A NEW APPROACH TO TOPOLOGICAL SINGULARITIES VIA A WEAK NOTION OF JACOBIAN FOR FUNCTIONS OF BOUNDED VARIATION

L. DE LUCA, R. SCALA, AND N. VAN GOETHEM

Abstract. We introduce a weak notion of 2×2-minors of gradients for a suitable subclass of BV functions. In the case of maps in $BV(\mathbb{R}^2;\mathbb{R}^2)$ such a notion extends the standard definition of Jacobian determinant to non-Sobolev maps.

We use this distributional Jacobian to prove a compactness and Γ-convergence result for a new model describing the emergence of topological singularities in two dimensions, in the spirit of Ginzburg-Landau and core-radius approaches. Within our framework, the order parameter is an SBV map u taking values in the unit sphere in $\mathbb{R}^2$ and the energy is given by the sum of the squared L^2 norm of the approximate gradient ∇u and of the length of (the closure of) the jump set of u multiplied by $\frac{1}{\varepsilon}$. Here, ε is a length-scale parameter. We show that, in the $|\log\varepsilon|$ regime, the distributional Jacobians converge, as $\varepsilon \to 0^+$, to a finite sum μ of Dirac deltas with weights multiple of π, and that the corresponding effective energy is given by the total variation of μ.

Keywords: Functions of Bounded Variation; Strict Convergence; Jacobian determinant; Topological Singularities; Γ-convergence; Ginzburg-Landau Model; Core-Radius Approach.

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Contents

Introduction

Topological singularities are ubiquitous in Physics and Materials Science: Vortices in superconductivity and superfluidity and (screw end edge) dislocations in single crystal plasticity are two paradigmatic examples [10, 33, 35, 40, 41, 42]. Loosely speaking, dislocations (resp., vortices) are identified as points around which the deformation gradient (resp., the order parameter) has non trivial circulation. In view of their central role in the theory of phase transitions, topological singularities have been the object of an extensive and intensive study in the last decades (see e.g. [12, 49]). Such an analysis focused basically on two main models: The core-radius approach (CR) and the Ginzburg-Landau model (GL). In the former, a finite distribution of topological singularities is identified by a sum of Dirac deltas (with integer weights) and the energy is assumed to be purely elastic "far enough" from such singularities. Hence, the *core parameter* ε is introduced, and the energy takes the form

$$\mathcal{E}_\varepsilon^{\mathrm{CR}}(\mu,u) := \frac{1}{2}\int_{\Omega_\varepsilon(\mu)} |\nabla u|^2 \,\mathrm{d}x + |\mu|(\Omega)\,. \tag{0.1}$$

In (0.1), μ represents the distribution of singularities, $\Omega \subset \mathbb{R}^2$ is the reference domain, and $\Omega_\varepsilon(\mu) := \Omega \setminus \bigcup_{\xi\in\operatorname{supp}\mu} \overline{B}_\varepsilon(\xi)$ is the domain deprived of the cores; moreover, $u \in H^1(\Omega_\varepsilon(\mu);\mathbb{S}^1)$ represents the order parameter that should be "compatible" with the measure μ, namely, it should satisfy $\deg(u,\partial A) = \mu(A)$ for every smooth domain $A \subset \Omega$ with $\partial A \subset \Omega_\varepsilon(\mu)$. Here and below, $\mathbb{S}^1$ denotes the set of unit vectors in $\mathbb{R}^2$. (CR) is mainly used to describe dislocations in semi-discrete models, i.e., models in which "far from the singularities" (in $\Omega_\varepsilon(\mu)$) the material is assumed to be continuous and elastic, whereas "close to the singularities" (in $\bigcup_{\xi\in\operatorname{supp}\mu} B_\varepsilon(\xi)$) the material resembles its "discrete" nature and the energy behaves like the cardinality of the atoms in the core. Here, (up to a scaling) ε represents an integer multiple of the lattice spacing.

For screw dislocations in antiplane elasticity it is well known that the displacement is a multiple of the purely axial Burgers vector times a phase (see, e.g. [53]). Let us denote by w a suitable normalization of this vertical displacement and introduce the order parameter $u := e^{2\pi\imath w}$. Then, the first addendum in the right-hand side of (0.1) is nothing but a suitable normalization of the stored elastic energy outside the cores. As for the "plastic term" $|\mu|(\Omega)$, it does not play any role in the asymptotic analysis as $\varepsilon \to 0^+$ and serves only to guarantee compactness; indeed, without such a term a distribution μ with $\Omega_\varepsilon(\mu) = \varnothing$ would pay zero energy.

We notice that the scalar field w cannot belong to $H^1(\Omega_\varepsilon(\mu))$, since its deformation gradient has a nontrivial circulation condition on a curve enclosing a singularity located at x^0; from a modeling point of view, the function w must be in $SBV^2(\Omega)$, hence showing a non zero jump on some one-dimensional set S_w which has x^0 as one of its endpoints. This jump set represents the (section of) a slip plane, while its amplitude $[w]$ corresponds to the circulation of the absolutely continuous gradient ∇w, which is quantized (i.e., it is integer valued), and represents the mismatch between the atomic lattices above and below S_w.

The most celebrated model used to describe topological singularities is, of course, (GL). In such a case, the energy depends only on the order parameter $u \in H^1(\Omega;\mathbb{R}^2)$ which is "penalized" (instead of "constrained") to take values in $\mathbb{S}^1$ and which is defined on the whole Ω. The most basic form of the energy is given by

$$\mathcal{E}^{\mathrm{GL}}_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_\Omega \left(1-|u|^2\right)^2 \,\mathrm{d}x\,, \tag{0.2}$$

where the parameter ε is called in this case *coherence length*. Note that the role of the measure μ in (CR) is actually played in the (GL) model by $Ju = \det\nabla u$ which is here a diffuse measure.

Let us focus on how the models (CR) and (GL) are able to detect the topological singularities. Loosely speaking, a prototypical topological singularity of degree $z \in \mathbb{Z}\setminus\{0\}$ at a point $x^0 \in \Omega$ can be thought of as the point singularity of a vectorial order parameter $\bar{u}_\varepsilon : \Omega \to \mathbb{R}^2$ which, in a ball $B_\delta(x^0)$ with $\varepsilon \ll \delta \ll 1$, winds around the center as $\left(\frac{x-x^0}{|x-x^0|}\right)^z$. The energy of $\bar{u}_\varepsilon$ diverges at order $|\log\varepsilon|$ as $\varepsilon \to 0^+$. As a consequence, to detect the effective energy cost of finitely many vortex singularities, one needs to study the (CR) and (GL) energy functionals at a logarithmic scaling. It has been proved (see e.g. [1, 38] and references therin) that a sequence $\{u_\varepsilon\}_\varepsilon$, along which these energy functionals are equi-bounded, has Jacobians Ju_ε that, up to a subsequence, converge in the flat sense (see Section 1) to an atomic measure $\pi\mu = \pi\sum_{i=1}^I z^i\delta_{x^i}$ (here the x^i represent the positions of the limiting vortices and $z^i \in \mathbb{Z}\setminus\{0\}$ are their multiplicities). The Γ-limit of $\frac{\mathcal{E}^{\mathrm{GL}}_\varepsilon}{|\log\varepsilon|}$ with respect to this convergence is then given by $\pi|\mu|$. The same analysis has been developed within the (CR) approach in [47]; moreover, in [5] it is proven that the two models are *asymptotically equivalent (in terms of* Γ*-convergence)* at energy regime $|\log\varepsilon|^p$ with $p \geq 1$. Variants of the models (CR) and (GL) have been intensively studied; we recall, for instance, discrete linear models for screw dislocations and vortices [3, 47, 6, 23] (see also [2, 4] for the case of periodic inhomogeneous materials) and semidiscrete and discrete models for edge dislocations [46, 29, 24, 7].

In this paper, we are interested in a generalization of the models (CR) and (GL) from a different perspective; specifically, we aim at providing a meaningful energy functional which depends on order parameter u defined on the whole domain Ω, taking values in $\mathbb{S}^1$ and whose approximate gradient is square-integrable in Ω. From the point of view of screw dislocations, this means

that the jump $[w]$ of the scalar displacement w cannot[1] be an entire multiple of the Burgers vector (i.e., equivalently, be integer-valued) on the whole jump set S_w, since this would lead to a deformation gradient that is non square-integrable. Therefore, the jump $[w]$ of the displacement should exhibit a transition between integers in a "small" subset Σ_w of S_w, close to the singularity. In our model this transition has a length of the order ε and as we let $\varepsilon \to 0^+$ the displacement jump is increasingly "forced" to take integer values everywhere on S_w; in other words the total length of Σ_w vanishes asymptotically as $\varepsilon \to 0^+$, giving rise to isolated atomic singularities. These considerations lead us to consider the following energy

$$\widehat{\mathcal{G}}_\varepsilon(w) := \int_\Omega \frac{1}{2}|\nabla w|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{S_w} W([w])\,\mathrm{d}\mathcal{H}^1\,, \tag{0.3}$$

where the first term is the standard antiplane elastic energy associated to the displacement w, whereas the second one is a multi-well potential which is null on $\mathbb{Z}$ and positive elsewhere. As for instance, one can take the potential

$$W(t) := \mathrm{dist}^\alpha(t,\mathbb{Z})\,, \qquad\qquad \text{for some parameter } \alpha \geq 0\,,$$

and hence, passing to $u := e^{2\pi\imath w}$, one arrives at

$$W([w]) := |[u]|^\alpha\,. \tag{0.4}$$

For fixed ε, minimizing (0.3) involves a competition between the stored-elastic energy (first term in (0.3)) which might be large (if for instance boundary conditions for u are given on $\partial\Omega$) and the creation of a integer displacement mismatch within the otherwise regular crystal lattice (which has an energetical cost given by the second term in (0.3)).

