Multisorted dualisability: change of base

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Abstract. We prove that if a quasivariety $\mathcal{A}$ generated by a finite family $\mathcal{M}$ of finite algebras has a multisorted duality based on $\mathcal{M}$, then $\mathcal{A}$ has a multisorted duality based on any finite family of finite algebras that generates it.

1. Introduction

The basic theory of natural dualities applies to quasivarieties of the form $\mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a single finite algebra. By no means all finitely generated varieties take this form. However in the congruence distributive setting any finitely generated variety is expressible in the form $\mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite algebras, and this applies in particular to lattice-based varieties. This led Davey and Priestley [3] to extend the theory of natural dualities so that it applies to any quasivariety of the form $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M} = \{\mathbf{M}_1, \dots, \mathbf{M}_n\}$ is a finite indexed family of finite algebras. In bare outline, the generalised theory works with a multisorted alter ego $\underset{\sim}{\mathcal{M}} = \langle \dot{\bigcup}_{i=1}^{n} M_i; G, H, R, \mathcal{T}\rangle$, with a sort for each algebra in $\mathcal{M}$, and the dual structures are then multisorted Boolean topological structures of the same type. When there is such an alter ego $\underset{\sim}{\mathcal{M}}$ that yields a natural duality on $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ we say that there is a *multisorted duality for* $\mathcal{A}$ *based on* $\mathcal{M}$. If $G \cup H \cup R$ is finite, we say that the duality is *of finite type*. We refer to Chapter 7 of the text by Clark and Davey [1] for an account of the requisite theory.

Assume that $\mathcal{A} = \mathbb{ISP}(\mathcal{M}) = \mathbb{ISP}(\mathcal{N})$, for finite families of finite algebras $\mathcal{M}$ and $\mathcal{N}$. If $\mathcal{A}$ has a multisorted duality for $\mathcal{A}$ based on $\mathcal{M}$, must $\mathcal{A}$ have a multisorted duality for $\mathcal{A}$ based on $\mathcal{N}$? This natural question was answered positively for the single-sorted case by Davey and Willard [4] and Saramago [6]. In this short paper we give a positive answer for the multisorted case. The functor D that maps each algebra in $\mathcal{A}$ to its dual is analogous to a logarithm with base $\mathcal{M}$. Indeed, dualities like Priestley duality that are truly logarithmic, in that they send finite products to disjoint union, are amongst the most useful

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natural dualities—see [1, Chapter 6]. Thus this theorem provides a *change of base* result for multisorted natural dualities.

Theorem 1.1. *Let* $\mathcal{A} = \mathbb{ISP}(\mathcal{M}) = \mathbb{ISP}(\mathcal{N})$*, where* $\mathcal{M}$ *and* $\mathcal{N}$ *are finite families of finite algebras. If there is a multisorted duality* [*of finite type*] *for* $\mathcal{A}$ *based on* $\mathcal{M}$*, then there is a multisorted duality* [*of finite type*] *for* $\mathcal{A}$ *based on* $\mathcal{N}$*.*

This theorem allows us immediately to resolve another question; indeed, it was this very natural question that initially led us to investigate change of base. Suppose we have a finite algebra $\mathbf{M}$ such that the quasivariety $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$ has no duality based on $\mathbf{M}$ and suppose that this quasivariety can be alternatively expressed as $\mathbb{ISP}(\mathcal{N})$, where $\mathcal{N}$ is a finite family of finite algebras. Could $\mathcal{A}$ have a multisorted duality based on $\mathcal{N}$? Corollary 1.2 tells us that non-dualisability of $\mathbf{M}$ cannot be circumvented by invoking the multisorted theory.

Corollary 1.2. *Let* $\mathbf{M}$ *be a finite algebra and define* $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$*. Assume that* $\mathcal{M}$ *is a finite family of finite algebras from* $\mathcal{A}$ *such that* $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$*. If* $\mathcal{A}$ *has no natural duality based on* $\mathbf{M}$*, then* $\mathcal{A}$ *has no multisorted natural duality based on* $\mathcal{M}$*.*

We stress that Theorem 1.1 applies to *indexed families* $\mathcal{M} = \{\mathbf{M}_1, \dots, \mathbf{M}_n\}$ and $\mathcal{N} = \{\mathbf{N}_1, \dots, \mathbf{N}_m\}$ of finite algebras. For example, it is quite possible that $\mathcal{M} = \{\mathbf{M}\}$ and $\mathcal{N} = \{\mathbf{N}_1, \mathbf{N}_2\}$ with $\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{M}$. In practice, it is sometimes worthwhile to work with a multisorted duality even though a single-sorted duality is available. For example, this may make the description of free algebras more transparent, as it did for the variety of Kleene algebras which provided the motivating example for the multisorted theory developed by Davey and Priestley [3].

