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Commuting conversions vs. the standard conversions of the "good" connectives

Abstract. Commuting conversions were introduced in the natural deduction calculus as *ad hoc* devices for the purpose of guaranteeing the subformula property in normal proofs. In a well known book, Jean-Yves Girard commented harshly on these conversions, saying that 'one tends to think that natural deduction should be modified to correct such atrocities.' We present an embedding of the intuitionistic predicate calculus into a second-order predicative system for which there is no need for commuting conversions. Furthermore, we show that the *redex* and the *conversum* of a commuting conversion of the original calculus translate into equivalent derivations by means of a series of bidirectional applications of standard conversions.

Keywords: Natural deduction. Commuting conversions. Predicative quantifiers.

Introduction

In [1], the first author showed how to embed the intuitionistic propositional calculus into *atomic* $\mathsf{PSOL}^{\mathsf{i}}$, a calculus with only two connectives: the conditional and the second-order universal quantifier. The word 'atomic' is justified by the restriction of the elimination rule of the second-order universal quantifier to atomic instantiations. The proof of the correctness of the embedding is straightforward once one hits upon the idea that the embedding might work at all. It is somewhat surprising that this simple idea has not arisen before in Proof Theory (recently, we learned that Tor Sandqvist rediscovered the embedding – see [4]).

The embedding is made possible by a phenomenon dubbed *instantiation overflow*. We easily extend the embedding to the intuitionistic predicate calculus. In this extended case, it consists of an embedding into a second-order calculus whose connectives are the conditional and first and second-order universal quantifiers. Observe that the *bad* connectives (cf. p. 80 of [2]) $\perp$, $\vee$ and $\exists$ are conspicuously absent. The main part of the present paper is devoted to answering the following question: how do the commuting conversions of the intuitionistic predicate calculus (these are conversions associated with the 'bad' connectives $\perp$, $\vee$ and $\exists$) translate via the above embedding? The main theorem of the paper shows that the *redex* and the *conversum* of a commuting conversion translate into equivalent derivations by means of a series of bidirectional applications of standard conversions.

1. Preliminaries

In this section, we describe the calculus *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ (an acronym for *quantifier second-order logic*). The language of this calculus is based on a pure first-order language. It has, furthermore, second-order sentential variables, $F, G, H, \ldots$ and a corresponding second-order universal quantifier. The second-order universal quantifier together with the first-order universal quantifier and the conditional are the sole primitive logical connectives. *Atomic formulas* are either second-order variables or expressions of the form $P(t_1, \ldots, t_n)$, where P is a n-ary relational symbol and $t_1, \ldots, t_n$ are first-order terms. The class of formulas is the smallest set containing the atomic formulas and closed under the conditional and first and second-order universal quantifiers. I.e.: if A and B are formulas, then $(A \to B)$, $\forall_1 x.A$ and $\forall_2 F.A$, with x a first-order variable and F a second-order variable, are also formulas. In the sequel, we usually omit the subscripts of $\forall_1$ and $\forall_2$.

The logic of *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ is *intuitionistic logic* (or, if one prefers, *minimal logic*, since we are in a setting without $\perp$ as a primitive symbol of the language) and the proof system used is framed in the *natural deduction calculus*. Natural deduction in *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ has the usual *introduction* rules for the conditional and the universal quantifiers:

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\ \to\mathrm{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall x.A}\ \forall_1\mathrm{I} \qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall F.A}\ \forall_2\mathrm{I}$$

where x and F do not occur free in any undischarged hypothesis (respectively). It also has *elimination* rules:

$$\dfrac{\begin{matrix}\vdots \\ A \to B\end{matrix} \qquad \begin{matrix}\vdots \\ A\end{matrix}}{B}\ \to\mathrm{E} \qquad \dfrac{\begin{matrix}\vdots \\ \forall x.A\end{matrix}}{A^x_t}\ \forall_1\mathrm{E} \qquad \dfrac{\begin{matrix}\vdots \\ \forall F.A\end{matrix}}{A^F_S}\ \forall_2\mathrm{E}$$

with t a term (free for x in A), S an *atomic* formula (free for F in A), and A^α_β results from A by replacing all the free occurrences of α by β.

Observe that in the second-order elimination rule $\forall_2$E, the instantiation of F is restricted to atomic formulas. This explains why we dub our calculus *atomic*. As we will see, this restriction is not as severe as one might first be led to think. A phenomenon, dubbed *instantiation overflow*, ensures that for formulas A with a certain structure, we can instantiate $\forall F.A$ by *any* formula of the language whatsoever.

2. The embedding

We now define the embedding of the intuitionistic predicate calculus into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$. The embedding follows a definition that Prawitz gave for the impredicative setting (see [3]):

$$\bot := \forall F.F$$
$$A \wedge B := \forall F((A \to (B \to F)) \to F)$$
$$A \vee B := \forall F((A \to F) \to ((B \to F) \to F))$$
$$\exists x.A := \forall F(\forall x(A \to F) \to F)$$

where F is a second-order variable which does not occur in A or B.

