Internal wave drag in stratified flow over mountains on a beta plane

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ABSTRACT: The impact of the variation of the Coriolis parameter \( f \) on the drag exerted by internal Rossby-gravity waves on elliptical mountains is evaluated using linear theory, assuming constant wind and static stability and a beta-plane approximation. Previous calculations of inertia-gravity wave drag are thus extended in an attempt to establish a connection with existing studies on planetary wave drag, developed primarily for fluids topped by a rigid lid. It is found that the internal wave drag for zonal westerly flow strongly increases relative to that given by the calculation where \( f \) is assumed to be a constant, particularly at high latitudes and for mountains aligned meridionally. Drag increases with mountain width for sufficiently wide mountains, reaching values much larger than those valid in the non-rotating limit. This occurs because the drag receives contributions from a low wavenumber range, controlled by the beta effect, which accounts for the drag amplification found here. This drag amplification is shown to be considerable for idealized analogues of real mountain ranges, such as the Himalayas and the Rocky mountains, and comparable to the barotropic Rossby wave drag addressed in previous studies. Copyright © 2008 Royal Meteorological Society

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1. Introduction

The drag force exerted on mountains by internal gravity and inertia-gravity waves has been reasonably well studied (Smith, 1979a; Miranda and James, 1992; Grisogono et al., 1993; Ólafsson and Bougeault, 1997). To the authors’ knowledge, however, there are no calculations of the drag in the case of orographic internal Rossby-gravity waves in a continuously stratified atmosphere, where the variation of the Coriolis parameter with latitude is important. This may be due to the fact that standard internal wave theory uses the Boussinesq approximation, a Cartesian coordinate system and a continuous spectrum of wavenumbers. As pointed out by Smith (1979b), these approximations are no longer accurate for Rossby-gravity waves. This is because the vertical wavelength of these waves is large, being affected by the vertical density variation in the atmosphere, and horizontally these waves are of planetary scale, being affected by the spherical geometry of the Earth and possessing a discrete wavenumber spectrum. Additionally, the orographic drag associated with these waves occurs at large scales resolved by even the models with the coarsest horizontal resolutions such as General Circulation Models (GCMs), so in general it does not need to be parameterized.

Despite this, most of the studies addressing Rossby-gravity waves and associated drag (e.g. Janowitz, 1977; McCartney, 1975; Thompson and Flierl, 1993) use a beta-plane approximation as this is the simplest way to retain the essential features of Rossby waves while neglecting other geometrical complications. As a rule, these studies also assume that the fluid under consideration is bounded above by a rigid lid or a sharp density interface. These studies are often concerned with oceanic flows (e.g. Janowitz, 1975), although this approximation may also be viewed as a rough representation of the finite extent of the troposphere (Janowitz, 1977). Consequently, internal waves are either entirely absent (when constant-density fluids are considered, e.g. Ingersoll, 1969; Thompson and Flierl, 1993) or discarded as insignificant compared to the barotropic mode (Janowitz, 1977), a result that relies crucially on the existence of the rigid lid. The adequacy of this latter choice will be re-examined in the present study.

Smith (1979c) qualitatively mentioned the relevance of the beta effect on the drag produced by internal Rossby-gravity waves in a study where he analysed the effects of the rigid lid approximation and the two-dimensional approximation. The impacts of these approximations on the surface drag were not addressed, but it is clear that unlike a continuously decreasing density (which in fact can be incorporated into Boussinesq models through a simple change of variables) the assumption of a rigid lid in a stratified atmosphere will modify the internal wave
drag considerably. In this situation, wave reflections at the top of the domain will surely induce resonances, in much the same way as described for pure gravity waves in Teixeira et al. (2005) and Teixeira and Miranda (2005).

The assumptions of the Boussinesq approximation, semi-infinite continuously stratified atmosphere and a continuous wave spectrum may not be strictly accurate in a geophysical context. However, they constitute leading-order approximations to the problem of internal Rossby-gravity waves, and are not much more unrealistic than the assumption of a rigid lid or of topographic barriers with sharp or even vertical edges, such as used by Ingersoll (1969), McCartney (1975) or Thompson and Flierl (1993). On the other hand, these assumptions have the great advantage of facilitating comparison with previous models of gravity and inertia-gravity waves, which adopted them, and for which any connection with existing Rossby-gravity wave drag studies is hard to establish. We therefore believe the assumptions listed and adopted here are acceptable for tackling our goal: an inclusion of Rossby wave effects in the existing mesoscale drag context.