Our proof of Theorem 1.1 is constructive. Hence it is possible to start with a dualising set R for $\mathcal{A}$ based on $\mathcal{M}$ and apply the constructions used in the proof to produce a dualising set R' for $\mathcal{A}$ based on $\mathcal{N}$. We illustrate this in Example 2.3: starting from Priestley's duality for the class $\mathcal{D}$ of bounded distributive lattices, which is based on the two-element chain, we derive a duality for $\mathcal{D}$ based on the three-element chain.

2. Proof of the theorem

Theorem 1.1 follows from the two lemmas proved below. For notational simplicity, we shall restrict to the case in which the dualising sets contain no operations or partial operations. No loss of generality is involved: any (partial) operation may be replaced by its graph (see, for example, [1, Lemma 2.1.2]).

Lemma 2.1 (**Remove an Algebra**)**.** *Let* $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$ *be the quasivariety generated by a finite family* $\mathcal{M} = \{\mathbf{M}_0, \mathbf{M}_1, \dots, \mathbf{M}_n\}$ *of finite algebras and assume that the algebra* $\mathbf{M}_0$ *can be embedded into a product of algebras from*

$\mathcal{M}^* := \{\mathbf{M}_1, \dots, \mathbf{M}_n\}$. *If there is a multisorted duality [of finite type] for* $\mathcal{A}$ *based on* $\mathcal{M}$*, then there is a multisorted duality [of finite type] for* $\mathcal{A}$ *based on* $\mathcal{M}^*$.

Proof. First note that since $\mathbf{M}_0$ embeds into a product of algebras from $\mathcal{M}^*$, we have $\mathcal{A} = \mathbb{ISP}(\mathcal{M}) = \mathbb{ISP}(\mathcal{M}^*)$. It also follows that there is a finite set U of homomorphisms $u\colon \mathbf{M}_0 \to \mathbf{N}_u$, with $\mathbf{N}_u \in \mathcal{M}^*$, such that U separates the points of $\mathbf{M}_0$. Consequently the natural homomorphism $\mu\colon \mathbf{M}_0 \to \prod_{u\in U} \mathbf{N}_u$ given by $\mu(m) := \langle u(m)\rangle_{u\in U}$, for all $m \in M_0$, is an embedding. We shall use the set U and the embedding μ to transfer the duality based on $\mathcal{M}$ to a duality based on $\mathcal{M}^*$.

Several occurrences of $m \in \mathbf{M}_0$ replaced by $m \in M_0$

Let R be a set of $\mathcal{M}$-sorted algebraic relations that yields a duality on $\mathcal{A}$ based on $\mathcal{M}$ and let $r \in R$ be k-ary. We can use U to encode r as an $\mathcal{M}^*$-sorted relation r^*. If r does not involve the sort $\mathbf{M}_0$, then simply define r^* to be r. If r does involve the sort $\mathbf{M}_0$, then we may assume without loss of generality that r is a subuniverse of

$$\mathbf{M}_0^t \times \mathbf{M}_{i_1} \times \cdots \times \mathbf{M}_{i_\ell}, \text{ with } \mathbf{M}_{i_j} \in \mathcal{M}^*, \text{ for all } j \in \{1, \dots, \ell\},$$

where $k = t+\ell$. Define r^* to be the $(|U|t+\ell)$-ary $\mathcal{M}^*$-sorted algebraic relation

$$r^* := \big\{ \big(\langle u(a_1)\rangle_{u\in U}, \dots, \langle u(a_t)\rangle_{u\in U}, a_{t+1}, \dots, a_k\big) \bigm| (a_1, \dots, a_k) \in r \big\}.$$

Now define $R^* := \{r^* \mid r \in R\} \cup \{s\}$, where $s := \mu(M_0)$. We claim that R^* yields a duality on $\mathcal{A}$ based on $\mathcal{M}^*$.