As observed, this embedding works fine in the impredicative calculus (i.e., where the elimination rule $\forall_2$E is unrestricted). Prawitz's embedding immerses the intuitionistic predicate calculus into impredicative second-order logic, a much stronger system from the proof-theoretic point of view. Note, furthermore, that in this system it does not make sense to define the notion of subformula because the instantiations of $\forall F.A$ can be arbitrarily complex. On the other hand, there is a perfectly natural definition of subformula within *atomic* $\mathsf{QSOL}^{\mathsf{i}}$: just say that the (proper) subformulas of $\forall F.A$ are the formulas A^F_S, where S is an atomic formula (free for F in A).

We claim that the above embedding is already operative into the strict predicative theory *atomic* $\mathsf{QSOL}^{\mathsf{i}}$. To see this, we need to ensure that in *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ the rules for $\bot$, $\wedge$, $\vee$ and $\exists$ remain valid after translated according to Prawitz's definition. The following result is instrumental:

PROPOSITION 1 (Instantiation overflow). *In* atomic $\mathsf{QSOL}^{\mathsf{i}}$, instantiation overflow *is available for every formula of the type above, i.e. from*

- $\forall F.F$
- $\forall F((A \to (B \to F)) \to F)$
- $\forall F((A \to F) \to ((B \to F) \to F))$
- $\forall F(\forall x(A \to F) \to F)$,

where F is a second-order variable which does not occur in A or B, we can deduce

- C
- $(A \to (B \to C)) \to C$

- $(A \to C) \to ((B \to C) \to C)$
- $\forall x(A \to C) \to C$,

for any *formula* C*, respectively. (In the last case,* x *must not occur in* C*.)*

Proof: The first three cases can be studied in a similar way to [1], where the study is effected within the context of the propositional calculus. Although here we have a richer language (with more formulas), the same strategy, by induction on the complexity of the formula C, works. It remains to study the fourth case. Suppose that we have $\forall F(\forall x(A \to F) \to F)$. We must show that it is possible to deduce $\forall x(A \to C) \to C$, for any formula C (in which x does not occur). The proof proceeds by induction on the complexity of C.

If C is an atomic formula, the result is immediate, applying the $\forall_2$E rule.

Let us study the case $C := C_1 \to C_2$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall F(\forall x(A \to F) \to F)}{\forall x(A \to C_2) \to C_2}\ \text{I.H.} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(A \to (C_1 \to C_2))]}{A \to (C_1 \to C_2)} \quad [A]}{C_1 \to C_2} \quad [C_1]}{C_2}}{A \to C_2}}{\forall x(A \to C_2)}}{C_2}}{C_1 \to C_2}}{\forall x(A \to (C_1 \to C_2)) \to (C_1 \to C_2)}$$

The double line is used to indicate that we are hiding a portion of the proof (between the two lines). In the present situation that proof exists by the induction hypothesis (I.H.).

We now present the discussion of the case $C := \forall y.C_1$ (the case $\forall G.C_1$ is similar, and we omit it). Suppose, without loss of generality, that y does not occur in A. We have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall F(\forall x(A \to F) \to F)}{\forall x(A \to C_1) \to C_1}\ \text{I.H.} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(A \to \forall y C_1)]}{A \to \forall y C_1} \quad [A]}{\forall y C_1}}{C_1}}{A \to C_1}}{\forall x(A \to C_1)}}{C_1}}{\forall y C_1}}{\forall x(A \to \forall y C_1) \to \forall y C_1}$$

■

The previous inductive proof provides an algorithmic *method* for obtaining instantiation overflows for the four types of formulas studied. We call these types *logical types*. We refer to the method of instantiation above as the *canonical* way of disclosing the portion of the proof hidden when using an instantiation overflow.

THEOREM 1. *The introduction and elimination rules of natural deduction for the connectives of the intuitionistic predicate calculus are valid in* atomic $\mathsf{QSOL}^{\mathsf{i}}$ *when translated according to Prawitz's definition.*

Proof: The rules for $\to$ and $\forall_1$ are primitive in *atomic* $\mathsf{QSOL}^{\mathsf{i}}$. The validity of the rules for $\bot$, $\wedge$ and $\vee$ can be established as in [1]. Therefore, $\exists_1$I and $\exists_1$E are the only rules requiring attention. The first is immediate:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix} \qquad \dfrac{[\forall x(A\to F)]}{A\to F}}{F}}{\forall x(A\to F)\to F}}{\forall F(\forall x(A\to F)\to F)} \qquad := \exists x.A$$

The second uses the previous proposition, i.e. from $\forall F(\forall x(A\to F)\to F)$ we can deduce $\forall x(A\to C)\to C$, for any formula C where x does not occur.

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{matrix}}{\forall x(A\to C)\to C}\,(=) \qquad \dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{A\to C}}{\forall x(A\to C)}}{C}$$

■

By a *canonical translation* of an intuitionistic proof in the predicate calculus into a proof in *atomic* $\mathsf{QSOL}^{\mathsf{i}}$, we mean a translation, rule-by-rule, according to the proof of the theorem above.