Notice that, although potentials as in (0.4) are largely used in the context of crack mechanics [14], with the jump term meaning the energetic cost of crack creation or propagation, the role of W in our case is a bit different. Indeed, here, W penalizes the "wrong" mismatch of atomic lattices through the interface S_w. Therefore, passing to $u := e^{2\pi\imath w}$, the functional $\widehat{\mathcal{G}}_\varepsilon$ in (0.3) takes the form

$$\mathcal{G}_\varepsilon(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\int_{S_u} |[u]|^\alpha\,\mathrm{d}\mathcal{H}^1\,. \tag{0.5}$$

Clearly, in order to describe the presence of topological singularities, the functional $\mathcal{G}_\varepsilon$ cannot be restricted to Sobolev functions, but should be defined on the set of $\mathbb{S}^1$-valued SBV^2 maps.

In this paper we focus on the case $\alpha = 0$ and define

$$\mathcal{F}_\varepsilon(u) := \int_\Omega \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\mathcal{H}^1(\overline{S}_u)\,. \tag{0.6}$$

We point out that, at least under some geometric assumptions on Ω and prescribing the set S_u accordingly[2], one can relate the elastic energy to the $H^{\frac{1}{2}}$-norm of the jump of u, whence leading to 1d models which have been already investigated (see [28]; related models for different lattice mismatches can be found in [26, 27] and references therein). Related to this setting is [31, 19] where a similar analysis is done in 2d.

Therefore we aim at studying the asymptotic behavior of $\frac{\mathcal{F}_\varepsilon}{|\log\varepsilon|}$ as $\varepsilon \to 0$. To this purpose, we first need to introduce a suitable notion of Jacobian for $\mathbb{S}^1$-valued SBV^2 maps. This task is accomplished in Section 2, where we provide an extension of the Jacobian determinant for maps in $BV(\Omega;\mathbb{R}^2)\cap L^\infty(\Omega;\mathbb{R}^2)$ (see [17, 18, 13] and the references therein for fine properties of the Jacobian determinant in Sobolev spaces). For a map $u : \Omega \to \mathbb{R}^2$, the distributional Jacobian determinant $\mathrm{Det}(\nabla u)$, introduced in [43], is defined as the distribution

$$\langle \mathrm{Det}(\nabla u), \varphi\rangle_\Omega := \int_\Omega \nabla\varphi\cdot\lambda_u\,\mathrm{d}x, \tag{0.7}$$

[1] if it would, in presence of an isolated singularity, the gradient of u would only be in L^p with $1 \leq p < 2$.

[2] e.g. when Ω is a square, S_u the horizontal axis of Ω, and prescribing periodic boundary conditions of u.

where

$$\lambda_u := \frac{1}{2}\Big(- u^1 \frac{\partial u^2}{\partial x_2} + u^2 \frac{\partial u^1}{\partial x_2}, u^1 \frac{\partial u^2}{\partial x_1} - u^2 \frac{\partial u^1}{\partial x_1}\Big); \tag{0.8}$$

it is well-defined under suitable summability assumptions on u ensuring that $\lambda_u \in L^1(\Omega;\mathbb{R}^2)$. Moreover, denoting by $j(u)$ the *current* associated to u, i.e., $j(u) := \frac{1}{2}(u^1\nabla u^2 - u^2\nabla u^1)$, one has $j(u)^\perp = \lambda_u$ and it is easy to check that

$$\mathrm{Det}(\nabla u) = -\mathrm{Div}\lambda_u = \mathrm{curl} j(u) = \frac{1}{2}\mathrm{curl}(\nabla w),$$

holds in the sense of distributions, where ∇w is the approximate gradient of w. In general $\mathrm{Det}(\nabla u)$ does not coincide with the pointwise Jacobian determinant $\det(\nabla u)$, but they coincide under suitable assumptions on u (see [45]). When $u \in BV(\Omega;\mathbb{R}^2)$ it is possible to extend definition (0.7) if one shows that λ_u is a Radon measure uniquely determined by u. Such a fact is trivial for smooth maps. Hence, arguing by approximation, it is possible to show that if $\{v_k\}_{k\in\mathbb{N}}$ is a sequence of smooth functions approaching u w.r.t. the strict topology of BV, then the distribution λ_u is uniquely determined as the limit of the distributions λ_{v_k}. This is done in Section 2, and follows as a consequence of the results by Jerrard and Jung who showed the existence and uniqueness of a so-called *minimal lifting* of $\mathrm{D}u$, for all $u \in BV(\Omega;\mathbb{R}^2)$ (see Theorem 1.1 and [37] for details).

We emphasize that in this planar case with maps with values in $\mathbb{R}^2$, using the aforementioned result of Jerrard and Jung, an extension of distributional determinant for maps of bounded variation has been recently obtained in [44], under the additional assumption that a map u has finite relaxed graph area (where relaxation is done w.r.t. the strict topology of BV). This means that u is approximable strictly in BV by smooth maps v_k with uniformly bounded graphs area (see [11, 44] for the relaxed area functional). This implies that the Cartesian current in $\Omega\times\mathbb{R}^2$ obtained as the limit of the graphs of the approximating v_k is uniquely determined, which guarantees, in turn, that the Jacobian of u is always a Radon measure with finite total variation (see [44] for details).

However, requiring only that $u \in BV(\Omega;\mathbb{R}^2)\cap L^\infty(\Omega;\mathbb{R}^2)$, the weak Jacobian determinant Ju of u (defined in (2.35)) is, in general, not a measure, but a mere distribution whose flat norm is controlled by the total variation of λ_u. Moreover, although we will employ the notion of weak Jacobian only for $\mathbb{S}^1$-valued maps defined on a planar domain, we point out that the definition of weak Jacobian can be extended to more general (possibly unbounded) functions with bounded variation $u : U \subset \mathbb{R}^n \to \mathbb{R}^m$, also in dimension and codimension greater than two, allowing to give a notion of weak 2×2-minors of $\mathrm{D}u$. This is stated and proved in Theorem 2.5, which also shows that the notion of weak Jacobian extends the distributional determinant in the case of unbounded Sobolev maps.

Once introduced our weak notion of Jacobian determinant, our main result is Theorem 3.1, where we state compactness and Γ-convergence for the functionals $\frac{\mathcal{F}_\varepsilon}{|\log\varepsilon|}$. As in the classical (CR) and (GL) models, the compactness property shows that, if $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ with

$$\sup_{\varepsilon>0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} < +\infty\,, \tag{0.9}$$

then, up to a subsequence, the weak Jacobians Ju_ε converge in the flat sense to a measure $\pi\mu := \pi\sum_{n=1}^N z^n\delta_{x^n}$ with $z^n \in \mathbb{Z}\setminus\{0\}$. The starting point of the compactness analysis is that $\mathcal{H}^1(\overline{S}_{u_\varepsilon}) \le C\varepsilon|\log\varepsilon|$, which guarantees that $\overline{S}_{u_\varepsilon}$ can be covered by a finite (for fixed ε) family of balls such that the sum of its radii vanishes as $\varepsilon\to 0$. Starting from this family one can use the *ball construction* technique introduced in [48, 36] in order to provide lower bounds for the (CR) and (GL) models. However, such a tool is not enough in order to guarantee compactness in our case. Indeed, in (GL) the corresponding bound as in (0.9), implies that $|Ju_\varepsilon|(\Omega) \le C|\log\varepsilon|$, which in turn allows the authors of [8] to prove that the flat distance between the diffuse Jacobian of u_ε and the atomic measure provided by the ball construction tends to zero. In order to bypass the control on the total variation of the Jacobians (which in our case are not even measures), we adopt the following strategy. In order to avoid to estimate the flat distance of the singularities

accumulating at the boundary, in our main Theorem 3.1 we assume that the jump sets of the maps u_ε are all contained in a fixed set $\Omega' \subset\subset \Omega$. In Theorem 3.5 we use this setting to provide the same result with an applied displacement boundary condition on $\partial\Omega$ but this time with possibly $S_u \cap \partial\Omega \neq \varnothing$. Moreover, in order to deal with zero degree clusters contained in Ω, we provide an iterative machinery that allows to modify the balls provided by the ball construction in order to get a "good bound" of $|\mathrm{D}u_\varepsilon|$ on the boundary of small regions containing these clusters. From a physical standpoint these zero degree clusters are related to the so-called *statistically-stored* dislocations in single crystals, as opposed to the (*geometrically necessary*) dislocations required to accommodate the incompatible deformations.

Some comments on our approach are in order. A natural question is whether in (0.6) one can drop the closure on S_u, replacing $\mathcal{H}^1(\overline{S}_u)$ with $\mathcal{H}^1(S_u)$. This creates some problems in the proof of compactness since, in such a case, the jump set, whose $\mathcal{H}^1$ measure vanishes as $\varepsilon \to 0^+$, can be dense in Ω. In terms of our proof, this means that S_{u_ε} can be covered by a family of *countably many*, not necessarily *finitely many*, balls whose sum of the radii vanishes as $\varepsilon \to 0^+$. Since the ball construction [48, 36] requires that the number of initial balls is finite, the problem above requires an extension of the ball construction procedure to the case of countably many initial balls.

Another issue concerns the case $\alpha > 0$ in (0.5). Also such a problem cannot be studied in full analogy with the one considered here, in view of the lack of a-priori control of the $\mathcal{H}^1$-measure of the jump set in that case. Furthermore, we do believe that our approach could be extended to higher energy regimes as well as to the vectorial case of edge dislocations. This last issue is much more delicate, since it requires the extension of our approach to maps with vectorial lifting. Finally, an extension of our approach to the 3d setting [21] deserves future investigations; in such a case, techniques coming from the theory of integral and Cartesian currents could be exploited, as successfully done in [34, 20, 50, 51, 52, 30].

Plan of the paper. The paper is organized as follows: In Section 1 we introduce the notation and preliminary notions used in the sequel. In Section 2 we extend the notion of distributional Jacobian determinant to some classes of maps of bounded variation (see Theorems 2.2 and 2.5). In Section 3 we finally pass to the Γ-convergence analysis of the functional (0.6) and prove our main result Theorem 3.1.