We calculate internal wave drag exerted on an elliptical mountain on a beta-plane using linear theory, assuming for simplicity that the atmosphere has constant zonal incoming wind and buoyancy (Brunt–Väisälä) frequency. This work can be regarded as an extension of the study of Smith (1979a) or the linearized calculations of Miranda et al. (1983) and Bougeault, 1997; Teixeira and Miranda, 2006), but where the effects of anisotropic mountain ranges. The incoming wind and buoyancy (Brunt–Väisälä) frequency.

This work is organised as follows. In Section 2 the model, including the governing wave equation and the radiation boundary condition, is introduced. We present the main results in Section 3, namely the dependence of the drag on the dimensionless parameters controlling this problem, and drag calculations for more realistic conditions using simple approximations to real orography. Finally, Section 4 contains the main conclusions of this study.

2. The vertical structure equation

Flow over a mountain with an elliptical horizontal cross-section is considered (cf. Phillips, 1984; Ölafsson and Bougeault, 1997; Teixeira and Miranda, 2006). For simplicity, the main axes of the ellipse are assumed to be aligned in the zonal (x) and meridional (y) directions. This kind of orography is able to approximate, to a certain extent, real anisotropic mountain ranges. The incoming flow is assumed to be constant and in the zonal direction. This flow automatically satisfies the conservation of absolute (or potential) vorticity; if the incoming flow was constant and simultaneously had a meridional component, it would violate this constraint. The flow is also assumed to be steady, in geostrophic and hydrostatic equilibrium and having a westerly orientation, since it is known that Rossby waves only exist for westerly flow.

The basic equations of Thuburn (2006, Equations (1)–(6)) are adopted here, but the Boussinesq approximation is further imposed on them. These are the inviscid and adiabatic equations of motion for a beta-plane expressed in spectral space in the horizontal directions. It would initially seem questionable to expand the flow variables as Fourier integrals along x and y as is traditionally done in gravity and inertia-gravity wave theories, because of the dependence of f (the Coriolis parameter) on y. However, Thuburn (2006) (see also Thuburn and Woolings, 2005) developed an approach for this purpose, which respects the conservation of energy and is thus self-consistent.

The linearized equations of motion take the form:

\[ iUk\hat{u} - f_0\hat{v} - i\frac{k\beta}{k^2 + l^2}\hat{u} = -i\frac{k}{\rho_0}\hat{p}, \]  
\[ iUk\hat{v} + f_0\hat{u} - i\frac{k\beta}{k^2 + l^2}\hat{v} = -i\frac{l}{\rho_0}\hat{p}, \]  
\[ -\frac{\hat{p}}{\rho_0} + \hat{b} = 0, \]  
\[ iU\hat{b} + N^2\hat{w} = 0, \]  
\[ ik\hat{u} + lN^2\hat{\omega} = 0. \]

where \( U \) and \( N \) are the wind velocity and Brunt–Väisälä frequency of the incoming flow, respectively, \( \rho_0 \) is a constant reference density, and \( (\hat{u}, \hat{v}, \hat{w}) \) are the Fourier transforms of the velocity, pressure and buoyancy perturbations, respectively. \( \hat{b} \) is the acceleration of gravity, \( \hat{\theta} \) is the Fourier transform of the potential temperature perturbation and \( \theta_0 \) is a reference potential temperature (assumed to be constant).

In Equations (1) and (2), the Coriolis parameter is expressed as \( f = f_0 + \beta y \) (consistent with the beta-plane approximation), where \( \beta \) is its meridional derivative. (k, l) is the horizontal wavenumber of the waves and the primes denote differentiation with respect to height, \( z \). The third terms in (1) and (2) correspond to the beta effect, in accordance with the arguments presented by Thuburn and Woolings (2005).

