A NOTE ON PROFINITE COMPLETIONS AND CANONICAL EXTENSIONS

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Abstract. J. Harding has proved that the profinite limit of an algebra **A** in a finitely generated variety of monotone lattice expansions coincides with its canonical extension. In this note we drop the monotonicity of the additional operations and prove the same result.

Given a lattice **L** and its canonical extension $(e, \mathbf{C})$, the canonical extension of $\mathbf{L}^n$ is $(e^n, \mathbf{C}^n)$,where $e^n : \mathbf{L}^n \to \mathbf{C}^n$ is the embedding defined by $e^n(a_1, \ldots, a_n) = (e(a_1), \ldots, e(a_n))$. The closed elements and the open elements of $\mathbf{C}^n$ are respectively the n-tuples of closed elements and the n-tuples of open elements of **C**. If f is an n-ary operation on a lattice **L**, f^σ and f^π are defined to be the σ- and π-extensions of f when regarded as a map from L^n to L(see [1] for further details):

$$f^\sigma(c_1, \ldots, c_n) =$$
$$\bigvee\{\bigwedge\{e(f(x_1, \ldots, x_n)) \mid \forall i \in \{1, \ldots, n\}\ x_i \in L,\ p_i \leqslant e(x_i) \leqslant q_i\} \mid$$
$$\forall i \in \{1, \ldots, n\}\ p_i \in K,\ q_i \in O,\ p_i \leqslant c_i \leqslant q_i\}$$
$$f^\pi(c_1, \ldots, c_n) =$$
$$\bigwedge\{\bigvee\{e(f(x_1, \ldots, x_n)) \mid \forall i \in \{1, \ldots, n\}\ x_i \in L,\ p_i \leqslant e(x_i) \leqslant q_i\} \mid$$
$$\forall i \in \{1, \ldots, n\}\ p_i \in K,\ q_i \in O,\ p_i \leqslant c_i \leqslant q_i\},$$

where K and O are respectively the set of closed elements and the set of open elements of **C**.

The operation f is said to be smooth if $f^\sigma = f^\pi$.

For a lattice expansion **A**, the (dual) canonical extension of **A** is defined to be the canonical extension of its lattice reduct endowed with the (π-extensions) σ-extensions of the additional operations of **A**. When all the additional operations f are monotone, i.e. in each coordinate either preserve or reverse the order, the definition we give for f^σ and f^π may be reduced to the formulae that John Harding uses in [2]. Since we are considering lattice expansions whose operations do not need to be monotone, we must use the formulae provided above instead.

Theorem 0.1. *For every variety $\mathcal{V}$ of lattice expansions, the first condition below implies the second, and for varieties of finite type the two conditions are equivalent.*

(1) $\mathcal{V}$ is finitely generated.
(2) Profinite completions coincide with canonical extensions on $\mathcal{V}$.

Proof. Let **A** be an algebra in a finitely generated variety $\mathcal{V}$ of lattice expansions. Let Φ be the set of all congruences of **A** of finite index. The profinite completion of **A** is the subalgebra of the direct product $\prod_{\theta \in \Phi} A/\theta$ formed by all elements $\widehat{a}$ that

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satisfy the following condition:

$$\text{if } \pi_\theta(\widehat{a}) = b/\theta \text{ and } \theta \subseteq \theta' \text{ then } \pi_{\theta'}(\widehat{a}) = b/\theta'.$$

From now on, for every $\widehat{a} \in \widehat{A}$, we choose an element in $\pi_\theta(\widehat{a})$ and denote it by $\widehat{a}^\theta$; note that $\pi_\theta(\widehat{a}) = \widehat{a}^\theta/\theta$.

In [2] Harding proves that the lattice reduct of $\widehat{\mathbf{A}}$ is the canonical extension of the lattice reduct of $\mathbf{A}$. Following his proof, we only need to show that any additional operation f of $\mathbf{A}$ is smooth and its canonical extension is the corresponding additional operation $\widehat{f}$ of $\widehat{A}$.

Let f be an additional operation of $\mathbf{A}$. By Lemma 4.2 of [1], we have that $f^\sigma \leqslant f^\pi$. We claim that $f^\pi \leqslant \widehat{f} \leqslant f^\sigma$. Assume that n is the arity of f and let $\widehat{a}_1, \ldots, \widehat{a}_n \in \widehat{\mathbf{A}}$ and $\theta \in \Phi$. Recall that e is now the embedding that assigns $(x/\theta)_{\theta\in\Phi}$ to each $x \in A$ and the joins and meets in $\widehat{A}$ are defined coordinatewise. We have that

$$\begin{aligned}&\pi_\theta(f^\pi(\widehat{a}_1, \ldots, \widehat{a}_n)) = \\ &\bigwedge\{\bigvee\{f(x_1, \ldots, x_n)/\theta \mid \forall i \in \{1, \ldots, n\}\ x_i \in A,\ \widehat{p}_i \leqslant e(x_i) \leqslant \widehat{q}_i\} \mid \\ &\qquad\qquad \forall i \in \{1, \ldots, n\}\ \widehat{p}_i \in K,\ \widehat{q}_i \in O,\ \widehat{p}_i \leqslant \widehat{a}_i \leqslant \widehat{q}_i\},\end{aligned}$$