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1. Preliminary results

In this section we collect some preliminary notions on the flat norm of measures and currents, as well as some properties of BV functions that will be used throughout the paper.

Multi-indices. Let $m, n \geq 2$ be two fixed integers. Let $I \subset \{1, \dots, n\}$ and $J \subset \{1, \dots, m\}$ be multi-indices (in particular, they have a specific order). We denote by $\widehat{I}$ the ordered set $\{1, \dots, n\} \setminus I$ and $\widehat{J} = \{1, \dots, m\} \setminus J$. We also denote by $\sigma(I, \widehat{I})$ and $\sigma(J, \widehat{J})$ the signs of the permutations $(I, \widehat{I}) \in S(n)$ and $(J, \widehat{J}) \in S(m)$, where $(I, \widehat{I})$ and $(J, \widehat{J})$ are seen as the sets $\{1, \dots, n\}$ and $\{1, \dots, m\}$ with a precise order. We will usually deal with the case in which $I = \{i, i'\}$ and $J = \{j, j'\}$ are multi-indices consisting of exactly to distinct elements.

Flat norm of Radon measures. Let $n \geq 1$ be an integer and let $U \subset \mathbb{R}^n$ be a bounded and open set. We denote by $\mathcal{M}_b(U)$ the space of Radon measures on U with finite total variation. If $\mu \in \mathcal{M}_b(U)$, we denote by $|\mu|(U)$ the total variation of μ. We recall that a sequence $\mu_k \in \mathcal{M}_b(U)$ converges tightly to $\mu \in \mathcal{M}_b(U)$ if μ_k converges to μ weakly* as measure, and $|\mu_k|(U) \to |\mu|(U)$.

We also introduce the concept of flat norm of a measure μ, denoted by $\|\mu\|_{\mathrm{flat}}$, as

$$\|\mu\|_{\mathrm{flat}} := \sup_{\substack{\varphi\in C_c^{0,1}(U)\\ \|\varphi\|_{C^{0,1}(U)}\le 1}} \int_U \varphi \,\mathrm{d}\mu\,. \tag{1.1}$$

Here and below, the Lipschitz norm $\|\varphi\|_{C^{0,1}(U)}$ is defined by

$$\|\varphi\|_{C^{0,1}(U)} := \|\varphi\|_{L^\infty(U)} + \sup_{\substack{x,y\in U\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|}\,.$$

By a density argument we easily see that the supremum in (1.1) can be equivalently computed among smooth and compactly supported (in U) functions φ with $\|\varphi\|_{C^{0,1}(U)}\le 1$.

Flat norm of k-currents. Let $n\ge 2$ be an integer and let $U\subset\mathbb{R}^n$ be an open set. For every $k\in\mathbb{N}$ with $0\le k\le n$, we denote by $\mathcal{D}^k(U)$ the topological vector space of smooth and compactly supported k-forms on U, and by $\mathcal{D}_k(U)$ its dual, i.e., the space of k-currents on U. The canonical basis of 1-vectors in $\mathbb{R}^n$ is denoted by $\{e_1,\dots,e_n\}$, and its dual basis of 1-covectors by $\{\mathrm{d}x^1,\dots,\mathrm{d}x^n\}$.

The mass $|T|$ of a current $T\in\mathcal{D}_k(U)$ is defined as

$$|T| = \sup\{\langle T,\omega\rangle :\ \omega\in\mathcal{D}^k(U), \|\omega\|_{L^\infty}\le 1\}\,.$$

As done in (1.1) for measures, we define the *flat norm* of a current $T\in\mathcal{D}_k(U)$ in U by

$$\|T\|_{\mathrm{flat},U} := \sup_{\substack{\omega\in\mathcal{D}^k(U)\\ \|\omega\|_{F,U}\le 1}} \langle T,\omega\rangle, \tag{1.2}$$

where

$$\|\omega\|_{F,U} := \|\omega\|_{L^\infty(U)} + \|\mathrm{d}\omega\|_{L^\infty(U)}\,.$$

In the special case that T is a 0-current and has finite mass, then it can be standardly identified with a measure, and the flat norm of T coincides with the flat norm of the measure T defined in (1.1).

Jacobian for $\mathbb{S}^1$-valued Sobolev maps in $\mathbb{R}^2$. Let $U\subset\mathbb{R}^2$ be a bounded and open set. Given a map $u\in W^{1,1}(U;\mathbb{R}^2)\cap L^\infty(U;\mathbb{R}^2)$ we recall that $Ju=\mathrm{Det}(\nabla u)$ is defined by (0.7). Notice that $\mathrm{Det}(\nabla u)$ is well-defined also if $u\in W^{1,\frac{4}{3}}(U;\mathbb{R}^2)$, since by Sobolev embedding theorem we have $\lambda_u\in L^1(U;\mathbb{R}^2)$ (see (0.8)).

In the sequel we will use the fact that a function $u\in H^1(U;\mathbb{S}^1)$ satisfies $\mathrm{Det}(\nabla u)=0$ in the sense of distributions. Moreover, if $u\in H^1(U\setminus\overline{B};\mathbb{S}^1)$, where $B\subset U$ is a ball, then, integrating by parts,

$$\int_{U\setminus\overline{B}} \lambda_u\cdot\nabla\varphi\,\mathrm{d}x = \int_{\partial B}\lambda_u\cdot\boldsymbol{\nu}\varphi\,\mathrm{d}\mathcal{H}^1 = \int_{\partial B} j(u)\cdot\boldsymbol{\tau}\varphi\,\mathrm{d}\mathcal{H}^1, \qquad \forall\varphi\in C_c^\infty(U),$$

where $\boldsymbol{\nu}$ is the inner normal vector to ∂B, $\boldsymbol{\tau}=-\boldsymbol{\nu}^\perp$ is the counter-clockwise tangent normal vector to ∂B, and $j(u)=\frac{1}{2}(u^1\nabla u^2-u^2\nabla u^1)=-\lambda_u^\perp$. Notice that $j(u)\cdot\boldsymbol{\tau}=\frac{1}{2}(u^1\frac{\partial u^2}{\partial\boldsymbol{\tau}}-u^2\frac{\partial u^1}{\partial\boldsymbol{\tau}})$ on ∂B.

We recall that $\deg(u,\partial B)\in\mathbb{Z}$ is defined as

$$\deg(u,\partial B) := \frac{1}{\pi}\int_{\partial B} j(u)\cdot\boldsymbol{\tau}\,\mathrm{d}\mathcal{H}^1 = \frac{1}{\pi}\int_{\partial B}\lambda_u\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^1\,, \tag{1.3}$$

whenever $u\in H^{\frac{1}{2}}(\partial B;\mathbb{S}^1)$.

Vector-valued BV functions. Here we recall some basic facts on vector-valued BV-functions that will be used throughout the paper. We refer to the monograph [9] for a complete analysis of BV-functions.

Let $m, n \geq 2$ be two integers and let $\Omega \subset \mathbb{R}^n$ be an open set. Let moreover $u \in BV(\Omega;\mathbb{R}^m)$. Then, there is an $\mathcal{H}^{n-1}$-rectifiable set $S_u \subset \Omega$ such that $\mathcal{H}^{n-1}$-a.e. $x \in \Omega \setminus S_u$ is a Lebesgue point for u. For every such $x \in \Omega \setminus S_u$, we denote by $\overline{u}(x)$, the Lebesgue value of u at x, so that $\overline{u}$ turns out to be well-defined $\mathcal{H}^{n-1}$-a.e. on $\Omega \setminus S_u$. The set S_u has the following property: For $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ there exists a unit vector $\boldsymbol{\nu}(x) \in \mathbb{R}^n$, which is orthogonal to the approximate tangent space to S_u at x, and there exist $u^+(x), u^-(x) \in \mathbb{R}^m$ such that the maps $\boldsymbol{\nu}, u^+, u^-$ are $\mathcal{H}^{n-1}$-measurable on S_u, and

$$u^+(x) = \operatorname{aplim}_{y\to x,\ (y-x)\cdot\boldsymbol{\nu}(x)>0} u(y), \qquad u^-(x) = \operatorname{aplim}_{y\to x,\ (y-x)\cdot\boldsymbol{\nu}(x)<0} u(y)\,,$$

for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$.

The distributional gradient $\mathrm{D}u$ of u reads as $\mathrm{D}u = \mathrm{D}^D u + \mathrm{D}^S u$, where $\mathrm{D}^S u$ is the jump part of the gradient provided by

$$\mathrm{D}^S u = [u] \otimes \boldsymbol{\nu}\, \mathrm{d}\mathcal{H}^{n-1}\,,$$

with $[u] := u^+ - u^-$ ($u^\pm$ being the traces of u on the two sides of S_u), and $\mathrm{D}^D u = \mathrm{D}u \llcorner (\Omega \setminus S_u)$ is the diffuse part of the gradient, which satisfies $\mathrm{D}^D u(A) = 0$ if $\mathcal{H}^{n-1}(A) < \infty$. In turn, $\mathrm{D}^D u = \mathrm{D}^a u + \mathrm{D}^c u$ splits into two mutually singular measures, $\mathrm{D}^a u$ which is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^n$, and the Cantor part $\mathrm{D}^c u$.

Eventually, we extend the definition of $\overline{u}$, which has been defined so far only $\mathcal{H}^{n-1}$-a.e. on $\Omega \setminus S_u$, by setting

$$\overline{u}(x) = \frac{1}{2}(u^+(x) + u^-(x)), \qquad x \in S_u\,. \tag{1.4}$$

In this way, $\overline{u}$ is well-defined $\mathcal{H}^{n-1}$-a.e. on S_u and, in general, $\overline{u}$ is well-defined $|\mathrm{D}u|$-a.e. in Ω. Furthermore, for every $\theta \in [0,1]$ and $\mathcal{H}^{n-1}$-a.e. $x \in S_u$ we define the function $\overline{u}^\theta$ as

$$\overline{u}^\theta(x) := \theta u^+(x) + (1-\theta)u^-(x)\,, \tag{1.5}$$

so that $\overline{u} \equiv \overline{u}^{\frac{1}{2}}$. Finally, we say that a sequence $\{u_k\}_{k\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^m)$ *strictly converges* to u in $BV(\Omega;\mathbb{R}^m)$ (as $k \to +\infty$), and we write $u_k \overset{\text{strict}}{\rightharpoonup} u$ (as $k \to +\infty$), if $u_k \to u$ (strongly) in $L^1(\Omega;\mathbb{R}^m)$ and $|\mathrm{D}u_k|(\Omega) \to |\mathrm{D}u|(\Omega)$ (as $k \to +\infty$).