3. Properties and advantages of *atomic* $\mathsf{QSOL}^{\mathsf{i}}$

The rules of elimination for the connectives $\bot$, $\vee$ and $\exists$ have been subjected to some criticism because they are not as natural and as well behaved as the other inferential rules. On the face of it, this is a curious line of criticism because $\vee$ and $\exists$ are the most characteristic connectives of *intuitionistic* logic. In the well known book 'Proofs and Types' ([2], p. 74), in a section entitled ‘Defects of the System,’ Jean-Yves Girard says that ‘the elimination rules [of these connectives] are very bad’ and adds that ‘what is catastrophic about them is the parasitic presence of a formula C which has no structural link with the formula which is eliminated.’ Moreover, in order to have normal proofs with the subformula property, there is the need to add some *ad hoc* conversions: the *commuting conversions* (also called *permutative conversions*). Girard adds that ‘the need to add these supplementary rules reveals an inadequacy of the syntax.’ Some pages afterwards (p. 80), apropos these conversions, it is said that ‘one tends to think that natural deduction should be modified to correct such atrocities.’ We take it that Girard is complaining against the artificiality of the commuting conversions, blaming the need for these extra conversions on the elimination rules of the connectives $\perp$, $\vee$ and $\exists$.

Girard writes that ‘if a connector has such bad rules, one ignores it (a very common attitude) or one tries to change the very spirit of natural deduction in order to be able to integrate it harmoniously with the others.’ Ignoring $\perp$, $\vee$ and $\exists$ is, actually, a very common attitude in presentations of the lambda calculus. We make the following proposal: embed the intuitionistic predicate calculus into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$, where there are no bad rules. We tentatively suggest that this is the right way to see the connectives $\perp$, $\vee$ and $\exists$ in Structural Proof Theory: through the lens of the above embedding. This is a very natural move and, after all, ‘the ($\perp$, $\vee$, $\exists$) fragment of the calculus is [not] etched on tablets of stone’ (cf. Girard, *ibidem*). Of course, the suggestion must be grounded on technical work. The present article does not address this question, but we point out that it has been shown that the disjunction property can be obtained within the new framework (see the final part of [1]).

The conversions of *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ are the standard ones, namely the usual proof-theoretic transformations (reductions) applied to an introduction rule followed immediately by an elimination rule of the same connective. For instance:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{\forall F.A}}{A^F_S} \quad\rightsquigarrow\quad \begin{matrix}\vdots\\ A^F_S\end{matrix}$$

where S is an *atomic* formula free for F in A, and on the right-hand side the proof above the formula A^F_S is obtained from the proof above A by replacing each suitable occurrence of the free variable F by S. To avoid syntactic rabble, we are not being totally precise at this juncture (syntactic matters concerning normalization are well understood: see, for instance, the discussion on proper parameters in [3]). The *redex* of the above conversion is the configuration on the left-hand side, whereas the configuration on the right-hand side is called the *conversum* of the conversion.

4. Translation of the commuting conversions

There are no commuting conversions in *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ (NB the connectives $\bot$, $\vee$ and $\exists$ are absent). A curious question may, nevertheless, be raised: *how are the commuting conversions of the intuitionistic predicate calculus translated into* atomic $\mathsf{QSOL}^{\mathsf{i}}$*?* In trying to answer this question, we discovered the following: the *redex* and the *conversum* of a commuting conversion translate into second-order derivations of *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ which are, in a certain precise sense, equivalent.

We remind the reader of the three types of commuting conversions (c.c.) of intuitionistic predicate calculus:

1) conversion $\bot$E

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{C}\,\bot\mathrm{E} \qquad \vdots}{D}\,\mathrm{r} \qquad \overset{\text{c.c.}}{\rightsquigarrow} \qquad \dfrac{\begin{matrix}\vdots\\ \bot\end{matrix}}{\mathrm{D}}\,\bot\mathrm{E}$$

2) conversion $\vee$E

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\,\vee\mathrm{E} \qquad \vdots}{D}\,\mathrm{r} \qquad \overset{\text{c.c.}}{\rightsquigarrow} \qquad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix} \quad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \vdots}{D}\,\mathrm{r} \quad \dfrac{\begin{matrix}[B]\\ \vdots\\ C\end{matrix} \quad \vdots}{D}\,\mathrm{r}}{D}\,\vee\mathrm{E}$$

3) conversion $\exists_1$E

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists xA\end{matrix} \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix}}{C}\,\exists_1\mathrm{E} \qquad \vdots}{D}\,\mathrm{r} \qquad \overset{\text{c.c.}}{\rightsquigarrow} \qquad \dfrac{\begin{matrix}\vdots\\ \exists xA\end{matrix} \quad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \vdots}{D}\,\mathrm{r}}{D}\,\exists_1\mathrm{E}$$

where r stands for an elimination rule with principal premise C.

The *redex* of a commuting conversion is the configuration on the left-hand side, whereas the configuration on the right-hand side is called the *conversum* of the conversion.

