsf{mr}\ A \wedge \boldsymbol{y}\ \mathsf{mr}\ B)^\circ \equiv (\boldsymbol{x},\boldsymbol{y}\ \mathsf{mr}\ A \wedge B)^\circ. \end{aligned}$$

The universal and existential quantifications also follow immediately from the way we define the translation and the interpretations, applying the induction hypothesis. Implication is treated as

$$\begin{aligned} |(A \to B)^\circ|^{\boldsymbol{g}} &\equiv |!(A^\circ \multimap B^\circ)|^{\boldsymbol{g}} \equiv\ !\forall x |A^\circ \multimap B^\circ|^{\boldsymbol{g}}_{\boldsymbol{x}} \equiv\ !\forall x(|A^\circ|^{\boldsymbol{x}} \multimap |B^\circ|^{\boldsymbol{g}\boldsymbol{x}}) \\ &\overset{\text{(IH)}}{\multimapboth}\ !\forall x((\boldsymbol{x}\ \mathsf{mr}\ A)^\circ \multimap (\boldsymbol{g}\boldsymbol{x}\ \mathsf{mr}\ B)^\circ) \multimapboth\ !\forall x!((\boldsymbol{x}\ \mathsf{mr}\ A)^\circ \multimap (\boldsymbol{g}\boldsymbol{x}\ \mathsf{mr}\ B)^\circ) \\ &\equiv (\forall x(\boldsymbol{x}\ \mathsf{mr}\ A \to \boldsymbol{g}\boldsymbol{x}\ \mathsf{mr}\ B))^\circ \equiv (\boldsymbol{g}\ \mathsf{mr}\ (A \to B))^\circ. \end{aligned}$$

whereas disjunction uses the auxiliary result above:

$$\begin{aligned} |(A \vee B)^\circ|^{\boldsymbol{x},\boldsymbol{y},z} &\overset{\text{(L2)}}{\multimapboth}\ !|(A \vee B)^\circ|^{\boldsymbol{x},\boldsymbol{y},z} \equiv\ !|A^\circ \oplus B^\circ|^{\boldsymbol{x},\boldsymbol{y},z} \equiv\ !(|A^\circ|^{\boldsymbol{x}} \diamond_z |B^\circ|^{\boldsymbol{y}}) \\ &\overset{\text{(IH)}}{\multimapboth}\ !((!(z = \mathrm{T}) \multimap (\boldsymbol{x}\ \mathsf{mr}\ A)^\circ) \,\&\, (!(z = \mathrm{F}) \multimap (\boldsymbol{y}\ \mathsf{mr}\ B)^\circ)) \\ &\multimapboth\ !(!(z = \mathrm{T}) \multimap (\boldsymbol{x}\ \mathsf{mr}\ A)^\circ) \otimes !(!(z = \mathrm{F}) \multimap (\boldsymbol{y}\ \mathsf{mr}\ B)^\circ) \\ &\equiv ((z = \mathrm{T} \to \boldsymbol{x}\ \mathsf{mr}\ A) \wedge (z = \mathrm{F} \to \boldsymbol{y}\ \mathsf{mr}\ B))^\circ \\ &\equiv (\boldsymbol{x},\boldsymbol{y},z\ \mathsf{mr}\ A \vee B)^\circ. \end{aligned}$$

That concludes the proof. □

4.2 Gödel's dialectica interpretation

Recall that Gödel's dialectica interpretation associates to each formula A a quantifier-free formula $A_D(\boldsymbol{x}; \boldsymbol{y})$ inductively, such that A is interpreted as $\exists \boldsymbol{x} \forall \boldsymbol{y} A_D(\boldsymbol{x}; \boldsymbol{y})$ (see [1], section 2.3). The next result shows the correspondence between the dialectica interpretation and the ILL^ω_r interpretation with $|!A|^{\boldsymbol{x}}_{\boldsymbol{y}} :\equiv \; !|A|^{\boldsymbol{x}}_{\boldsymbol{y}}$, via the simplified embedding $(\cdot)^+$ (cf. Proposition 6).

Theorem 3. $|A^+|^{\boldsymbol{x}}_{\boldsymbol{y}} \multimapboth (A_D(\boldsymbol{x}; \boldsymbol{y}))^+$.