Minimal lifting. Here we deal with maps $v : \Omega \to \mathbb{R}^m$, where $\Omega \subset \mathbb{R}^n$ is a bounded and open set. We denote by $x = (x_1;\dots;x_n) \in \mathbb{R}^n$ the coordinates in $\mathbb{R}^n$, and $y = (y^1;\dots;y^m)$ those in $\mathbb{R}^m$. Given $v = (v^1;\dots;v^m) \in C^1(\Omega;\mathbb{R}^m) \cap W^{1,1}(\Omega;\mathbb{R}^m)$, for every $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, we define $(\mu_v)^i_j \in \mathcal{M}_b(\Omega\times\mathbb{R}^m)$ as the only measure satisfying

$$\int_{\Omega\times\mathbb{R}^m} \phi(x,y)\, \mathrm{d}(\mu_v)^i_j = \int_\Omega \phi(x,v(x))\frac{\partial v^i}{\partial x_j}\,\mathrm{d}x\,, \qquad \text{for every } \phi \in C_{\mathrm{c}}(\Omega\times\mathbb{R}^m)\,. \tag{1.6}$$

In [37] the following result is proved.

Theorem 1.1. *Let $u \in BV(\Omega;\mathbb{R}^m)$. Then, for every $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, there exists a measure $(\mu_u)^i_j \in \mathcal{M}_b(\Omega\times\mathbb{R}^m)$ such that the following holds: If $\{v_k\}_{k\in\mathbb{N}} \subset C^1(\Omega;\mathbb{R}^m) \cap W^{1,1}(\Omega;\mathbb{R}^m)$ satisfies $v_k \overset{\text{strict}}{\rightharpoonup} v$ (as $k \to +\infty$), then $(\mu_{v_k})^i_j \overset{*}{\rightharpoonup} (\mu_u)^i_j$ in $\mathcal{M}_b(\Omega\times\mathbb{R}^m)$ and the $\mathbb{R}^{m\times n}$-valued measure $\mu_{v_k} := \big((\mu_{v_k})^i_j\big)_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ converges tightly to $\mu_u := \big((\mu_u)^i_j\big)_{\substack{i=1,\dots,m\\ j=1,\dots,n}}$ (as $k \to +\infty$). Furthermore, for every $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, it holds*

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^m} \phi(x,y)\, \mathrm{d}(\mu_u)^i_j &= \int_{\Omega\setminus S_u} \phi(x,\overline{u}(x))\, \mathrm{d}(\mathrm{D}^D_j u^i)(x)\\ &\quad + \int_{S_u}\int_0^1 \phi(x,\overline{u}^\theta(x))\, \mathrm{d}\theta\, \mathrm{d}(\mathrm{D}^S_j u^i)(x)\,,\end{aligned} \tag{1.7}$$

for every $\phi \in C_{\mathrm{c}}(\Omega\times\mathbb{R}^m)$.

As a consequence of Theorem 1.1, it turns out that the $\mathbb{R}^{m\times n}$-valued measure μ_u is uniquely determined by $u \in BV(\Omega;\mathbb{R}^m)$ and is called *minimal lifting* of $\mathrm{D}u$. Moreover, it easily follows that if $\{u_k\}_{k\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^m)$ satisfies $u_k \overset{\text{strict}}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^m)$ (as $k\to+\infty$), then $\mu_{u_k} \to \mu_u$ tightly.

2. Weak 2×2-minors of gradients and jacobian determinants of BV functions

We now see how the notion of minimal lifting provided by Theorem 1.1 allows to define, in a weak sense, the 2×2-minors of gradient of suitable BV-maps.

Let $m \geq 2$ and $n \geq 2$ be two integers and let $\Omega \subset \mathbb{R}^n$ be a bounded and open set. Given $v \in C^1(\Omega;\mathbb{R}^m) \cap W^{1,1}(\Omega;\mathbb{R}^m)$, testing (1.6) with $\phi(x,y) = \varphi(x)\psi(y)$, where $\varphi \in C_c(\Omega)$ and $\psi \in C_c(\mathbb{R}^m)$, we have

$$\int_{\Omega\times\mathbb{R}^m} \varphi(x)\psi(y)\ \mathrm{d}(\mu_v)^i_j = \int_\Omega \varphi(x)\psi(v(x))\frac{\partial v^i}{\partial x_j}\,\mathrm{d}x\,.$$

It follows that, if also $v \in L^\infty(\Omega;\mathbb{R}^m)$, for every $h = 1,\dots,m$, we can take $\psi := \psi^h$ with $\psi^h \in C_c(\mathbb{R}^m)$ such that $\psi^h(y) = y^h$ in $\overline{B}_{\|v\|_{L^\infty(\Omega;\mathbb{R}^m)}}(0) \subset \mathbb{R}^m$, thus obtaining

$$\int_{\Omega\times\mathbb{R}^m} \varphi(x)\psi^h(y)\ \mathrm{d}(\mu_v)^i_j = \int_\Omega \varphi(x)v^h(x)\frac{\partial v^i}{\partial x_j}\,\mathrm{d}x\,. \tag{2.1}$$

Now, let $u \in BV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ and $\{v_k\}_{k\in\mathbb{N}} \subset C^1(\Omega;\mathbb{R}^m) \cap W^{1,1}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ be such that $v_k \overset{\text{strict}}{\rightharpoonup} u$ as $k\to+\infty$, and $\|v_k\|_{L^\infty(\Omega;\mathbb{R}^m)} \leq C$ for all $k \geq 1$. Then, it is easy to see that for every $i,h = 1,\dots,m$ and for every $j = 1,\dots,n$, it holds

$$v^h_k(x)\frac{\partial v^i_k}{\partial x_j}\,\mathrm{d}x \overset{*}{\rightharpoonup} (\nu_u)^{i,h}_j \qquad \text{in } \mathcal{M}_b(\Omega) \text{ as } k\to+\infty\,, \tag{2.2}$$

where the measure $(\nu_u)^{i,h}_j$ is defined by the formula

$$\int_\Omega \varphi(x)\ \mathrm{d}(\nu_u)^{i,h}_j = \int_{\Omega\times\mathbb{R}^m} \varphi(x)y^h\ \mathrm{d}(\mu_u)^i_j \qquad \forall \varphi \in C_c(\Omega)\,, \tag{2.3}$$

for all $i,h \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$. More precisely, we have the following result.

Corollary 2.1. *Let $u \in BV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$, then there exists a unique measure $\nu_u \in \mathcal{M}_b(\Omega;\mathbb{R}^{m\times n\times m})$ such that, whenever $\{v_k\}_{k\in\mathbb{N}} \subset C^1(\Omega;\mathbb{R}^m) \cap W^{1,1}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ satisfies $\|v_k\|_{L^\infty(\Omega;\mathbb{R}^m)} \leq C < +\infty$ for all $k \geq 1$ and $v_k \overset{\text{strict}}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^m)$, then $v_k \otimes \nabla v_k \to \nu_u$, where $(\nu_u)^{i,h}_j$ is defined by (2.3) for every $i,h \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. Finally, if $\{u_k\}_{k\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ with*

$$\|u_k\|_{L^\infty(\Omega;\mathbb{R}^m)} \leq C, \tag{2.4}$$

for some constant $C > 0$, and $u_k \overset{\text{strict}}{\rightharpoonup} u$ in $BV(\Omega;\mathbb{R}^m)$, then

$$\nu_{u_k} \overset{*}{\rightharpoonup} \nu_u \qquad \text{in } \mathcal{M}_b(\Omega;\mathbb{R}^{m\times n\times m})\,. \tag{2.5}$$

Proof. The first part of the statement follows straightforwardly from the discussion above (see formulas (2.1), (2.2), (2.3)). Let us comment on the last claim, and assume $u_k \overset{\text{strict}}{\rightharpoonup} u$ with $\{u_k\}_{k\in\mathbb{N}} \subset BV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$. Let $\varphi \in C_c(\Omega)$, $i,h \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$. Let moreover $\psi^h \in C_c(\mathbb{R}^m)$ be such that $\psi^h = y^h$ inside $B_C(0) \subset \mathbb{R}^m$, where C is the constant in (2.4). By Theorem 1.1, we get

$$\lim_{k\to+\infty} \int_{\Omega\times\mathbb{R}^m} \varphi(x)\psi^h(y)\ \mathrm{d}(\mu_{u_k})^i_j = \int_{\Omega\times\mathbb{R}^m} \varphi(x)\psi^h(y)\ \mathrm{d}(\mu_u)^i_j\,. \tag{2.6}$$

On the other hand, in view of (2.4), by (2.3), we have that

$$\int_{\Omega\times\mathbb{R}^m} \varphi(x)\psi^h(y)\ \mathrm{d}(\mu_{u_k})^i_j = \int_\Omega \varphi(x)\ \mathrm{d}(\nu_{u_k})^{i,h}_j$$
$$\int_{\Omega\times\mathbb{R}^m} \varphi(x)\psi^h(y)\ \mathrm{d}(\mu_u)^i_j = \int_\Omega \varphi(x)\ \mathrm{d}(\nu_u)^{i,h}_j,$$

which, together with (2.6), implies that

$$\int_\Omega \varphi(x)\ \mathrm{d}(\nu_{u_k})^{i,h}_j \overset{k\to+\infty}{\longrightarrow} \int_\Omega \varphi(x)\ \mathrm{d}(\nu_u)^{i,h}_j\,.$$