Definition 1. *We say that two derivations of* atomic $\mathsf{QSOL}^{\mathsf{i}}$ *are* $\to\forall_1\forall_2$-equivalent *if one is obtained from the other by a finite series of standard conversions of* $\to$, $\forall_1$ *and* $\forall_2$ *in* both *directions.*

Theorem 2 (Main Theorem). *The canonical translations into* atomic $\mathsf{QSOL}^{\mathsf{i}}$ *of the* redex *and the* conversum *of a commuting conversion are* $\to\forall_1\forall_2$-*equivalent.*

The above result is not true for Prawitz's embedding of the intuitionistic predicate calculus into impredicative second-order logic (see [5]). For example, if A and B are atomic formulas, the c.c.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \bot\end{array}}{A\to B}\,\bot\mathrm{E} \qquad \begin{array}{c}\vdots\\ A\end{array}}{B}\to\mathrm{E} \qquad \overset{\text{c.c.}}{\rightsquigarrow} \qquad \dfrac{\begin{array}{c}\vdots\\ \bot\end{array}}{B}\,\bot\mathrm{E}$$

translated into full (unrestricted) $\mathsf{QSOL}^{\mathsf{i}}$ has *redex*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall F.F\end{array}}{A\to B}\,\forall_2\mathrm{E} \qquad \begin{array}{c}\vdots\\ A\end{array}}{B}\to\mathrm{E} \qquad \text{and } \textit{conversum} \qquad \dfrac{\begin{array}{c}\vdots\\ \forall F.F\end{array}}{B}\,\forall_2\mathrm{E}$$

and these cannot be linked via standard conversions.

The correspondence works in *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ because the translation requires that $\forall_2$E-rules are instantiated only with atomic formulas. In the previous example, the *redex* becomes

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall F.F\end{array}}{B}\,\forall_2\mathrm{E}}{A\to B} \qquad \begin{array}{c}\vdots\\ A\end{array}}{B}\to\mathrm{E} \qquad \text{and the } \textit{conversum} \qquad \dfrac{\begin{array}{c}\vdots\\ \forall F.F\end{array}}{B}\,\forall_2\mathrm{E}$$

and the latter is obtained from the former via a single standard conversion.

The remaining part of this article is dedicated to the proof of the above Main Theorem. The proof proceeds in two steps. In this section, we take care of the first step (which is of independent interest). We need the following auxiliary notion:

Definition 2. *Let be given a logical type of the form* $\forall F.A$ *and a derivation*

$$\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\forall F.A}$$

such that the subderivation above A *only has second-order elimination rules* $\forall_2$E *applied to logical types. Let* D *be a formula free for* F *in* A*. We call the following proof-transformation a* standard block-conversion *(considering* D *as a block):*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{\forall F.A}}{\overline{A^F_D}} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\\ A^F_D\end{array}$$

where, on the left hand-side the double line hides the canonical way of overflowing instantiation and, on the right-hand side, the configuration above the formula A^F_D *is obtained from the proof above* A *by replacing each suitable occurrence of the free variable* F *by* D *and by* inserting *the canonical justifications of the instantiation overflows.*

As usual, the *redex* of a standard block-conversion is the configuration on the left-hand side, whereas the configuration on the right-hand side is called the *conversum* of the block-conversion. The following proposition is the first step towards proving the Main Theorem. We believe that it has also independent interest because it uses conversions in *one* direction only:

Proposition 2. *From the canonical translation of the* redex *of a commuting conversion into* atomic $\mathsf{QSOL}^{\mathsf{i}}$ *we can obtain, through successive applications of standard conversions* and *standard block-conversions, the canonical translation of the* conversum *of the commuting conversion.*

Proof: We study in detail the case of the commuting conversion for $\exists_1$E. The other commuting conversions can be studied in a similar way.

We show that from the canonical translation of

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \exists x A\end{array} \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array}}{C}\,\exists_1\text{E} \qquad \begin{array}{c}\vdots\end{array}}{D}\,\text{r}$$

into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ we obtain, by means of standard conversions and standard block-conversions (where D is used as a block), the canonical translation of

$$\dfrac{\begin{matrix}\vdots\\ \exists x A\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}\vdots\\ \end{matrix}}{D}\,\mathrm{r}}{D}\,\exists_1\mathrm{E}$$

We study exhaustively all the possibilities for the formula C:

• C cannot be an atomic formula because r is an elimination rule with C as a principal premise.

• If C is the formula $\bot$, the c.c. is

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists x A\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\\ \bot\end{matrix}}{\bot}\,\exists_1\mathrm{E}}{D} \quad \overset{\text{c.c.}}{\rightsquigarrow} \quad \dfrac{\begin{matrix}\vdots\\ \exists x A\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ \bot\end{matrix}}{D}}{D}\,\exists_1\mathrm{E}$$

The formula-by-formula translation of the *redex* of the c.c. to *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ yields (for ease of notation, we ignore the translations of A and D)

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A \to F) \to F)\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\\ \forall F.F\end{matrix}}{\forall F.F}}{D}$$

We will be somewhat detailed in the discussion of this case. By effecting the canonical translation into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ we get:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A \to F) \to F)\end{matrix}}{\forall x(A \to F) \to F} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(A \to \forall F.F)]}{A \to \forall F.F} \quad [A]}{\forall F.F}}{F}}{A \to F}}{\forall x(A \to F)}}{F}}{\forall F.F}}{\forall x(A \to \forall F.F) \to \forall F.F} \qquad \dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\\ \forall F.F\end{matrix}}{A \to \forall F.F}}{\forall x(A \to \forall F.F)}}{\dfrac{\forall F.F}{D}}$$

Three standard conversions yield

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{matrix}}{\forall x(A\to F)\to F}\qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\\ \forall F.F\end{matrix}}{F}}{A\to F}}{\forall x(A\to F)}}{F}}{\forall F.F}}{D}\ (\text{double line})
$$

Applying a standard block-conversion (considering D as a block), we get

$$
\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{matrix}}{\forall x(A\to D)\to D}\ (\text{double line})\qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\\ \forall F.F\end{matrix}}{D}\ (\text{double line})}{A\to D}}{\forall x(A\to D)}}{D}
$$

which is the canonical translation to *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ of the *conversum* of the commuting conversion.