In Equation (3), the hydrostatic approximation was also adopted for the flow perturbations, which is acceptable at the large horizontal scales addressed here. These equations may be combined in order to eliminate all dependent variables except the Fourier transform of the vertical velocity perturbation, \( \hat{u} \), yielding an equation akin to the Taylor–Goldstein equation (cf. Teixeira and Miranda, 2006), but where the effects of \( f_0 \) and \( \beta \) are both retained:

\[ \hat{w}'' - \left\{ \frac{N^2(k^2 + l^2)}{k^2U^2} \left[ 1 - \frac{\beta}{U(k^2 + l^2)} \right] - f_0^2 \right\} \hat{w} = 0. \]
In the case of a neutrally stratified atmosphere, such as considered in the study of barotropic Rossby waves by Janowitz (1977), Equation (6) gives the trivial result that \( \hat{w} \) varies linearly with height. For \( N^2 > 0 \), since the coefficient multiplying \( \hat{\psi} \) in Equation (6) is constant, this equation will have solutions of the form

\[
\hat{w}(z) = \hat{w}(0)e^{imz},
\]

(7)

where \( m \) is a vertical wavenumber. By substituting Equation (7) into Equation (6), one obtains:

\[
m^2 = \frac{N^2(k^2 + l^2)
}{k^2U^2\left[1 - \frac{\beta}{U(k^2 + l^2)}\right] - f_0^2}.
\]

(8)

It may be checked that this definition is consistent with Equation (58) of Thuburn and Woolings (2005), giving the dispersion relation of planetary internal waves if \( \omega \) (the angular frequency) is replaced by \( -Uk \) and the non-Boussinesq terms are neglected. Note that Equation (8) reduces to the usual definition of \( m \) for inertia-gravity waves when \( \beta = 0 \), and to the \( m \) appropriate for pure gravity waves when both \( \beta = 0 \) and \( f_0 = 0 \).

Equation (6) is subject to two boundary conditions. The lower boundary condition (free-slip condition) requires that the flow be tangential to the topography at the surface, i.e.

\[
\hat{w}(z = 0) = iUk\hat{h},
\]

(9)

where \( \hat{h} \) is the Fourier transform of the surface elevation. The upper boundary condition requires that the wave energy radiates upwards (since the waves are generated where \( h \) is the Fourier transform of the surface elevation. This result will be used in the drag calculations that follow.

2.1. Wave drag

The drag exerted on the mountain is defined in Fourier space as (Teixeira and Miranda, 2006)

\[
(D_x, D_y) = 4\pi^2i \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (k, l)\hat{p}^*(z = 0)\hat{h} dk dl,
\]

(12)

where \( \hat{p} \) is the Fourier transform of the pressure perturbation and the asterisk denotes complex conjugate.

The pressure perturbation at the surface may be obtained in terms of \( \hat{w} \) by combining the equations of motion (1)–(5) and Equation (7). If the lower boundary condition (9) is additionally used, this yields:

\[
\hat{p}(z = 0) = i\rho_0 \left\{ \frac{(Uk)^2\left[1 - \frac{\beta}{U(k^2 + l^2)}\right] - f_0^2}{(k^2 + l^2)^2\left[1 - \frac{\beta}{U(k^2 + l^2)}\right]} \right\} \hat{m} h.
\]

(13)

On inserting Equation (13) into Equation (12), and using also Equations (8) and (11) to substitute \( m \), the RGWD becomes

\[
(D_x, D_y) = 4\pi^2\rho_0NU \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{(k, l)\hat{p}^*(z = 0)\hat{h}}{(k^2 + l^2)^2} \times \frac{1}{\text{sign}(k)\text{Re}\{\beta\}} \frac{k^2 \left[1 - \frac{\beta}{U(k^2 + l^2)}\right]^2 - f_0^2}{U^2} \frac{1}{\frac{1}{2}} \right) \right\} \frac{1}{2} dk dl,
\]

(14)

where ‘Re’ denotes the ‘real part’.