Proof simplified: set S replaced by a single relation s

Let $\mathbf{A}$ belong to $\mathcal{A}$ and let $\alpha\colon \dot{\bigcup}_{i=1}^n \mathcal{A}(\mathbf{A}, \mathbf{M}_i) \to \dot{\bigcup}_{i=1}^n M_i$ be a continuous $\mathcal{M}^*$-sorted map preserving the relations in R^*. We must prove that α is given by evaluation at some $a \in A$. To use the fact that R yields a duality on $\mathcal{A}$ based on $\mathcal{M}$, we need to extend α to a map $\widehat{\alpha}\colon \dot{\bigcup}_{i=0}^n \mathcal{A}(\mathbf{A}, \mathbf{M}_i) \to \dot{\bigcup}_{i=0}^n M_i$ preserving the relations in R.

Added explaination to the reader of what we need to do.

Let $x \in \mathcal{A}(\mathbf{A}, \mathbf{M}_0)$. By definition, $\langle u \circ x\rangle_{u\in U}$ belongs to the relation s on $\dot{\bigcup}_{i=1}^n \mathcal{A}(\mathbf{A}, \mathbf{M}_i)$. Since α preserves s, we have $\langle \alpha(u \circ x)\rangle_{u\in U} \in s$. Let $\sigma\colon s \to M_0$ be the inverse of μ. Hence we may define $\alpha_0\colon \mathcal{A}(\mathbf{A}, \mathbf{M}_0) \to M_0$ by $\alpha_0(x) := \sigma(\langle \alpha(u \circ x)\rangle_{u\in U})$, and then define $\widehat{\alpha}\colon \dot{\bigcup}_{i=0}^n \mathcal{A}(\mathbf{A}, \mathbf{M}_i) \to \dot{\bigcup}_{i=0}^n M_i$ by $\widehat{\alpha} := \alpha_0 \cup \alpha$.

The relations s_x replaced by the single relation s and the maps η_x replaced by a single map σ.

We claim that $\widehat{\alpha}$ is continuous and R-preserving. Since $\widehat{\alpha} = \alpha_0 \cup \alpha$ and α is continuous by assumption, we must establish the continuity of α_0. Let $m \in M_0$. Then

Stray right bracket removed from second line of display.

$$\begin{aligned}
\alpha_0^{-1}(m) &= \{\, x \in \mathcal{A}(\mathbf{A}, \mathbf{M}_0) \mid \sigma(\langle \alpha(u \circ x)\rangle_{u\in U}) = m \,\} \\
&= \{\, x \in \mathcal{A}(\mathbf{A}, \mathbf{M}_0) \mid \mu(m) = \langle \alpha(u \circ x)\rangle_{u\in U} \,\} \\
&= \bigcap_{u\in U} \{\, x \in \mathcal{A}(\mathbf{A}, \mathbf{M}_0) \mid \alpha(u \circ x) = u(m) \,\} \\
&= \bigcap_{u\in U} \{\, x \in \mathcal{A}(\mathbf{A}, \mathbf{M}_0) \mid u \circ x \in \alpha^{-1}(u(m)) \,\},
\end{aligned}$$

which is open in $\mathcal{A}(\mathbf{A}, \mathbf{M}_0)$ since the map $u \circ -\colon \mathcal{A}(\mathbf{A}, \mathbf{M}_0) \to \mathcal{A}(\mathbf{A}, \mathbf{N}_u)$ is continuous, for each $u \in U$, and the set $\alpha^{-1}(u(m))$ is open in $\mathcal{A}(\mathbf{A}, \mathbf{N}_u)$, as

α is continuous. Hence α_0 is continuous. Let $r \in R$ be k-ary and assume that $(x_1, \dots, x_k)$ belongs to the relation r on $\dot\bigcup_{i=0}^{n} \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_i)$. Note that, by assumption, we have $x_1, \dots, x_t \colon \mathbf{A} \to \mathbf{M}_0$. Thus $(x_1(a), \dots, x_k(a)) \in r$, for all $a \in A$, and hence

Reminder added about the codomain of $x_1, \dots, x_t$.