Proof. The proof is easily done by induction on the complexity of the formula A. Again the atomic formulas are checked trivially and the other formulas follow immediately by induction hypothesis using the definitions of the $(\cdot)^+$-translation and the interpretations. We illustrate with two cases: Conjunction

$$|(A \wedge B)^+|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} \equiv |A^+ \,\&\, B^+|^{\boldsymbol{x},\boldsymbol{v}}_{\boldsymbol{y},\boldsymbol{w}} \equiv |A^+|^{\boldsymbol{x}}_{\boldsymbol{y}} \,\&\, |B^+|^{\boldsymbol{v}}_{\boldsymbol{w}} \overset{\text{(IH)}}{\multimapboth} (A_D(\boldsymbol{x}; \boldsymbol{y}))^+ \,\&\, (B_D(\boldsymbol{v}; \boldsymbol{w}))^+$$
$$\equiv (A_D(\boldsymbol{x}; \boldsymbol{y}) \wedge B_D(\boldsymbol{v}; \boldsymbol{w}))^+ \equiv ((A \wedge B)_D(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{y}, \boldsymbol{w}))^+.$$

and disjunction:

$$\begin{aligned}
|(A \vee B)^+|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} &\equiv |A^+ \oplus B^+|^{\boldsymbol{x},\boldsymbol{v},z}_{\boldsymbol{y},\boldsymbol{w}} \equiv |A^+|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_z |B^+|^{\boldsymbol{v}}_{\boldsymbol{w}} \\
&\equiv (!(z = \mathrm{T}) \multimap |A^+|^{\boldsymbol{x}}_{\boldsymbol{y}}) \,\&\, (!(z = \mathrm{F}) \multimap |B^+|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\
&\overset{\text{(IH)}}{\multimapboth} (!(z = \mathrm{T}) \multimap (A_D(\boldsymbol{x}; \boldsymbol{y}))^+) \,\&\, (!(z = \mathrm{F}) \multimap (B_D(\boldsymbol{v}; \boldsymbol{w}))^+) \\
&\equiv (z = \mathrm{T} \to A_D(\boldsymbol{x}; \boldsymbol{y}))^+ \,\&\, (z = \mathrm{F} \to B_D(\boldsymbol{v}; \boldsymbol{w}))^+ \\
&\equiv ((z = \mathrm{T} \to A_D(\boldsymbol{x}; \boldsymbol{y})) \wedge (z = \mathrm{F} \to B_D(\boldsymbol{v}; \boldsymbol{w})))^+ \\
&\equiv ((A \vee B)_D(\boldsymbol{x}, \boldsymbol{v}, z; \boldsymbol{y}, \boldsymbol{w}))^+.
\end{aligned}$$

The other cases are treated similarly. □

Note that although $(\cdot)^+$ translates formulas from IL^ω into $\mathsf{ILL}^\omega_r + \mathrm{P}_\oplus + \mathrm{P}_\exists$, since these two principles are interpretable the verifying system is still ILL^ω_b. Let us argue that $\mathrm{P}_\oplus$ and $\mathrm{P}_\exists$ are interpretable, by showing that the interpretation of premise implies that of the conclusion (hence the identity and projection functions can be taken as realisers for the implication). It can be proved that

$$\forall x \sqsubset a\, (A(x) \,\&\, B) \multimap (\forall x \sqsubset a\, A(x) \,\&\, B) \text{ and}$$
$$\forall x \sqsubset a\, (B \multimap A(x)) \multimap (B \multimap \forall x \sqsubset a\, A(x))$$

when the variable x does not occur free in B. Also, $!(A \diamond_b B) \multimap !A \diamond_b !B$. So,

$$\begin{aligned}
|!(A \oplus B)|^{\boldsymbol{x},\boldsymbol{v},b}_{\boldsymbol{a},\boldsymbol{c}} &\equiv \; !\forall \boldsymbol{y} \sqsubset \boldsymbol{a}\, \forall \boldsymbol{w} \sqsubset \boldsymbol{c}\, (|A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_b |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\
&\multimap \; !(\forall \boldsymbol{y} \sqsubset \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_b \forall \boldsymbol{w} \sqsubset \boldsymbol{c}\, |B|^{\boldsymbol{v}}_{\boldsymbol{w}}) \\
&\multimap \; !\forall \boldsymbol{y} \sqsubset \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \diamond_b !\forall \boldsymbol{w} \sqsubset \boldsymbol{c}\, |B|^{\boldsymbol{v}}_{\boldsymbol{w}} \equiv |!A \;\oplus\; !B|^{\boldsymbol{x},\boldsymbol{v},b}_{\boldsymbol{a},\boldsymbol{c}}.
\end{aligned}$$

Similarly, $|!\exists z A|^{\boldsymbol{x},z}_{\boldsymbol{a}} \equiv \; !\forall \boldsymbol{y} \sqsubset \boldsymbol{a}\, |\exists z A|^{\boldsymbol{x},z}_{\boldsymbol{y}} \equiv \; !\forall \boldsymbol{y} \sqsubset \boldsymbol{a}\, |A|^{\boldsymbol{x}}_{\boldsymbol{y}} \equiv |!A|^{\boldsymbol{x}}_{\boldsymbol{a}} \equiv |\exists z !A|^{\boldsymbol{x},z}_{\boldsymbol{a}}$.