Since this is the type of orography used in most of the studies that the present calculations aim to extend (e.g. Phillips, 1984; Ólafsson and Bougeault, 1997), the mountain is assumed to have an elliptical horizontal cross-section and a bell-shaped profile:

\[
h = \frac{h_0}{\sqrt{\left(1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{3}{2}}}} \Rightarrow \hat{h} = \frac{h_0a b}{2\pi} e^{-\left(a^2k^2 + b^2l^2\right)} \frac{1}{2},
\]

(15)

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where $h_0$ is the maximum height, $a$ is the zonal half-width and $b$ is the meridional half-width. It may be noted that, for this kind of mountain or any other orography that is symmetric with respect to $y$, the corresponding Fourier transform is even in $l$. The integral of $D_l$ therefore cancels in Equation (14) by symmetry, because the integrand is odd in $l$. We will consequently focus on the zonal drag component $D_o$, hereafter referred to as $D$.

Given Equation (15), it is convenient to perform a change of variables in the integrals of Equation (14), adopting polar elliptical coordinates for the horizontal wavenumbers (cf. Teixeira and Miranda, 2006):

$$k = \frac{\kappa}{a} \cos \theta, \quad l = \frac{\kappa}{b} \sin \theta.$$  

(16)

Using Equation (15), Equation (14) may then be expressed as

$$D = \rho_0 N U b h_0^2 \int_0^{2\pi} \int_0^{+\infty} \frac{\kappa e^{-2\kappa} \cos \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^{\frac{3}{2}}} d\kappa d\theta.$$  

$$D = \rho_0 N U b h_0^2 \int_0^{2\pi} \int_0^{+\infty} \frac{k^2 \cos^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^{\frac{3}{2}}} \left[1 - \frac{N_\beta}{k^2(\cos^2 \theta + \gamma^2 \sin^2 \theta)}\right]^{\frac{1}{2}} - Ro^{-2} \right] \frac{d\kappa d\theta}{k^2(\cos^2 \theta + \gamma^2 \sin^2 \theta)},$$  

(17)

where $\gamma = a/b$ is the horizontal aspect ratio of the mountain, $Ro = U/(f_0 a)$ is a Rossby number and $N_\beta = \beta a^2/U$ is a dimensionless number quantifying the importance of the beta effect. A dimensionless number similar to $N_\beta$ was referred to as $\beta$ by Bannon (1980) and by McCartney (1975), for example.

In order to facilitate its interpretation, the drag is normalized by its value in the absence of rotation, which results from Equation (17) when $Ro^{-1} = N_\beta = 0$, i.e.

$$D_0 = \rho_0 N U b h_0^2 \int_0^{2\pi} \int_0^{+\infty} \frac{k^2 \cos^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^{\frac{3}{2}}} d\kappa d\theta.$$  

(18)

Another important reference value is the quasi-geostrophic drag, to which Equation (17) reduces when $Ro \ll 1$:

$$D_{QG} = \rho_0 N_{f0} a b h_0^2 \int_0^{2\pi} \int_0^{+\infty} \frac{\kappa e^{-2\kappa} \cos \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^{\frac{1}{2}}} \right] \frac{d\kappa d\theta}{k^2(\cos^2 \theta + \gamma^2 \sin^2 \theta)},$$  

(19)

While in Equation (18) all wavenumbers contribute to the drag, in Equation (19) only the wavenumbers which satisfy

$$\kappa < \kappa_1 = \frac{1}{2} \frac{N_\beta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^{\frac{1}{2}}},$$  

(20)

contribute. In Equation (17) the pattern is still more complex, with wavenumbers both above a certain threshold and contained in a lower interval contributing, namely

$$\kappa > \left\{ \kappa_1^2 + \frac{(\kappa_1 + \kappa_2)^2 - \kappa_1^4}{2} \right\}^{\frac{1}{2}},$$  

(21)

or

$$\left\{ \kappa_2^2 + \frac{(\kappa_1 + \kappa_2)^2 - \kappa_1^4}{2} \right\}^{\frac{1}{2}} < \kappa < \kappa_1.$$  

(22)

where

$$\kappa_2 = \frac{Ro^{-1}}{\sqrt{2|\cos \theta|}},$$  

(23)

The upper wavenumber range (which vanishes in the quasi-geostrophic limit) clearly corresponds to the drag produced by gravity waves, as in Smith (1979a), Miranda and James (1992, Appendix A) or Grisogono et al. (1993), while the lower wavenumber range accounts for the drag produced by internal Rossby waves. In contrast, for a non-stratified flow with a rigid lid, all contributions to the drag come from the singular wavenumber where the expression inside square brackets in Equation (19) is zero: the wavenumber of free barotropic Rossby waves (see Janowitz, 1977).