$$(\langle u(x_1(a))\rangle_{u\in U}, \dots, \langle u(x_t(a))\rangle_{u\in U}, x_{t+1}(a), \dots, x_k(a)) \in r^*,$$

for all $a \in A$. Thus $(\langle u \circ x_1\rangle_{u\in U}, \dots, \langle u \circ x_t\rangle_{u\in U}, x_{t+1}, \dots, x_k)$ belongs to the relation r^* on $\dot\bigcup_{i=1}^{n} \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_i)$, and consequently, as α preserves r^*,

$$(\langle \alpha(u \circ x_1)\rangle_{u\in U}, \dots, \langle \alpha(u \circ x_t)\rangle_{u\in U}, \alpha(x_{t+1}), \dots, \alpha(x_k)) \in r^*.$$

So, by the definition of r^*, there exists $(a_1, \dots, a_k) \in r$ with

$$\begin{aligned}(\langle \alpha(u \circ x_1)\rangle_{u\in U}, \dots, \langle \alpha(u \circ x_t)\rangle_{u\in U}, \alpha(x_{t+1}), \dots, \alpha(x_k))\\ = (\langle u(a_1)\rangle_{u\in U}, \dots, \langle u(a_t)\rangle_{u\in U}, a_{t+1}, \dots, a_k).\end{aligned}$$

For $i \in \{1, \dots, t\}$, we have $\langle \alpha(u \circ x_i)\rangle_{u\in U} = \langle u(a_i)\rangle_{u\in U} = \mu(a_i)$, and consequently $\sigma(\langle \alpha(u \circ x_i)\rangle_{u\in U}) = a_i$. This gives

$$\begin{aligned}&(\widehat{\alpha}(x_1), \dots, \widehat{\alpha}(x_k)) = (\alpha_0(x_1), \dots, \alpha_0(x_t), \alpha(x_{t+1}), \dots, \alpha(x_k))\\ &= (\sigma(\langle \alpha(u \circ x_1)\rangle_{u\in U}), \dots, \sigma(\langle \alpha(u \circ x_t)\rangle_{u\in U}), \alpha(x_{t+1}), \dots, \alpha(x_k))\\ &= (a_1, \dots, a_t, a_{t+1}, \dots, a_k) \in r.\end{aligned}$$

Hence $\widehat{\alpha}$ is continuous and R-preserving and so $\widehat{\alpha} = e_{\mathbf{A}}(a)$, for some $a \in A$. It follows at once that α is also given by evaluation at a. Thus, R^* yields a duality on $\boldsymbol{\mathcal{A}}$ based on $\mathcal{M}^*$. Moreover, if R is finite, then so is R^*. □

Lemma 2.2 (**Add an Algebra**). *Let $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\mathcal{M})$ be the quasivariety generated by a finite family $\mathcal{M} = \{\mathbf{M}_1, \dots, \mathbf{M}_n\}$ of finite algebras. Let $\mathbf{M}_0$ be a finite algebra in $\boldsymbol{\mathcal{A}}$ and define $\mathcal{M}^+ = \{\mathbf{M}_0, \mathbf{M}_1, \dots, \mathbf{M}_n\}$. If there is a multisorted duality [of finite type] for $\boldsymbol{\mathcal{A}}$ based on $\mathcal{M}$, then there is a multisorted duality [of finite type] for $\boldsymbol{\mathcal{A}}$ based on $\mathcal{M}^+$.*

Proof. Note that $\mathbb{ISP}(\mathcal{M}^+) = \mathbb{ISP}(\mathcal{M})$. As $\mathbf{M}_0 \in \mathbb{ISP}(\mathcal{M})$, there is a set U of homomorphisms $u \colon \mathbf{M}_0 \to \mathbf{N}_u$, with $\mathbf{N}_u \in \mathcal{M}$, that separates the points of $\mathbf{M}_0$. Let R be a set of $\mathcal{M}$-sorted algebraic relations that yields a duality on $\boldsymbol{\mathcal{A}}$ based on $\mathcal{M}$. We claim that $R^+ := R \cup U$ yields a duality on $\boldsymbol{\mathcal{A}}$ based on $\mathcal{M}^+$. Let $\mathbf{A}$ belong to $\boldsymbol{\mathcal{A}}$ and let $\alpha \colon \dot\bigcup_{i=0}^{n} \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_i) \to \dot\bigcup_{i=0}^{n} M_i$ be a continuous $\mathcal{M}^+$-sorted map that preserves the relations in R^+. The restriction of α to $\dot\bigcup_{i=1}^{n} \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_i)$ is a continuous $\mathcal{M}$-sorted map that preserves the relations in R and so is given by evaluation at some $a \in A$. It remains to prove that $\alpha(x) = x(a)$, for all $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_0)$. Let $u \in U$ and $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_0)$. Then $u \circ x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M}_i)$, for some $i \in \{1, \dots, n\}$, and so $\alpha(u \circ x) = (u \circ x)(a) = u(x(a))$. As α preserves u, we have $u(\alpha(x)) = \alpha(u(x)) = \alpha(u \circ x) = u(x(a))$. Since this holds for all $u \in U$ and U separates the points of $\mathbf{M}_0$, it follows that $\alpha(x) = x(a)$, as required. As $\mathbf{M}_0$ is finite, so is the set U. Consequently, R^+ is finite provided R is. □