- If C is a formula of the form $C_1 \vee C_2$, the c.c. is

$$
\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists xA\end{matrix}\qquad \begin{matrix}[A]\\ \vdots\\ C_1\vee C_2\end{matrix}}{C_1\vee C_2}\qquad \begin{matrix}[C_1]\\ \vdots\\ D\end{matrix}\qquad \begin{matrix}[C_2]\\ \vdots\\ D\end{matrix}}{D}
\quad\overset{\text{c.c.}}{\rightsquigarrow}\quad
\dfrac{\begin{matrix}\vdots\\ \exists xA\end{matrix}\qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ C_1\vee C_2\end{matrix}\qquad \begin{matrix}[C_1]\\ \vdots\\ D\end{matrix}\qquad \begin{matrix}[C_2]\\ \vdots\\ D\end{matrix}}{D}}{D}
$$

The formula-by-formula translation of the *redex* of the c.c. to *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ yields (for ease of notation, we ignore the translations of A, C_1, C_2 and D)

$$
\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{matrix}\qquad \begin{matrix}[A]\\ \vdots\\ \forall F((C_1\to F)\to((C_2\to F)\to F))\end{matrix}}{\forall F((C_1\to F)\to((C_2\to F)\to F))}\qquad \begin{matrix}[C_1]\\ \vdots\\ D\end{matrix}\qquad \begin{matrix}[C_2]\\ \vdots\\ D\end{matrix}}{D}\ (\text{dashed lines})
$$

Applying the canonical translation into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ we obtain, with the aid of *twelve* standard conversions, the configuration in Fig. 1 (see the appendix for this and other figures). By a standard block-conversion (considering D as a block), we get Fig. 2. With *two* more standard conversions we get Fig. 3, which is the canonical translation into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ of the *conversum* of the commuting conversion.

• If C is a formula of the form $C_1 \to C_2$, the c.c. is

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists xA\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\\ C_1\to C_2\end{matrix}}{C_1 \to C_2} \qquad \begin{matrix}\vdots\\ C_1\end{matrix}}{C_2} \quad \overset{\text{c.c.}}{\rightsquigarrow} \quad \dfrac{\begin{matrix}\vdots\\ \exists xA\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ C_1\to C_2\end{matrix} \qquad \begin{matrix}\vdots\\ C_1\end{matrix}}{C_2}}{C_2}$$

The formula-by-formula translation of the *redex* of the c.c. to *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ yields (for ease of notation, we ignore the translations of A, C_1 and C_2)

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A \to F) \to F)\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\\ C_1\to C_2\end{matrix}}{C_1 \to C_2} \qquad \begin{matrix}\vdots\\ C_1\end{matrix}}{C_2}$$

Applying the canonical translation into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$, we obtain Fig. 4 and with *four* standard conversions we get Fig. 5, which is the canonical translation into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ of the *conversum* of the commuting conversion.

• We omit the study of conjunction to make the paper shorter. Actually, conjunction is also a 'good' connective and could have been taken as primitive.

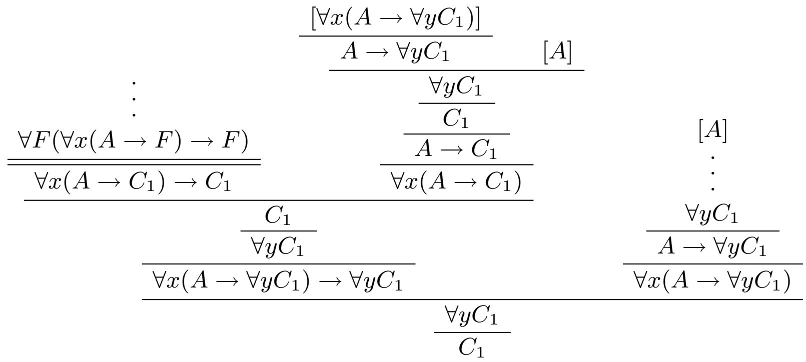
• If C is a formula of the form $\forall y C_1$, the c.c. is

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \exists xA\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\\ \forall y C_1\end{matrix}}{\forall y C_1}}{C_1} \quad \overset{\text{c.c.}}{\rightsquigarrow} \quad \dfrac{\begin{matrix}\vdots\\ \exists xA\end{matrix} \qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ \forall y C_1\end{matrix}}{C_1}}{C_1}$$

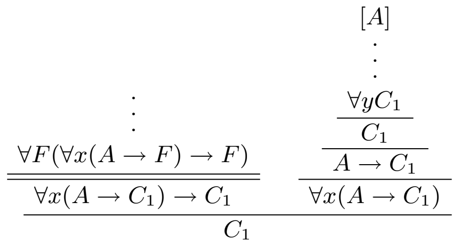
The formula-by-formula translation of the *redex* of the c.c. to *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ yields (for ease of notation, we ignore the translations of A and C_1)