By the arbitrariness of φ we get (2.5). □

In view of Corollary 2.1, for every $u \in BV(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$, we are allowed to adopt the following notation

$$[u^h \mathrm{D}_j u^i] := (\nu_u)^{i,h}_j, \quad i,h\in\{1,\dots,m\}, \quad j\in\{1,\dots,n\},$$

and

$$[u\otimes \mathrm{D}u] := \nu_u := \Big((\nu_u)^{i,h}_j\Big)_{\substack{i,h\in\{1,\dots,m\}\\ j\in\{1,\dots,n\}}}.$$

Let $u\in BV(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ and let us fix four indices $i,i'\in\{1,\dots,m\}$ and $j,j'\in\{1,\dots,n\}$. We set $I:=\{i,i'\}$, $J:=\{j,j'\}$, and consider the measures $\lambda^I_j,\lambda^I_{j'}\in\mathcal{M}_b(\Omega)$ given by

$$\begin{aligned}\lambda^I_j :=& \frac{1}{2}\big([u^i\mathrm{D}_j u^{i'}]-[u^{i'}\mathrm{D}_j u^i]\big) = \frac{1}{2}((\nu_u)^{i',i}_j-(\nu_u)^{i,i'}_j)\,,\\ \lambda^I_{j'} :=& \frac{1}{2}\big([u^i\mathrm{D}_{j'} u^{i'}]-[u^{i'}\mathrm{D}_{j'} u^i]\big) = \frac{1}{2}((\nu_u)^{i',i}_{j'}-(\nu_u)^{i,i'}_{j'})\,.\end{aligned} \tag{2.7}$$

We define the current $(T_u)_{I,J}\in\mathcal{D}_1(\Omega)$ as

$$\begin{aligned}(T_u)_{I,J} :=& \sigma^I_J(\lambda^I_j e_{j'}-\lambda^I_{j'}e_j)\\ =& \frac{\sigma^I_J}{2}\big([u^i\mathrm{D}_j u^{i'}]-[u^{i'}\mathrm{D}_j u^i]\big)e_{j'} - \frac{\sigma^I_J}{2}\big([u^i\mathrm{D}_{j'} u^{i'}]-[u^{i'}\mathrm{D}_{j'} u^i]\big)e_j,\end{aligned} \tag{2.8}$$

where $\sigma^I_J := \sigma(I,\widehat{I})\sigma(J,\widehat{J})\in\{-1,1\}$. The 1-current $(T_u)_{I,J}$ acts on 1-forms $\omega\in\mathcal{D}^1(\Omega)$ with $\omega=\sum_h \omega_h\,\mathrm{d}x^h$, $\omega_h\in C^\infty_{\mathrm{c}}(\Omega\times\mathbb{R}^m)$, as

$$\begin{aligned}(T_u)_{I,J}(\omega) &= \sigma^I_J\Big(\int_\Omega \omega_{j'}\,\mathrm{d}\lambda^I_j - \int_\Omega \omega_j\,\mathrm{d}\lambda^I_{j'}\Big)\\ &= \frac{\sigma^I_J}{2}\int_\Omega \omega_{j'}\,\mathrm{d}([u^i\mathrm{D}_j u^{i'}]-[u^{i'}\mathrm{D}_j u^i]) - \frac{\sigma^I_J}{2}\int_\Omega \omega_j\,\mathrm{d}([u^i\mathrm{D}_{j'} u^{i'}]-[u^{i'}\mathrm{D}_{j'} u^i]).\end{aligned} \tag{2.9}$$

A consequence of Corollary 2.1 is that $(T_u)_{I,J}$ is well-defined (for all $u\in BV(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$), it is a Radon measure in $\mathcal{M}_b(\Omega;\mathbb{R}^m)$, and is weakly* continuous with respect to strict convergence of u_k to u in $BV(\Omega;\mathbb{R}^m)$, provided that u_k are uniformly bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Therefore we arrive at the following result.

Theorem 2.2. *For all $u\in BV(\Omega;\mathbb{R}^m)\cap L^\infty(\Omega;\mathbb{R}^m)$ and for all $I=\{i,i'\}\subset\{1,\dots,n\}$, $J=\{j,j'\}\subset\{1,\dots,m\}$, let $\partial(T_u)_{I,J}\in\mathcal{D}_0(\Omega)$ denote the boundary of the current $(T_u)_{I,J}$ given by* (2.9)*. Then,*

$$\|\partial(T_u)_{I,J}\|_{\mathrm{flat},\Omega}\le C\|u\|_{L^\infty(\Omega;\mathbb{R}^m)}|\mathrm{D}u|(\Omega)\,, \tag{2.10}$$

for some universal constant $C>0$ (independent of u).

Proof. Let $\varphi\in C^\infty_{\mathrm{c}}(\Omega)$ and consider the 0-form $\omega=\varphi$, whose differential is $\mathrm{d}\omega=\sum_h \frac{\partial\varphi}{\partial x_h}\,\mathrm{d}x^h$. Hence, by definition of boundary of a current

$$\begin{aligned}\partial(T_u)_{I,J}(\omega) = (T_u)_{I,J}(\mathrm{d}\omega) &= \int_\Omega \frac{\partial\varphi}{\partial x_{j'}}\,\mathrm{d}\lambda^I_j - \int_\Omega \frac{\partial\varphi}{\partial x_j}\,\mathrm{d}\lambda^I_{j'}\\ &= \frac{\sigma^I_J}{2}\int_\Omega \frac{\partial\varphi}{\partial x_{j'}}\,\mathrm{d}([u^i\mathrm{D}_j u^{i'}]-[u^{i'}\mathrm{D}_j u^i]) - \frac{\sigma^I_J}{2}\int_\Omega \frac{\partial\varphi}{\partial x_j}\,\mathrm{d}([u^i\mathrm{D}_{j'} u^{i'}]-[u^{i'}\mathrm{D}_{j'} u^i])\,,\end{aligned} \tag{2.11}$$

from which (2.10) follows by definition of flat norm. □

We can then introduce the definition of weak 2×2-minors of $\mathrm{D}u$. In what follows, for every open and bounded set $U\subset\mathbb{R}^k$ (for some $k\in\mathbb{N}$) we denote by $\mathcal{D}(U)$ the family of test functions defined in U and by $\mathcal{D}'(U)$ its dual, i.e., the family of distributions.

Definition 2.3. *Let $u \in BV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ and let $I = \{i, i'\} \subset \{1,\dots,n\}$, $J = \{j, j'\} \subset \{1,\dots,m\}$. Then we define $M_I^J(\mathrm{D}u) \in \mathcal{D}'(\Omega)$ the 2×2-minor of $\mathrm{D}u$ with rows in I and columns in J as the distribution*

$$\begin{aligned}\langle M_I^J(\mathrm{D}u), \varphi\rangle_\Omega :&= \int_\Omega \frac{\partial\varphi}{\partial x_{j'}}\,\mathrm{d}\lambda_j^I - \int_\Omega \frac{\partial\varphi}{\partial x_j}\,\mathrm{d}\lambda_{j'}^I\\ &= \frac{\sigma_J^I}{2}\int_\Omega \frac{\partial\varphi}{\partial x_{j'}}\,\mathrm{d}([u^i\mathrm{D}_j u^{i'}] - [u^{i'}\mathrm{D}_j u^i]) - \frac{\sigma_J^I}{2}\int_\Omega \frac{\partial\varphi}{\partial x_j}\,\mathrm{d}([u^i\mathrm{D}_{j'}u^{i'}] - [u^{i'}\mathrm{D}_{j'}u^i]),\end{aligned} \tag{2.12}$$

for all $\varphi \in \mathcal{D}(\Omega)$, where λ_j^I and $\lambda_{j'}^I$ are defined in (2.7).

Notice that, if $v \in C^2(\Omega;\mathbb{R}^m) \cap W^{1,1}(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$, by (2.11), (2.12), and (2.1), we get

$$\begin{aligned}M_I^J(\nabla v) =& \frac{\sigma_J^I}{2}\int_\Omega \frac{\partial\varphi}{\partial x_{j'}}\Big(v^i\frac{\partial v^{i'}}{\partial x_j} - v_{i'}\frac{\partial v^i}{\partial x_j}\Big)\,\mathrm{d}x - \frac{\sigma_J^I}{2}\int_\Omega \frac{\partial\varphi}{\partial x_j}\Big(v^i\frac{\partial v^{i'}}{\partial x_{j'}} - v^{i'}\frac{\partial v^i}{\partial x_{j'}}\Big)\,\mathrm{d}x\\ &= \sigma_J^I\int_\Omega \varphi(x)\Big(\frac{\partial v^i}{\partial x_j}(x)\frac{\partial v^{i'}}{\partial x_{j'}}(x) - \frac{\partial v^i}{\partial x_{j'}}(x)\frac{\partial v^{i'}}{\partial x_j}(x)\Big)\,\mathrm{d}x,\end{aligned}$$

namely, the distribution $M_I^J(\nabla v)$ coincides with the (2×2)-subdeterminant of the matrix ∇v with rows in I and columns in J.