$\pi_\theta(\widehat{f}(\widehat{a}_1, \ldots, \widehat{a}_n)) = f(\widehat{a}_1^\theta, \ldots, \widehat{a}_n^\theta)/\theta$ and

$$\begin{aligned}&\pi_\theta(f^\sigma(\widehat{a}_1, \ldots, \widehat{a}_n)) = \\ &\bigvee\{\bigwedge\{f(x_1, \ldots, x_n)/\theta \mid \forall i \in \{1, \ldots, n\}\ x_i \in A,\ \widehat{p}_i \leqslant e(x_i) \leqslant \widehat{q}_i\} \mid \\ &\qquad\qquad \forall i \in \{1, \ldots, n\}\ \widehat{p}_i \in K,\ \widehat{q}_i \in O,\ \widehat{p}_i \leqslant \widehat{a}_i \leqslant \widehat{q}_i\}.\end{aligned}$$

For every $\widehat{a}_i$, $\widehat{a}_i = \bigvee\{\widehat{p} \in K \mid \widehat{p} \leqslant \widehat{a}_i\}$, $\widehat{a}_i = \bigwedge\{\widehat{q} \in O \mid \widehat{a}_i \leqslant \widehat{q}\}$. Since A/θ is finite, the sets $\{\widehat{p}^\theta/\theta \mid \widehat{p} \in K,\ \widehat{p} \leqslant \widehat{a}_i\}$ and $\{\widehat{q}^\theta/\theta \mid \widehat{q} \in O,\ \widehat{a}_i \leqslant \widehat{q}\}$ are finite. The sets K and O are closed under finite joins and meets (see [1]) which implies the existence of $\widehat{c}_i \in K$ and $\widehat{o}_i \in O$ such that $\widehat{c}_i \leqslant \widehat{a}_i$, $\widehat{a}_i \leqslant \widehat{o}_i$,

$$\widehat{a}_i^\theta/\theta = \bigvee\{\widehat{p}^\theta/\theta \in K \mid \widehat{p} \leqslant \widehat{a}_i\} = \widehat{c}_i^\theta/\theta \text{ and}$$

$$\widehat{a}_i^\theta/\theta = \bigwedge\{\widehat{q}^\theta/\theta \in O \mid \widehat{a}_i \leqslant \widehat{q}\} = \widehat{o}_i^\theta/\theta.$$

Observe that for every $x \in A$ we have

$$\begin{aligned}\widehat{c}_i \leqslant e(x) \leqslant \widehat{o}_i &\Rightarrow \widehat{c}_i^\theta/\theta \leqslant x/\theta \leqslant \widehat{o}_i^\theta/\theta \\ &\Rightarrow \widehat{a}_i^\theta/\theta \leqslant x/\theta \leqslant \widehat{a}_i^\theta/\theta \Rightarrow x/\theta = \widehat{a}_i^\theta/\theta\end{aligned}$$

Consequently

$$\begin{aligned}&\pi_\theta(f^\pi(\widehat{a}_1, \ldots, \widehat{a}_n)) \\ &\leqslant \bigvee\{f(x_1/\theta, \ldots, x_n/\theta) \mid \forall i \in \{1, \ldots, n\}\ x_i \in A,\ \widehat{c}_i \leqslant e(x_i) \leqslant \widehat{o}_i\} \\ &= f(\widehat{a}_1^\theta/\theta, \ldots, \widehat{a}_n^\theta/\theta) \\ &= \bigwedge\{f(x_1/\theta, \ldots, x_n/\theta) \mid \forall i \in \{1, \ldots, n\}\ x_i \in A,\ \widehat{c}_i \leqslant e(x_i) \leqslant \widehat{o}_i\} \\ &\leqslant \pi_\theta(f^\sigma(\widehat{a}_1, \ldots, \widehat{a}_n))\end{aligned}$$

Thus $\pi_\theta(f^\pi(\widehat{a}_1, \ldots, \widehat{a}_n)) \leqslant \pi_\theta(\widehat{f}(\widehat{a}_1, \ldots, \widehat{a}_n)) \leqslant \pi_\theta(f^\sigma(\widehat{a}_1, \ldots, \widehat{a}_n))$.

The proof that condition (2) implies condition (1), when $\mathcal{V}$ is of finite type, is identical to the proof presented in [2]. $\square$

In the presented proof we take a general arity for f instead of assuming, without loss of generality, that the operations are unary, as Harding does in [2]. The unique reason we do so is the fact that this proof is not much more complicated than the one we would get for arity 1.

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References

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