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A \to F) \to F)\end{matrix} \qquad \begin{matrix}[A]\\ \vdots\\ \forall y C_1\end{matrix}}{\forall y C_1}}{C_1}$$

Applying the canonical translation into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$, we obtain:

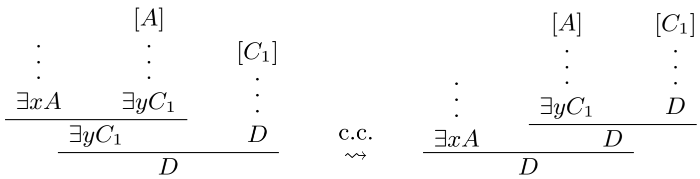


We now apply *four* standard conversions and get

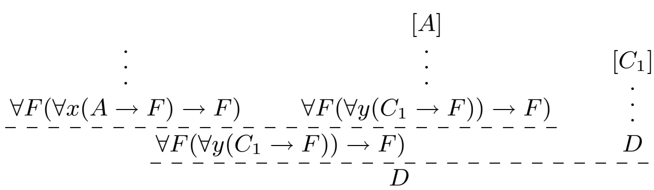


which is the canonical translation into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ of the *conversum* of the commuting conversion.

- Finally, suppose that C is a formula of the form $\exists y C_1$. Its c.c. is



The formula-by-formula translation of the *redex* of the c.c. to *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ yields (for ease of notation, we ignore the translations of A, C_1 and D)



Applying the canonical translation into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ we obtain, with the aid of *six* standard conversions,

$$
\dfrac{
\dfrac{\overline{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{array}}{\forall x(A\to F)\to F} \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ \forall F(\forall y(C_1\to F)\to F)\end{array}}{\forall y(C_1\to F)\to F} \quad [\forall y(C_1\to F)]}{F}}{A\to F}}{\forall x(A\to F)}}{F}}{\forall y(C_1\to F)\to F}}{\forall F(\forall y(C_1\to F)\to F)}}}{\forall y(C_1\to D)\to D}
\quad
\dfrac{\dfrac{\begin{array}{c}[C_1]\\ \vdots\\ D\end{array}}{C_1\to D}}{\forall y(C_1\to D)}
}{D}
$$

Now, by a standard block-conversion (considering D as a block), we get:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{array}}{\overline{\forall x(A\to D)\to D}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ \forall F(\forall y(C_1\to F)\to F)\end{array}}{\overline{\forall y(C_1\to D)\to D}} \quad [\forall y(C_1\to D)]}{D}}{A\to D}}{\forall x(A\to D)}}{D}}{\forall y(C_1\to D)\to D}
\quad
\dfrac{\dfrac{\begin{array}{c}[C_1]\\ \vdots\\ D\end{array}}{C_1\to D}}{\forall y(C_1\to D)}
}{D}
$$

Applying a further standard conversion, we have

$$
\dfrac{
\dfrac{\begin{array}{c}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{array}}{\overline{\forall x(A\to D)\to D}}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ \forall F(\forall y(C_1\to F)\to F)\end{array}}{\overline{\forall y(C_1\to D)\to D}} \quad \dfrac{\dfrac{\begin{array}{c}[C_1]\\ \vdots\\ D\end{array}}{C_1\to D}}{\forall y(C_1\to D)}}{D}}{A\to D}}{\forall x(A\to D)}
}{D}
$$

which is the canonical translation into *atomic* $\mathsf{QSOL}^{\mathsf{i}}$ of the *conversum* of the commuting conversion.

The proof of the proposition is finished. ■

5. Analyzing block-conversions

In order to finish the proof of the Main Theorem, it is enough to show that the standard block-conversions that were *actually used* in the proof of Proposition 2 enjoy the following property: their *redexes* and *conversa* are $\to\forall_1\forall_2$-equivalent.

We study, in detail, the first standard block-conversion used in the proof (case $C :=\bot$).

The standard block-conversion has *redex*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{array}}{\forall x(A\to F)\to F}\qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ \forall F.F\end{array}}{F}}{A\to F}}{\forall x(A\to F)}}{\dfrac{F}{\dfrac{\forall F.F}{D}}}$$

and *conversum*

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{array}}{\forall x(A\to D)\to D}\qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ \forall F.F\end{array}}{D}}{A\to D}}{\forall x(A\to D)}}{D}$$

The study is done by induction on the complexity of the formula D.

When D is an atomic formula, the standard block-conversion is, in fact, a standard conversion. Graphically, denoting by Δ_1 and Δ_2 the *redex* and the *conversum* of the standard block-conversion respectively and by arrows the standard conversions, we have: $\overset{\Delta_1}{\bullet}\xrightarrow{1}\overset{\Delta_2}{\bullet}$

If $D := D_1 \to D_2$ the *redex* (Δ_1) of the standard block-conversion has the form

$$\dfrac{\dfrac{\underline{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{array}}{\forall x(A\to F)\to F}\qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ \forall F.F\end{array}}{F}}{A\to F}}{\forall x(A\to F)}}{F}}{\forall F.F}}}{D_2}}{D_1\to D_2}$$

By induction hypothesis we obtain the following derivation (denoted by Δ)

$$\dfrac{\dfrac{\dfrac{\underline{\begin{array}{c}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{array}}}{\forall x(A\to D_2)\to D_2}\qquad \dfrac{\dfrac{\dfrac{\underline{\begin{array}{c}[A]\\ \vdots\\ \forall F.F\end{array}}}{D_2}}{A\to D_2}}{\forall x(A\to D_2)}}{D_2}}{D_1\to D_2}$$