4.3 Diller-Nahm interpretation

The Diller-Nahm interpretation differs from Gödel's dialectica interpretation since it allows finite sets to witness the negative content of an implication. Formally, the Diller-Nahm interpretation is defined inductively as

$$
\begin{array}{ll}
(A_{\mathsf{at}})_{dn}(;) & :\equiv A_{\mathsf{at}} \\
(A \wedge B)_{dn}(\boldsymbol{x}, \boldsymbol{v}; \boldsymbol{y}, \boldsymbol{w}) & :\equiv A_{dn}(\boldsymbol{x}; \boldsymbol{y}) \wedge B_{dn}(\boldsymbol{v}; \boldsymbol{w}) \\
(A \vee B)_{dn}(\boldsymbol{x}, \boldsymbol{v}, z; \boldsymbol{y}, \boldsymbol{w}) & :\equiv (z = \mathrm{T} \to A_{dn}(\boldsymbol{x}; \boldsymbol{y})) \wedge (z = \mathrm{F} \to B_{dn}(\boldsymbol{v}; \boldsymbol{w})) \\
(A \to B)_{dn}(\boldsymbol{f}, \boldsymbol{g}; \boldsymbol{x}, \boldsymbol{w}) & :\equiv \forall y \in \boldsymbol{fxw} A_{dn}(\boldsymbol{x}; \boldsymbol{y}) \to B_{dn}(\boldsymbol{gx}; \boldsymbol{w}) \\
(\forall z A)_{dn}(\boldsymbol{f}; \boldsymbol{y}, z) & :\equiv A_{dn}(\boldsymbol{f}z; \boldsymbol{y}) \\
(\exists z A)_{dn}(\boldsymbol{x}, z; \boldsymbol{y}) & :\equiv A_{dn}(\boldsymbol{x}; \boldsymbol{y}).
\end{array}
$$

Next we show that the Diller-Nahm interpretation of IL^{ω} corresponds to the interpretation of ILL_r^{ω} with $|!A|_{\boldsymbol{a}}^{\boldsymbol{x}} :\equiv !\forall \boldsymbol{y} \in \boldsymbol{a}\, |A|_{\boldsymbol{y}}^{\boldsymbol{x}}$.

Theorem 4. $|A^+|_{\boldsymbol{y}}^{\boldsymbol{x}} \multimapboth (A_{dn}(\boldsymbol{x}; \boldsymbol{y}))^+$.

Proof. The proof, by induction on the structure of A, is almost entirely similar to the one concerning Gödel's interpretation. The only case which needs attention is that of implication, which we analyse below.

$$
\begin{array}{rcl}
|(A \to B)^+|_{\boldsymbol{x},\boldsymbol{w}}^{\boldsymbol{f},\boldsymbol{g}} & \equiv & |!A^+ \multimap B^+|_{\boldsymbol{x},\boldsymbol{w}}^{\boldsymbol{f},\boldsymbol{g}} \;\equiv\; |!A^+|_{\boldsymbol{fxw}}^{\boldsymbol{x}} \multimap |B^+|_{\boldsymbol{w}}^{\boldsymbol{gx}} \\
& \equiv & !\forall \boldsymbol{y} \in \boldsymbol{fxw} |A^+|_{\boldsymbol{y}}^{\boldsymbol{x}} \multimap |B^+|_{\boldsymbol{w}}^{\boldsymbol{gx}} \\
& \overset{\text{(IH)}}{\multimapboth} & !\forall \boldsymbol{y} \in \boldsymbol{fxw} (A_{dn}(\boldsymbol{x}; \boldsymbol{y}))^+ \multimap (B_{dn}(\boldsymbol{gx}; \boldsymbol{w}))^+ \\
& \equiv & !(\forall \boldsymbol{y} \in \boldsymbol{fxw} A_{dn}(\boldsymbol{x}; \boldsymbol{y}))^+ \multimap (B_{dn}(\boldsymbol{gx}; \boldsymbol{w}))^+ \\
& \equiv & (\forall \boldsymbol{y} \in \boldsymbol{fxw} A_{dn}(\boldsymbol{x}; \boldsymbol{y}) \to B_{dn}(\boldsymbol{gx}; \boldsymbol{w}))^+ \\
& \equiv & ((A \to B)_{dn}(\boldsymbol{f}, \boldsymbol{g}; \boldsymbol{x}, \boldsymbol{w}))^+.
\end{array}
$$

Note that the $(\cdot)^+$ translation of $\forall y \in a A$ is $\forall y \in a A^+$, as we can see below:

$$
\begin{array}{rl}
(\forall y \in a\, A)^+ & \equiv (\forall y (y \in a \to A))^+ \\
& \equiv \forall y (!(y \in a)^+ \multimap A^+) \;\equiv\; \forall y (!(y \in a) \multimap A^+) \;\equiv\; \forall y \in a\, A^+.
\end{array}
$$

That concludes the proof. □

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