The proof of Theorem 1.1 is now an easy exercise. Assume that there is a multisorted duality for $\boldsymbol{\mathcal{A}}$ based on the family $\boldsymbol{\mathcal{M}}$. By successively applying Lemma 2.2 to the algebras in $\boldsymbol{\mathcal{N}}$, it follows that there is a multisorted duality for $\boldsymbol{\mathcal{A}}$ based on the family $\boldsymbol{\mathcal{M}} \mathbin{\dot\cup} \boldsymbol{\mathcal{N}}$. Now by successively applying Lemma 2.1 to remove the algebras in $\boldsymbol{\mathcal{M}}$ we conclude that there is a multisorted duality for $\boldsymbol{\mathcal{A}}$ based on the family $\boldsymbol{\mathcal{N}}$.

We end with the promised application to bounded distributive lattices.

Example 2.3. We wish to start with Priestley duality for the class $\boldsymbol{\mathcal{D}}$ of bounded distributive lattices [5] and use it to obtain a duality for $\boldsymbol{\mathcal{D}}$ based on the three-element chain $\mathbf{3}$. Thus we begin, as Priestley did, with

$$\boldsymbol{\mathcal{M}} = \{\mathbf{2}\} \text{ and } R = \{\leqslant\},$$

where $\mathbf{2}$ is the two-element bounded lattice and $\leqslant$ is the order on it.

Now apply Lemma 2.2 to add the algebra $\mathbf{M}_0 := \mathbf{3}$. Let $3 = \{0, a, 1\}$, and take $U = \{f, g\}$, where $f, g \colon \mathbf{3} \to \mathbf{2}$ are given by $f(a) = 0$ and $g(a) = 1$. We obtain

$$\boldsymbol{\mathcal{M}}^{+} = \{\mathbf{3}, \mathbf{2}\} \text{ and } R^{+} = \{\leqslant_{\mathbf{2}}, f, g\},$$

where $\leqslant_{\mathbf{2}}$ is the order relation on $\mathbf{2}$.

Next apply Lemma 2.1 to remove $\mathbf{M}_0 := \mathbf{2}$ from $\{\mathbf{2}, \mathbf{3}\}$. Take $U = \{u\}$, where $u \colon \mathbf{2} \to \mathbf{3}$ is the inclusion. We first replace f and g by $r^f := \{00, 0a, 11\}$ and $r^g := \{00, 1a, 11\}$, respectively—note that, as in the proof of Lemma 2.1, the M_0-coordinates are now at the front. We then obtain

Structure on $\boldsymbol{\mathcal{M}}^{+*}$ simplified due to new proof of Lemma 2.1.

$$\boldsymbol{\mathcal{M}}^{+*} = \{\mathbf{3}\} \text{ and } R^{+*} = \{\leqslant_{\mathbf{2}}, r^f, r^g, \{0, 1\}\},$$

where $\leqslant_{\mathbf{2}}$, r^f and r^g are now regarded as a subsets of 3×3. The relations r^f and r^g can be replaced by f and g viewed as endomorphisms of $\mathbf{3}$, while the relation $\{0, 1\} = \mathrm{fix}(f)$ can be removed without destroying the duality. We conclude that $\underset{\sim}{\mathbf{3}} := \langle \{0, a, 1\}; f, g, \leqslant_{\mathbf{2}}, \mathcal{T} \rangle$ yields a duality on $\boldsymbol{\mathcal{D}}$ based on $\mathbf{3}$. This result was first proved by Davey, Haviar and Priestley [2]. They went on to prove that $\leqslant_{\mathbf{2}}$ could be removed without destroying the duality, that is, $\mathbf{3}$ is endodualisable.

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