The *conversum* (Δ_2) of the standard block-conversion has the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\underline{\begin{array}{c}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{array}}}{\forall x(A\to D_2)\to D_2}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(A\to(D_1\to D_2))]}{A\to(D_1\to D_2)}\quad [A]}{D_1\to D_2}\quad [D_1]}{D_2}}{A\to D_2}}{\forall x(A\to D_2)}}{D_2}}{D_1\to D_2}}{\forall x(A\to(D_1\to D_2))\to(D_1\to D_2)}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\underline{\begin{array}{c}[A]\\ \vdots\\ \forall F.F\end{array}}}{D_2}}{D_1\to D_2}}{A\to(D_1\to D_2)}}{\forall x(A\to(D_1\to D_2))}}{D_1\to D_2}$$

Applying *four* standard conversions we obtain the derivation Δ. The scheme is $\overset{\Delta_1}{\bullet}\overset{IH}{\longleftrightarrow}\overset{\Delta}{\bullet}\overset{4}{\longleftarrow}\overset{\Delta_2}{\bullet}$

If $D := \forall y D_1$ (the case $D := \forall G D_1$ is similar, and we omit it) the *redex* (Δ_1) of the standard block-conversion has the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{matrix}}{\forall x(A\to F)\to F}\qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\\ \forall F.F\end{matrix}}{F}}{A\to F}}{\forall x(A\to F)}}{F}}{\forall F.F}}{D_1}}{\forall y D_1}$$

By induction hypothesis we have the following derivation (denoted by Δ)

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{matrix}}{\forall x(A\to D_1)\to D_1}\qquad \dfrac{\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\\ \forall F.F\end{matrix}}{D_1}}{A\to D_1}}{\forall x(A\to D_1)}}{D_1}}{\forall y D_1}$$

The *conversum* (Δ_2) of the standard block-conversion has the form

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall F(\forall x(A\to F)\to F)\end{matrix}}{\forall x(A\to D_1)\to D_1}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(A\to\forall y D_1)]}{A\to\forall y D_1}\quad [A]}{\forall y D_1}}{D_1}}{A\to D_1}}{\forall x(A\to D_1)}}{D_1}}{\forall y D_1}}{\forall x(A\to\forall y D_1)\to\forall y D_1}\qquad \dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\\ \forall F.F\end{matrix}}{D_1}}{\forall y D_1}}{A\to\forall y D_1}}{\forall x(A\to\forall y D_1)}}{\forall y D_1}$$

Applying *four* standard conversions we obtain the derivation Δ. Again, the diagram is $\overset{\Delta_1}{\bullet}\overset{IH}{\longleftrightarrow}\overset{\Delta}{\bullet}\overset{4}{\longleftarrow}\overset{\Delta_2}{\bullet}$

The other standard block-conversions can be examined in an entirely analogous way. We will just indicate the number and direction of the standard conversions needed to establish the equivalences.

In the second standard block-conversion used in the proof of Proposition 2 (case $C := C_1 \vee C_2$), the graphics are:
$\overset{\Delta_1}{\bullet}\overset{1}{\longrightarrow}\overset{\Delta_2}{\bullet}$, $\overset{\Delta_1}{\bullet}\overset{IH}{\longleftrightarrow}\overset{\Delta}{\bullet}\overset{2}{\longrightarrow}\bullet\overset{6}{\longleftarrow}\overset{\Delta_2}{\bullet}$ and $\overset{\Delta_1}{\bullet}\overset{IH}{\longleftrightarrow}\overset{\Delta}{\bullet}\overset{2}{\longrightarrow}\bullet\overset{6}{\longleftarrow}\overset{\Delta_2}{\bullet}$

for D an atomic formula, $D := D_1 \to D_2$ and $D := \forall y D_1$ respectively.

In the third standard block-conversion used in the proof of Proposition 2 (case $C := \exists y C_1$), the graphics are:

$\overset{\Delta_1}{\bullet} \overset{1}{\longrightarrow} \overset{\Delta_2}{\bullet}$, $\overset{\Delta_1}{\bullet} \overset{IH}{\longleftrightarrow} \overset{\Delta}{\bullet} \overset{1}{\longrightarrow} \bullet \overset{5}{\longleftarrow} \overset{\Delta_2}{\bullet}$ and $\overset{\Delta_1}{\bullet} \overset{IH}{\longleftrightarrow} \overset{\Delta}{\bullet} \overset{1}{\longrightarrow} \bullet \overset{5}{\longleftarrow} \overset{\Delta_2}{\bullet}$

for D an atomic formula, $D := D_1 \to D_2$ and $D := \forall y D_1$ respectively. ■