3. Results and discussion

It should first be noted that when $N_\beta = 0$, e.g. at the poles of the Earth, Equation (17) reduces to

$$D = \rho_0 N U b h_0^2 \int_0^{2\pi} \int_0^{+\infty} \frac{\kappa e^{-2\kappa} \cos \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^{\frac{1}{2}}} \right] \frac{d\kappa d\theta}{k^2(\cos^2 \theta + \gamma^2 \sin^2 \theta)}.$$  

(24)

For the case of a circular mountain ($\gamma = 1$), Equation (24) is equivalent to Miranda and James (1992, Equation (A15)). In this case, the incoming flow should be understood as not zonal (which would not make sense at the pole), but rather having orientation. This is admissible because the absence of the beta effect removes the anisotropy present in the original problem formulation. A lower cutoff for the horizontal wavenumbers that contribute to the drag is set by the condition

$$\kappa > \frac{Ro^{-1}}{|\cos \theta|},$$  

(25)

arising from the requirement that the expression within brackets in the second line of Equation (24) be positive. For this reason, as the mountain becomes wider (as $Ro^{-1}$ increases) the drag decreases rapidly (approximately exponentially) as in Miranda and James (1992, Appendix...
and reduce large errors in geophysically realistic conditions, we integral in Equation (17) gives contributions to the drag, on the spherical Earth. This aspect is, again, especially crete nature of the zonal wavenumbers that can exist for higher latitudes should therefore be less reliable because of the assumed continuous spectrum.

Another, more serious, limitation of the present approach is that it does not take into account the discrete nature of the zonal wavenumbers that can exist on the spherical Earth. This aspect is, again, especially relevant in the lowest wavenumber range for which the integral in Equation (17) gives contributions to the drag, Equation (22). While this limitation can probably introduce large errors in geophysically realistic conditions, we decide to keep the continuous formulation in the present study. This has the advantage of facilitating the treatment of different orography shapes and the study of asymptotic drag regimes, avoiding the obscuring discontinuities in drag behaviour brought about by a discrete approach. The continuous approach can be considered suitable for a generic planet with a very large radius, but qualitatively the behaviour of the RGWD is similar to that occurring when the wavenumbers are discrete.

A rough idea of the impact of this aspect may be obtained by noting that Equation (22) may be expressed in terms of \( k \) (the zonal wavenumber), defining an interval for which this quantity contributes to the drag. We may also note that in the spherical Earth, the zonal wavenumbers that can exist are given by

\[
k = \frac{n}{R_E \cos \phi},
\]

where \( R_E \) is the radius of the Earth, \( n \) is an integer number and \( \phi \) is the latitude. Figure 1 shows how many of these discrete wavenumbers fit into the interval for \( k \) defined by Equation (22), therefore contributing to the RGWD, as a function of \( \theta \) (the wavenumber angle) and latitude. A circular mountain (\( \gamma = 1 \)) is considered, and \( R_E = 6.37 \times 10^6 \) m and \( U = 10 \) m s\(^{-1} \) are used as reference values. It is also noted that \( f_0 = 2 \Omega \sin \phi \) and \( \beta = 2 \Omega R_E \cos \phi \), where \( \Omega \) is the angular velocity of rotation of the Earth. It is seen that the amount of possible discrete wavenumbers is zero both for \( \phi = 0^\circ \) and \( \phi = 90^\circ \), or when \( \theta \) is very large (\( \theta \approx 90^\circ \)) because the amplitude of the interval defined by Equation (22) tends to zero in these limits. For \( \theta = 0^\circ \), the amount of discrete wavenumbers is a maximum and goes from 6 at \( \phi = 30^\circ \) to 3 at \( \phi = 60^\circ \), reaching an absolute maximum of 7 slightly below \( \phi = 30^\circ \). The results to be presented at

\[
\varepsilon = \frac{\beta b}{f_0} = \frac{N_\beta}{\gamma R_0},
\]

which must be at most of the order 1 for the beta-plane approximation to be valid. For this reason, the results are deemed to be unreliable in the region with the grey shading (where \( \varepsilon > 1 \)). The short-dashed lines illustrate conditions for flows with \( U = 10 \) m s\(^{-1} \), at latitudes of 30°, 40°, 50° and 60°. These relations are independent of the ridge width \( a \), because they have been obtained eliminating this quantity between \( R_0^{-1} \) and \( N_\beta \).