2.1. **An extension of the 2×2-minors for more general BV maps.** In this section we refine the result in Theorem 2.2 weakening the hypothesis that $u \in L^\infty(\Omega;\mathbb{R}^m)$. To this purpose, we preliminarily notice that, if $u \in BV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$, we can take $\phi(x,y) = \varphi(x)y^h$ in (1.7), with $\varphi \in C_c(\Omega)$ and $h \in \{1,\dots,m\}$. From this expression, using (2.3), we obtain

$$\begin{aligned}\int_\Omega \varphi(x)\,\mathrm{d}(\nu_u)_j^{i,h} &= \int_{\Omega\times\mathbb{R}^m} \varphi(x)y^h\,\mathrm{d}(\mu_u)_j^i\\ &= \int_{\Omega\setminus S_u} \varphi(x)\overline{u}^h(x)\,\mathrm{d}(\mathrm{D}_j^D u^i) + \int_{S_u}\varphi(x)\int_0^1 (\overline{u}^\theta)^h(x)\,\mathrm{d}\theta\,\mathrm{d}(\mathrm{D}_j^S u^i)\\ &= \int_{\Omega\setminus S_u} \varphi(x)\overline{u}^h(x)\,\mathrm{d}(\mathrm{D}_j u^i) + \int_{S_u}\varphi(x)\overline{u}^h(x)\,\mathrm{d}(\mathrm{D}_j u^i),\end{aligned} \tag{2.13}$$

where $\overline{u}^\theta = ((\overline{u}^\theta)^1;\dots;(\overline{u}^\theta)^m)$ is defined in (1.5). Let now $u \in BV(\Omega;\mathbb{R}^m)$ and assume that

(P) For some $i, i' \in \{1,\dots,m\}$ with $i \neq i'$ and $j, j' \in \{1,\dots,n\}$ with $j \neq j'$, it holds
$$\begin{aligned} y^i &\in L^1(\Omega\times\mathbb{R}^m, (\mu_u)_j^{i'}) \cap L^1(\Omega\times\mathbb{R}^m, (\mu_u)_{j'}^{i'}),\\ y^{i'} &\in L^1(\Omega\times\mathbb{R}^m, (\mu_u)_j^{i}) \cap L^1(\Omega\times\mathbb{R}^m, (\mu_u)_{j'}^{i}),\end{aligned}$$
where $L^1(\Omega\times\mathbb{R}^m,\mu)$ denotes the space of L^1 functions with respect to the measure μ.

We emphasize that if $u \in BV(\Omega;\mathbb{R}^m) \cap L^\infty(\Omega;\mathbb{R}^m)$ then property (P) is readily satisfied, since $\operatorname{supp}\mu_u$ is bounded in $\overline{\Omega}\times\mathbb{R}^m$. In Theorem 2.5 below, we will show that property (P) is enough to ensure the well-posedness of the definition of the current (2.8) as well as to guarantee that it and its boundary have controlled flat norm. To this purpose, we preliminarily notice that, in view of (P), by (1.7), for every $\phi \in C_c(\Omega\times\mathbb{R}^m)$ of the form $\phi(x,y) = \varphi(x)\psi(y)|y^h|$ with $h \in \{i, i'\}$, it holds

$$\begin{aligned}&\int_{\Omega\times\mathbb{R}^m} \varphi(x)\psi(y)|y^h|\,\mathrm{d}(\mu_u)_j^i\\ &= \int_{\Omega\setminus S_u} \varphi(x)\psi(\overline{u}(x))|\overline{u}^h(x)|\,\mathrm{d}(\mathrm{D}_j u^i) + \int_{S_u}\varphi(x)\int_0^1 \psi(\overline{u}^\theta(x))|(\overline{u}^h)^\theta(x)|\,\mathrm{d}\theta\,\mathrm{d}(\mathrm{D}_j u^i)\\ &\le \|\psi\|_{L^\infty(\mathbb{R}^m)}\int_{\Omega\setminus S_u}|\overline{u}^h(x)|\,\mathrm{d}|\mathrm{D}_j u^i| + \|\psi\|_{L^\infty(\mathbb{R}^m)}\int_{S_u}\int_0^1 |(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}|\mathrm{D}_j u^i|.\end{aligned} \tag{2.14}$$

Therefore, we can take the supremum of the left-hand side among all $\varphi \in C_c(\Omega;[-1,1])$ and $\psi \in C_c(\mathbb{R}^m;[-1,1])$, and by standard density arguments we get

$$\int_{\Omega\times\mathbb{R}^m} |y^h|\,\mathrm{d}|(\mu_u)^i_j| \le \int_{\Omega\setminus S_u} |\overline{u}^h(x)|\,\mathrm{d}|\mathrm{D}_j u^i| + \int_{S_u}\int_0^1 |(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}|\mathrm{D}_j u^i|. \tag{2.15}$$

In particular, if $u \in BV(\Omega;\mathbb{R}^m)$ satisfies (P), then the measure $y^h\cdot(\mu_u)^i_j$ (for $h\in\{i,i'\}$) has finite total variation which is bounded by the right-hand side of (2.15). Now we show that (2.15) holds actually with equality.

Lemma 2.4. *Let $u \in BV(\Omega;\mathbb{R}^m)$ and let $i,i'\in\{1,\dots,m\}$ with $i\neq i'$ and $j,j'\in\{1,\dots,n\}$ with $j\neq j'$ be such that* (P) *holds. Then for $h\in\{i,i'\}$ it holds*

$$\int_{\Omega\times\mathbb{R}^m} |y^h|\,\mathrm{d}|(\mu_u)^i_j| = \int_{\Omega\setminus S_u} |\overline{u}^h(x)|\,\mathrm{d}|\mathrm{D}_j u^i| + \int_{S_u}\int_0^1 |(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}|\mathrm{D}_j u^i|. \tag{2.16}$$

In particular it follows

$$\overline{u}^h \in L^1(\Omega\setminus S_u, |\mathrm{D}^D_j u^i|), \qquad \overline{u}^h \in L^1(S_u, |\mathrm{D}^S_j u^i|)\,, \tag{2.17}$$

where the function $\overline{u}$ is defined on S_u by (1.4).

Proof. Fix h,i,j and $\varepsilon>0$, and choose a compact set $S^\varepsilon_u\subset S_u$ such that

$$|\mathrm{D}_j u^i|(S_u\setminus S^\varepsilon_u) = |\mathrm{D}^S_j u^i|(S_u\setminus S^\varepsilon_u) < \frac{\varepsilon}{2}\,. \tag{2.18}$$

Thanks to the compactness of S^ε_u, for all $\delta>0$ small enough, we can choose an open neighborhood U^ε_δ of S^ε_u such that

$$|\mathrm{D}^D_j u^i|(U^\varepsilon_\delta) < \frac{\varepsilon}{2}\,, \tag{2.19}$$

and $U^\varepsilon_\delta \searrow S^\varepsilon_u$ as $\delta\to 0$. Notice that, by (2.18), for all $\delta>0$ small enough we have

$$|\mathrm{D}^S_j u^i|(S_u\setminus U^\varepsilon_\delta) \le |\mathrm{D}^S_j u^i|(S_u\setminus S^\varepsilon_u) < \frac{\varepsilon}{2}\,, \tag{2.20}$$

whereas, by (2.19), we get

$$|\mathrm{D}_j u^i|(U^\varepsilon_\delta\setminus S_u) = |\mathrm{D}^D_j u^i|(U^\varepsilon_\delta\setminus S_u) < \frac{\varepsilon}{2}\,. \tag{2.21}$$

Let $\varphi_{U^\varepsilon_\delta}\in C_c(U^\varepsilon_\delta;[-1,1])$ and $\varphi^c_{U^\varepsilon_\delta}\in C_c(\Omega\setminus\overline{U^\varepsilon_\delta};[-1,1])$ and set $\varphi := \varphi_{U^\varepsilon_\delta}+\varphi^c_{U^\varepsilon_\delta}$. Moreover, let $\psi\in C_c(\mathbb{R}^m;[-1,1])$. By the equality in (2.14), using that $\sup\{|y|: y\in\operatorname{supp}\psi\} =: \widehat{C}(\psi) < +\infty$, we obtain

$$\begin{aligned}
&\int_{\Omega\times\mathbb{R}^m} |y^h|\,\mathrm{d}|(\mu_u)^i_j| \ge \int_{\Omega\times\mathbb{R}^m} \varphi(x)\psi(y)|y^h|\,\mathrm{d}(\mu_u)^i_j \\
&= \int_{U^\varepsilon_\delta\setminus S_u} \varphi_{U^\varepsilon_\delta}(x)\psi(\overline{u}(x))|\overline{u}^h(x)|\,\mathrm{d}(\mathrm{D}_j u^i) + \int_{\Omega\setminus(U^\varepsilon_\delta\cup S_u)} \varphi^c_{U^\varepsilon_\delta}(x)\psi(\overline{u}(x))|\overline{u}^h(x)|\,\mathrm{d}(\mathrm{D}_j u^i) \\
&\quad + \int_{U^\varepsilon_\delta\cap S_u} \varphi_{U^\varepsilon_\delta}(x)\int_0^1 \psi(\overline{u}^\theta(x))|(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}(\mathrm{D}_j u^i) \\
&\quad + \int_{(\Omega\setminus U^\varepsilon_\delta)\cap S_u} \varphi^c_{U^\varepsilon_\delta}(x)\int_0^1 \psi(\overline{u}^\theta(x))|(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}(\mathrm{D}_j u^i) \\
&\ge -|\mathrm{D}_j u^i|(U^\varepsilon_\delta\setminus S_u)\,\widehat{C}(\psi) + \int_{\Omega\setminus(U^\varepsilon_\delta\cup S_u)} \varphi^c_{U^\varepsilon_\delta}(x)\psi(\overline{u}(x))|\overline{u}^h(x)|\,\mathrm{d}(\mathrm{D}_j u^i) \\
&\quad - |\mathrm{D}^S_j u^i|(S_u\setminus S^\varepsilon_u)\,\widehat{C}(\psi) + \int_{U^\varepsilon_\delta\cap S^\varepsilon_u} \varphi_{U^\varepsilon_\delta}(x)\int_0^1 \psi(\overline{u}^\theta(x))|(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}(\mathrm{D}_j u^i) \\
&\quad - |\mathrm{D}^S_j u^i|(S_u\cap(\Omega\setminus U^\varepsilon_\delta))\,\widehat{C}(\psi)\,.
\end{aligned} \tag{2.22}$$

Therefore, taking in the right hand side of (2.22) the supremum over $\varphi_{U^\varepsilon_\delta} \in C_c(U^\varepsilon_\delta;[-1,1])$, $\varphi^c_{U^\varepsilon_\delta} \in C_c(\Omega\setminus\overline{U}^\varepsilon_\delta;[-1,1])$, and assuming $\psi\geq 0$, by (2.18)-(2.21), we deduce