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Appendix

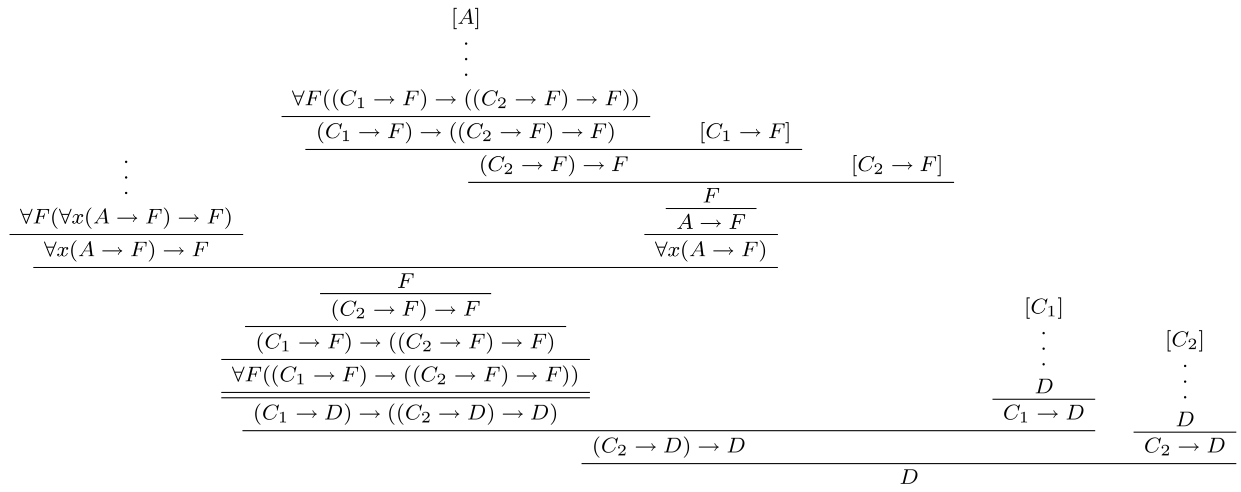

Fig. 1

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\begin{array}{c}\vdots\\ \forall F(\forall x(A \to F) \to F)\end{array}}{\forall x(A \to D) \to D}
      \quad
      \dfrac{\dfrac{\dfrac{
        \dfrac{
          \dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ \forall F((C_1 \to F) \to ((C_2 \to F) \to F))\end{array}}{(C_1 \to D) \to ((C_2 \to D) \to D)} \quad [C_1 \to D]}{(C_2 \to D) \to D}
          \quad [C_2 \to D]
        }{D}
      }{A \to D}}{\forall x(A \to D)}
    }{\dfrac{\dfrac{D}{(C_2 \to D) \to D}}{(C_1 \to D) \to ((C_2 \to D) \to D)}}
    \quad
    \dfrac{\begin{array}{c}[C_1]\\ \vdots\\ D\end{array}}{C_1 \to D}
  }{(C_2 \to D) \to D}
  \quad
  \dfrac{\begin{array}{c}[C_2]\\ \vdots\\ D\end{array}}{C_2 \to D}
}{D}
$$

Fig. 2

$$
\dfrac{
  \dfrac{\begin{array}{c}\vdots\\ \forall F(\forall x(A \to F) \to F)\end{array}}{\forall x(A \to D) \to D}
  \quad
  \dfrac{\dfrac{\dfrac{
    \dfrac{
      \dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\\ \forall F((C_1 \to F) \to ((C_2 \to F) \to F))\end{array}}{(C_1 \to D) \to ((C_2 \to D) \to D)} \quad \dfrac{\begin{array}{c}[C_1]\\ \vdots\\ D\end{array}}{C_1 \to D}}{(C_2 \to D) \to D}
      \quad \dfrac{\begin{array}{c}[C_2]\\ \vdots\\ D\end{array}}{C_2 \to D}
    }{D}
  }{A \to D}}{\forall x(A \to D)}
}{D}
$$

Fig. 3

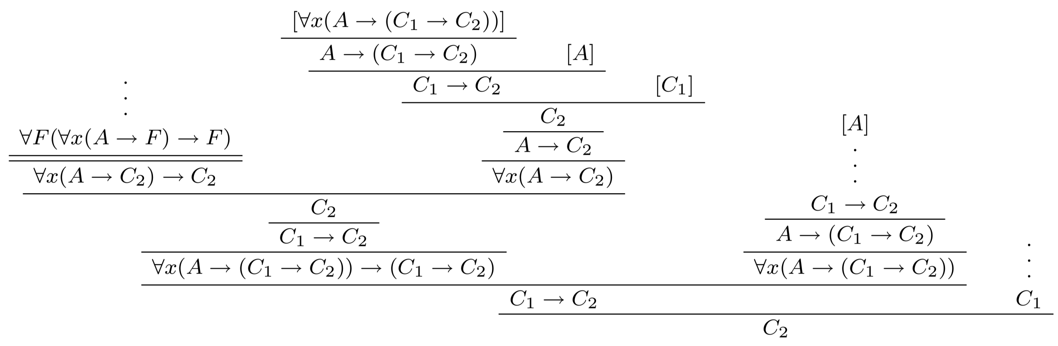

Fig. 4

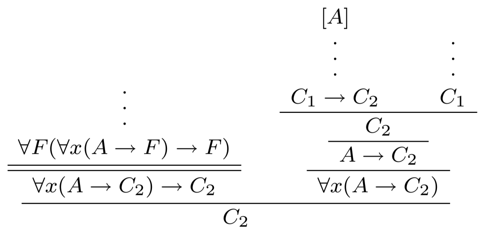

Fig. 5