As can be seen, for this range of latitudes and values of \( R_0^{-1} \) and \( N_\beta \), the short-dashed lines generally stay outside the shaded region, indicating that the beta-plane approximation is essentially valid. For lower values of \( U \), this condition is satisfied even more closely, while for higher \( U \) or for lower latitudes, the opposite happens. It can be seen that the drag is \( \sim 1 \) for low values of \( R_0^{-1} \) and \( N_\beta \), as expected. On the other hand, when \( R_0^{-1} \) is large, the drag takes high values. These values are largest when \( N_\beta = O(1) \), being somewhat smaller.
when either \( N_\beta \ll 1 \) or \( N_\beta \gg 1 \). The existence of high values of the drag due to the beta effect is a well-known result since early studies on planetary waves, and is illustrated neatly, for example, in Thompson and Flierl (1993) for barotropic Rossby waves in a neutrally stratified and vertically bounded fluid. In the present study, this drag amplification can be explained by making the quasi-gestrophic approximation, which corresponds to taking \( Ro \to +\infty \) in Equation (17). The RGWD then takes the asymptotic form of Equation (19), which when normalized by Equation (18) is proportional to \( Ro \), growing indefinitely as this parameter increases.

When \( N_\beta \) is small, the drag takes values below 1 for \( Ro \to 0 = O(1) \). This is a manifestation of the processes that dominate the mesoscale drag in the studies of Smith (1979a) or Miranda and James (1992, Appendix A). However, this is not immediately recognisable, because the drag behaviour as \( N_\beta \) tends to zero (Equation (24)) is approached here gradually, particularly for high \( Ro \). In fact, it can be shown from Equation (19) that, in the limits \( Ro \to +\infty \) and \( N_\beta \to 0 \), the normalized RGWD is asymptotically

\[
\frac{D_{\text{RGW}}}{D_0} \sim 4 \, Ro^{-1} \, N_\beta, \tag{28}
\]

for a circular mountain \((\gamma = 1)\). It suffices to note that when \( N_\beta \) is small, the exponential in Equation (19) is \( \sim 1 \) and the remaining terms may be integrated directly, yielding the desired result.

In this same limit, the barotropic Rossby wave drag calculated by McCartney (1975) and Janowitz (1977, Equation (6)) when normalized by Equation (18) for \( \gamma = 1 \), can be shown to take the approximate form:

\[
\frac{D_B}{D_0} \approx \frac{U}{NH} \, Ro^{-2} \, N_\beta^{\frac{1}{2}}, \tag{29}
\]

where \( H \) is the depth of the troposphere in Janowitz’s notation. From Equations (28) and (29), the ratio of the internal and barotropic drags (ignoring proportionality constants) may be estimated as follows:

\[
\frac{D_{\text{QRG}}}{D_0} \approx \frac{NH}{U} \, Ro^{-1} \, N_\beta^{\frac{1}{2}} = \frac{NH}{f_0} \left( \frac{\beta}{U} \right)^{\frac{1}{2}}, \tag{30}
\]

(see analogous relation near the top of pg. 805 in Janowitz, 1977). Taking values of these parameters typical of midlatitudes, such as \( H = 10^4 \, m \), \( f_0 = 10^{-4} \, s^{-1} \), \( N = 10^{-2} \, s^{-1} \), \( U = 10 \, m \, s^{-1} \) and \( \beta = 10^{-11} \, s^{-1} \, m^{-1} \), this ratio takes the value 1, indicating that internal wave drag may well be of comparable magnitude to the barotropic wave drag. This most likely also happens when \( N_\beta = O(1) \) (see discussion in Janowitz, 1977), which partially motivates the relevance of the present calculations. In other parameter ranges, the relative magnitude of internal and barotropic drag depends more strongly on the shape of the orography, so it is more difficult to evaluate.