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^m}|y^h|\,\mathrm{d}|(\mu_u)^i_j| \geq &-2\varepsilon\,\widehat{C}(\psi)+\int_{\Omega\setminus(U^\varepsilon_\delta\cup S_u)}|\psi(\overline{u}(x))\overline{u}^h(x)|\,\mathrm{d}|\mathrm{D}_j u^i|\\ &+\int_{U^\varepsilon_\delta\cap S^\varepsilon_u}\int_0^1|\psi(\overline{u}^\theta(x))(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}|\mathrm{D}_j u^i|\,,\end{aligned}\tag{2.23}$$

for all $\psi\in C_c(\mathbb{R}^m;[0,1])$. Letting $\delta\to 0$ in (2.23), for every $\psi\in C_c(\mathbb{R}^m;[0,1])$ we get

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^m}|y^h|\,\mathrm{d}|(\mu_u)^i_j| \geq &-2\varepsilon\,\widehat{C}(\psi)+\int_{\Omega\setminus S_u}|\psi(\overline{u}(x))\overline{u}^h(x)|\,\mathrm{d}|\mathrm{D}_j u^i|\\ &+\int_{S^\varepsilon_u}\int_0^1|\psi(\overline{u}^\theta(x))(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}|\mathrm{D}_j u^i|\end{aligned}$$

and, by the arbitrariness of $\varepsilon>0$, we conclude

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^m}|y^h|\,\mathrm{d}|(\mu_u)^i_j| \geq &\int_{\Omega\setminus S_u}|\psi(\overline{u}(x))\overline{u}^h(x)|\,\mathrm{d}|\mathrm{D}_j u^i|\\ &+\int_{S_u}\int_0^1|\psi(\overline{u}^\theta(x))(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}|\mathrm{D}_j u^i|\,,\end{aligned}\tag{2.24}$$

for every $\psi\in C_c(\mathbb{R}^m;[0,1])$. Let us choose a non-increasing compactly supported continuous function $\Psi:[0,\infty)\to[0,1]$ such that $\Psi(0)=1$, and, for every $M>0$ we set $\psi_M(\cdot)=\Psi(\frac{\cdot}{M})$. Taking $\psi(y):=\psi_M(|y^h|)$ in (2.24), and letting $M\to+\infty$, by Fatou Lemma we get

$$\int_{\Omega\times\mathbb{R}^m}|y^h|\,\mathrm{d}|(\mu_u)^i_j|\geq\int_{\Omega\setminus S_u}|\overline{u}^h(x)|\,\mathrm{d}|\mathrm{D}_j u^i|+\int_{S_u}\int_0^1|(\overline{u}^\theta)^h(x)|\,\mathrm{d}\theta\,\mathrm{d}|\mathrm{D}_j u^i|\,,$$

which, together with (2.15), proves (2.16). Finally, the first formula in (2.17) is straightforward, whereas the latter follows from the inquality $\int_0^1|(\overline{u}^\theta)^h|\,\mathrm{d}\theta\geq|\overline{u}^h|$. □

We will make use of property (2.17) in order to prove the following theorem.

Theorem 2.5. *Let $u\in BV(\Omega;\mathbb{R}^m)$, let $I=\{i,i'\}\subset\{1,\dots,m\}$, and $J=\{j,j'\}\subset\{1,\dots,n\}$ with $i\neq i'$ and $j\neq j'$. Assume that u satisfies hypothesis* (P) *for i,i', and j,j'. Then for all $i,h\in I$, $j\in J$ there exists a unique measure $(\nu_u)^{i,h}_j$ enjoying* (2.13) *and having finite total variation. As a consequence, the current $(T_u)_{I,J}\in\mathcal{D}_1(\Omega)$ given by* (2.8) *is well-defined, has finite mass, and*

$$\|\partial(T_u)_{I,J}\|_{\mathrm{flat},\Omega}\leq|(\lambda_u)^I_j|(\Omega)+|(\lambda_u)^I_{j'}|(\Omega)\,,$$

where the measures $(\lambda_u)^I_j$ and $(\lambda_u)^I_{j'}$ are defined in (2.7). *Moreover, if $\{u_k\}_{k\in\mathbb{N}}\subset BV(\Omega;\mathbb{R}^m)$ is a sequence of maps satisfying* (P) *and converging strictly to u in $BV(\Omega;\mathbb{R}^m)$ with*

$$\sup_{k\in\mathbb{N}}\Big\{\sum_{h=j,j'}\big(|y^i(\mu_{u_k})^{i'}_h|(\Omega\times\mathbb{R}^m)+|y^{i'}(\mu_{u_k})^i_h|(\Omega\times\mathbb{R}^m)\big)\Big\}<C\,,\tag{2.25}$$

for some $C>0$, then, as $k\to+\infty$, $(\nu_{u_k})^{i,h}_j\overset{*}{\rightharpoonup}(\nu_u)^{i,h}_j$ in $\mathcal{M}_b(\Omega)$ for all $i,h\in I$, $j\in J$, and $(T_{u_k})_{I,J}$ converges to $(T_u)_{I,J}$ in $\mathcal{D}_1(\Omega)$.

Proof. Fix $i\in I$, $h=i'$, and $j\in J$. We start by showing that the measures $(\nu_u)^{i,h}_j$ and $(\nu_u)^{i,h}_{j'}$ provided by (2.13) are well-defined and have finite total variation in $\Omega\times\mathbb{R}^m$.

Step 1: For every $N>0$ let $\eta_N:\mathbb{R}\to[-N,N]$ be the function defined by $\eta_N(t):=(-N)\vee t\wedge N$. We introduce the standard truncation of u at level N as $u_N:=(\eta_N(u^h))_{h=1,\dots,m}$. For all $N>0$, since $u^h_N\in L^\infty(\Omega)$ and in view of (2.13), the measure $(\nu^{i,i'}_j)_N$ defined by

$$(\nu^{i,i'}_j)_N:=(\nu_{u_N})^{i,i'}_j=[u^{i'}_N\mathrm{D}_j u^i_N]\,,$$

is well-defined as a Radon measure in $\mathcal{M}_b(\Omega)$. By (2.3) and (2.13), for all $\varphi \in C_c(\Omega)$, we can write

$$\int_\Omega \varphi(x)\, \mathrm{d}(\nu_j^{i,i'})_N = \int_{\Omega\setminus S_u} \varphi(x)\overline{u}_N^{i'}(x)\, \mathrm{d}(\mathrm{D}_j u_N^i) + \int_{S_u} \varphi(x)\overline{u}_N^{i'}(x)\, \mathrm{d}(\mathrm{D}_j u_N^i) \tag{2.26}$$

By [32, Theorem 7, pag. 486] η_N is differentiable at $u^i(x)$ for $\mathrm{D}_j u^i$-a.e. $x \in \Omega \setminus S_u$, and

$$\mathrm{D}_j u_N^i = \begin{cases} \mathrm{D}_j \eta_N(u^i)\mathrm{D}_j u^i & \text{in } \Omega\setminus S_u\,, \\ \big((u_N^i)^+ - (u_N^i)^-\big)\boldsymbol{\nu}^j \cdot \mathcal{H}^{n-1} & \text{in } S_u\,, \end{cases}$$

where

$$\mathrm{D}_j \eta_N(u^i) = \begin{cases} 1 & \text{in } \Omega_N := \{x \in \Omega\setminus S_u : |u^i(x)| < N\} \\ 0 & \text{in } R_N := \{x \in \Omega\setminus S_u : |u^i(x)| \geq N\}\,, \end{cases}$$

with

$$\Omega_N \nearrow (\Omega \setminus S_u) \qquad \text{as } N \to +\infty \text{ up to a } |\mathrm{D}_j u^i|\text{-negligible set.} \tag{2.27}$$

Hence, we can write

$$\mathrm{D}_j u_N^i = \begin{cases} \mathrm{D}_j u^i & \text{on } \Omega_N \\ 0 & \text{on } R_N. \end{cases}$$

Step 2: We claim that, as $N \to +\infty$, $(\nu_j^{i,i'})_N \overset{*}{\rightharpoonup} \nu_j^{i,i'}$ in $\mathcal{M}_b(\Omega)$ where $\nu_j^{i,i'}$ is uniquely determined by the formula

$$\begin{aligned} \int_\Omega \varphi(x)\, \mathrm{d}\nu_j^{i,i'} &= \int_{\Omega\times\mathbb{R}^m} \varphi(x) y^{i'}\, \mathrm{d}(\mu_u)_j^i \\ &= \int_{\Omega\setminus S_u} \varphi(x)\overline{u}^{i'}(x)\, \mathrm{d}(\mathrm{D}_j u^i) + \int_{S_u} \varphi(x)\overline{u}^{i'}(x)\, \mathrm{d}(\mathrm{D}_j u^i)\,, \end{aligned} \tag{2.28}$$

for all $\varphi \in C_c(\Omega)$. To prove this, we start by observing that for every $N > 0$, (2.26) implies

$$\begin{aligned} |(\nu_j^{i,i'})_N|(\Omega) &\leq \int_{\Omega\setminus S_u} |\overline{u}_N^{i'}|\, \mathrm{d}|\mathrm{D}_j u_N^i| + \int_{S_u} \frac{1}{2}|(u_N^{i'})^+ + (u_N^{i'})^-|\, \mathrm{d}|\mathrm{D}_j u_N^i| \\ &\leq \int_{\Omega\setminus S_u} |\overline{u}^{i'}|\, \mathrm{d}|\mathrm{D}_j u^i| + \int_{S_u} \frac{1}{2}|(u^{i'})^+ + (u^{i'})^-|\, \mathrm{d}|\mathrm{D}_j u^i| < \infty\,, \end{aligned} \tag{2.29}$$

where the last inequality is a consequence of property (P) (see (2.17)) and the last but one inequality follows from

$$|\overline{u}_N^h| \leq |\overline{u}^h|\,, \qquad |(u_N^{i'})^+ + (u_N^{i'})^-| \leq |(u^{i'})^+ + (u^{i'})^-| \qquad \text{for all } N > 0\,, h \in \{i, i'\}\,. \tag{2.30}$$