3.2. RGWD as a function of mountain width

It is of geophysical relevance to consider the variation of the RGWD with the mountain width for realistic values of \( Ro^{-1} \) and \( N_\beta \). We first note the definitions of these two dimensionless parameters as functions of \( f_0 \) and \( \beta \), and the relation of \( f_0 \) and \( \beta \) with \( \phi \), presented before. The situations considered will be those represented as the short-dashed lines in Figure 2, but for mountains with different aspect ratios.

Figure 3(a) shows the normalized RGWD given by Equation (17) over Equation (18) as a function of zonal wind. The magnitude of the incoming wind is \( U = 10 \, m \, s^{-1} \), latitudes range from \( \phi = 30^\circ \) to \( \phi = 60^\circ \) and \( \gamma = 0.25 \), i.e. a meridional ridge perpendicular to the wind is considered. The RGWD for a latitude \( \phi = 90^\circ \), which is given by Equation (24) and reduces to the linear result of Miranda and James (1992) when \( \gamma = 1 \), is also represented. It can be seen that, as latitude increases between 30° and 60°, the RGWD tends to be amplified progressively more as the mountain width increases, attaining values more than 10 times larger than that of the drag without rotation for \( a = 1000 \, km \). A qualitatively similar behaviour was observed for very different conditions (a bounded, neutrally stratified flow over a cylindrical mountain) by Thompson and Flierl (1993). The RGWD attains a minimum between \( a = 100 \, km \) and \( a = 200 \, km \), due to the competing effects of \( f_0 \) and \( \beta \). On the other hand, when \( a > 200 \, km \), the drag
is many orders of magnitude larger than that given by the expression of Miranda and James (1992, Appendix A). In Figure 3(b) and (c), similar results are shown but for \( \gamma = 1 \) (a circular mountain) and \( \gamma = 4 \) (a zonally oriented mountain, parallel to the flow), respectively. It can be seen that, as could perhaps be anticipated, the drag increase is more modest for wind along the mountain than across the mountain, for a similar zonal half-width. This happens because the lower \( \gamma \) is, the larger is the meridional extent of the mountain, therefore leading to a greater deflection of the flow and the generation of more intense Rossby-gravity waves.

As latitude increases, a point must be reached where the RGWD starts decreasing toward its low polar values (due to the narrowing of the wavenumber interval as defined in Equation (22), which contributes to the drag in Equation (17)). The latitude where the drag is stationary with \( \phi \) before it begins decreasing is around \( \phi = 50 \) or \( \phi = 60 \) in Figure 3(c), but appears to be higher for lower values of \( \gamma \), as demonstrated by Figures 3(a) and (b).

### 3.3. Representative orography

In order to better illustrate the practical relevance of the calculations developed here, two examples of the RGWD produced by approximations to real mountain ranges will be presented: the Himalayas and the Rocky Mountains. The surface elevation for both mountain ranges was taken from the US Navy elevation database. The Himalayas are arbitrarily defined as the region between 62–107°E longitude and between 23–40°N latitude. The Rocky mountains, on the other hand, are assumed to exist between 116–101°W longitude and 32–47°N latitude. The surface elevation distribution contained in these boxes is then centred and adjusted, through a least-squares fit, to an idealized mountain with Gaussian shape:

\[
h = h_0 \exp \left( -\frac{a^2}{a_0^2} - \frac{y^2}{b_0^2} \right). \tag{31}\]

The Fourier transform of this surface elevation then replaces Equation (15) in the drag expressions.

This type of orography is used here rather than the bell-shaped mountain employed previously, since the latter was judged to be too spiky to provide a reasonable fit to the mountain ranges under consideration. Note also that Equation (31) still assumes a meridional or zonal orientation of the mountain ranges. This is an acceptable assumption for the Himalayas, which are approximately meridional. Finally, consistent with the beta-plane approximation, the geometry of the level terrain was approximated as Cartesian, with zonal displacements converted from degrees to distances assuming a mean latitude \( \phi_0 \).

Table I shows that, for a wind speed \( U = 10 \) m s\(^{-1}\) or \( U = 20 \) m s\(^{-1}\), the impact of the beta effect is quite important for the RGWD exerted on both mountain ranges, making it assume values considerably larger than the non-rotating limit. It is also interesting to note that the quasi-geostrophic approximation is quite accurate, surprisingly better for the Rocky Mountains than for the Himalayas. This may be due to the following reasons. Although \( R^{-1} \) is larger for the Himalayas than for the Rockies, \( N^{-1} \gamma_0 \) is also larger by a bigger factor. When Equation (17) reduces to Equation (19) in the quasi-geostrophic approximation, the term that must be neglected involves \( N^{-1} \gamma_0 \), therefore the larger this...
of zonal wavenumbers could replace the continuous one treatment are possible. In particular, a discrete spectrum obviously, many improvements and extensions to this are probably of more qualitative than quantitative value. They a leading-order treatment of a problem which appears into account in the global angular momentum budget. These waves seem to have been mostly overlooked (except for brief references by Smith 1979c). It would presumably be more difficult (and probably less necessary) to abandon the beta-plane approximation in favour of an explicit treatment of the spherical geometry of the Earth. Nevertheless, all these developments (some of which have already been pursued elsewhere for neutrally stratified fluids topped by a rigid lid or a free surface) remain open to further investigation. We must emphasize, however, that our goal was merely to give a flavour of the inclusion of internal Rossby-gravity waves in the existing mesoscale drag context.

4. Concluding remarks

The present study has shown that the drag due to internal Rossby-gravity waves existing in a stably stratified westerly flow is not only much larger than the equivalent inertia-gravity wave drag (Smith, 1979a; Ólafsson and Bougeault, 1997), but also larger than the equivalent pure gravity wave drag (Phillips, 1984). Additionally, it is typically comparable with the barotropic wave drag existing in a neutral but vertically bounded atmosphere (Janowitz, 1977). The drag attains a maximum enhancement for large $Ro^{-1}$ and $N^2 \beta$ of order 1. It is also larger for meridional mountains than for zonal mountains, as would perhaps be expected. The RGWD is found to increase with mountain width and latitude, at least for $\phi$ between 30° and 60°.

This force, along with the barotropic Rossby wave drag, may significantly enhance the zonal torque exerted on the Earth by Rossby-gravity waves, contributing to a deceleration of the jet-streams or the mean westerly flow in general in midlatitudes. Admittedly, internal Rossby-gravity waves are resolved by most meteorological models running at the current resolutions except for brief references by Smith 1979c and Janowitz 1977), but their existence must be taken into account in the global angular momentum budget.

The calculations presented in this paper aim to provide a leading-order treatment of a problem which appears to the authors not to have been addressed before. They are probably of more qualitative than quantitative value. Obviously, many improvements and extensions to this treatment are possible. In particular, a discrete spectrum of zonal wavenumbers could replace the continuous one used here. The Boussinesq approximation could also be abandoned by using a procedure analogous to that described, for example, by Smith (1979b, pg. 95). It would presumably be more difficult (and probably less necessary) to abandon the beta-plane approximation in favour of an explicit treatment of the spherical geometry of the Earth. Nevertheless, all these developments (some of which have already been pursued elsewhere for neutrally stratified fluids topped by a rigid lid or a free surface) remain open to further investigation. We must emphasize, however, that our goal was merely to give a flavour of the inclusion of internal Rossby-gravity waves in the existing mesoscale drag context.

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References


Table I. Input and output parameters for the Himalayas and the Rocky Mountains: mean latitude, zonal half-width, meridional half-width, aspect ratio, beta-plane small parameter, incoming wind velocity, inverse Rossby number, beta-effect number, drag normalized by non-rotating value and drag normalized by quasi-geostrophic value.

<table>
<thead>
<tr>
<th>Range</th>
<th>$\phi_0$ (°N)</th>
<th>$a$ (km)</th>
<th>$b$ (km)</th>
<th>$\gamma$</th>
<th>$\varepsilon$</th>
<th>$U$ (m s$^{-1}$)</th>
<th>$Ro^{-1}$</th>
<th>$N^2 \beta$</th>
<th>$D/D_0$</th>
<th>$D/D_{QG}$</th>
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<tr>
<td>Himalayas</td>
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<td>959</td>
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<td>0.2318</td>
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<td>6.77</td>
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<td>1297</td>
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