By (2.29), up to subsequences,

$$(\nu_j^{i,i'})_N \overset{*}{\rightharpoonup} \nu_j^{i,i'}\,, \qquad \text{in } \mathcal{M}_b(\Omega) \text{ as } N \to +\infty\,, \tag{2.31}$$

for some $\nu_j^{i,i'} \in \mathcal{M}_b(\Omega)$. Let $\varphi \in C_c(\Omega)$ be fixed; by (2.26), for every $N > 0$ it holds

$$\begin{aligned} \int_\Omega \varphi(x)\, \mathrm{d}(\nu_j^{i,i'})_N &= \int_{\Omega_N} \varphi(x)\overline{u}_N^{i'}(x)\, \mathrm{d}(\mathrm{D}_j u^i) \\ &\quad + \int_{S_u} \varphi(x)\frac{1}{2}((u_N^{i'})^+ + (u_N^{i'})^-)((u_N^i)^+ - (u_N^i)^-)\boldsymbol{\nu}^j\, \mathrm{d}\mathcal{H}^{n-1}\,; \end{aligned} \tag{2.32}$$

therefore, using once again (2.17), (2.30), (2.27) and the fact that

$$|(u_N^i)^+ - (u_N^i)^-| \leq |(u^i)^+ - (u^i)^-| \qquad \mathcal{H}^1\text{-a.e. on } S_u, \text{ for every } N > 0\,,$$

by the Dominated Convergence Theorem, we conclude that the right hand side of (2.32) tends to (as $N \to +\infty$)

$$\int_{\Omega\setminus S_u} \varphi(x)\overline{u}^{i'}(x)\, \mathrm{d}(\mathrm{D}_j u^i) + \int_{S_u} \varphi(x)\frac{1}{2}((u^{i'})^+ + (u^{i'})^-)\, \mathrm{d}(\mathrm{D}_j u^i)\,.$$

This fact, together with (2.31) and (2.32), implies that

$$\int_\Omega \varphi(x)\ \mathrm{d}\nu_j^{i,i'} = \int_{\Omega\setminus S_u} \varphi(x)\overline{u}^{i'}(x)\ \mathrm{d}(\mathrm{D}_j u^i) + \int_{S_u} \varphi(x)\overline{u}^{i'}(x)\ \mathrm{d}(\mathrm{D}_j u^i)\,,$$

for all $\varphi \in C_c(\Omega)$. To conclude the claim it remains to show that also the first equality in (2.28) is satisfied. By (2.3), for every $N > 0$, we have

$$\int_\Omega \varphi(x)\ \mathrm{d}(\nu_j^{i,i'})_N = \int_{\Omega\times\mathbb{R}^m} \varphi(x) y^{i'}\ \mathrm{d}(\mu_{u_N})_j^i \qquad \forall \varphi \in C_c(\Omega)\,, \tag{2.33}$$

and, by Theorem 1.1, $(\mu_{u_N})_j^i \overset{*}{\rightharpoonup} (\mu_u)_j^i$ in $\mathcal{M}_b(\Omega\times\mathbb{R}^m)$. It follows that $y^{i'}(\mu_{u_N})_j^i \rightharpoonup y^{i'}(\mu_u)_j^i$ as distributions in $\mathcal{D}(\Omega\times\mathbb{R}^m)$. Moreover, by (2.29) and (2.33), we deduce that the sequence $\{|y^{i'}(\mu_{u_N})_j^i|(\Omega)\}_N$ is uniformly bounded, and hence, up to subsequences, $y^{i'}(\mu_{u_N})_j^i \overset{*}{\rightharpoonup} \tau$ in $\mathcal{M}_b(\Omega\times\mathbb{R}^m)$ for some $\tau \in \mathcal{M}_b(\Omega\times\mathbb{R}^m)$. Therefore, $\tau = y^{i'}(\mu_u)_j^i$, the whole sequence $\{y^{i'}(\mu_{u_N})_j^i\}_N$ converges to $y^{i'}(\mu_u)_j^i$ (by the Urysohn property), and

$$\int_\Omega \varphi(x)\ \mathrm{d}\nu_j^{i,i'} = \int_{\Omega\times\mathbb{R}^m} \varphi(x) y^{i'}\ \mathrm{d}(\mu_u)_j^i \qquad \forall \varphi \in C_c(\Omega)\,,$$

thus concluding the proof of (2.28) .

Step 3: It remains to show the last claim in the statement. By (2.25), up to a subsequence, $y^{i'}(\mu_{u_k})_j^i \overset{*}{\rightharpoonup} \tau$ for some $\tau \in \mathcal{M}_b(\Omega\times\mathbb{R}^m)$. Arguing as in Step 2, we get that $\tau = y^{i'}(\mu_u)_j^i$ and that the whole sequence $y^{i'}(\mu_{u_k})_j^i$ converges to τ. As a consequence, we deduce that $(\nu_{u_k})_j^{i,i'} \overset{*}{\rightharpoonup} (\nu_u)_j^{i,i'}$, explicitly given by

$$\int_{\Omega\times\mathbb{R}^m} \varphi(x) y^{i'}\ \mathrm{d}(\mu_u)_j^i = \int_{\Omega\setminus S_u} \varphi(x)\overline{u}^{i'}(x)\ \mathrm{d}(\mathrm{D}_j u^i) + \int_{S_u} \varphi(x)\overline{u}^{i'}(x)\ \mathrm{d}(\mathrm{D}_j u^i) = \int_\Omega \varphi(x)\ \mathrm{d}(\nu_u)_j^{i,i'}\,,$$

for all $\varphi \in C_c(\Omega)$. By the very definitions of λ_j^I and $\lambda_{j'}^I$ in (2.7) and of T_u in (2.8), the claim easily follows. □

We can then give the following definition, which extends Definition 2.3.

Definition 2.6. *Let $I = \{i, i'\} \subset \{1,\dots,n\}$, $J = \{j, j'\} \subset \{1,\dots,m\}$ with $i \neq i'$ and $j \neq j'$, and let $u \in BV(\Omega;\mathbb{R}^m)$ be such that* (P) *holds for the indeces in I and J. Then we define $M_I^J(Du) \in \mathcal{D}'(\Omega)$ the 2×2-minor of Du with rows in I and columns in J as the distribution in* (2.12).

Remark 2.7. As a byproduct of the proof of Theorem 2.5, we deduce that, if $u \in BV(\Omega;\mathbb{R}^m)$ satisfies hypothesis (P), then the measures $y^i(\mu_u)_j^{i'}$ and $(\nu_u)_j^{i',i}$ take the form

$$\begin{aligned}\int_{\Omega\times\mathbb{R}^m} \phi(x,y) y^i\ \mathrm{d}(\mu_u)_j^{i'} = &\int_{\Omega\setminus S_u} \phi(x,\overline{u}(x))\overline{u}^i(x)\ \mathrm{d}(\mathrm{D}_j u^{i'}) \\ &+ \int_{S_u}\int_0^1 \phi(x,\overline{u}^\theta)(\overline{u}^\theta)^i(x)\ \mathrm{d}\theta\ \mathrm{d}(\mathrm{D}_j u^{i'}),\end{aligned}$$

for all $\phi \in C_c(\Omega\times\mathbb{R}^m)$, and

$$\int_{\Omega\times\mathbb{R}^m} \varphi(x) y^i\ \mathrm{d}(\mu_u)_j^{i'} = \int_{\Omega\setminus S_u} \varphi(x)\overline{u}^i(x)\ \mathrm{d}(\mathrm{D}_j u^{i'}) + \int_{S_u} \varphi(x)\overline{u}^i(x)\ \mathrm{d}(\mathrm{D}_j u^{i'}),$$

for all $\varphi \in C_c(\Omega)$. If $u \in L^p(\Omega;\mathbb{R}^m) \cap W^{1,q}(\Omega;\mathbb{R}^m)$, with $\frac{1}{p} + \frac{1}{q} = 1$ (including the case $p = \infty$, $q = 1$), property (P) is readily satisfied, and then the previous formulas read

$$\int_{\Omega\times\mathbb{R}^m} \phi(x,y) y^i\ \mathrm{d}(\mu_u)_j^{i'} = \int_\Omega \phi(x,u(x)) u^i(x) \frac{\partial u^{i'}}{\partial x_j}(x)\ \mathrm{d}x,$$

for all $\phi \in C_c(\Omega\times\mathbb{R}^m)$, and

$$\int_\Omega \varphi(x)\ \mathrm{d}(\nu_u)_j^{i',i} = \int_{\Omega\times\mathbb{R}^m} \varphi(x) y^i\ \mathrm{d}(\mu_u)_j^{i'} = \int_\Omega \varphi(x) u^i(x) \frac{\partial u^{i'}}{\partial x_j}(x)\ \mathrm{d}x,$$

for all $\varphi \in C_c(\Omega)$. In particular, the measure $(\nu_u)_j^{i',i}$ belongs to $L^1(\Omega)$ and coincides almost everywhere with the function $u^i \frac{\partial u^{i'}}{\partial x_j}$.

2.2. **The 2×2 dimensional case.** Let us now restrict ourselves to the two dimensional case $n = 2$, namely $\Omega \subset \mathbb{R}^2$ and let also $m = 2$. In this case the only possible choice for I and J is $I = J = \{1, 2\}$, so assume $u \in BV(\Omega; \mathbb{R}^2)$ satisfies property (P) for $I = J = \{1, 2\}$. The current $(T_u)_{I,J}$ will be simply denoted by T_u and we denote the distribution $M_I^J(\mathrm{D}u)$ by Ju. We also set $(\lambda_u)_1^{1,2} =: \lambda_1$, and $(\lambda_u)_2^{1,2} =: \lambda_2$. Explicitly,

$$\begin{aligned}(T_u)(\omega) &= \int_\Omega \omega_2 \, \mathrm{d}\lambda_1 - \int_\Omega \omega_1 \, \mathrm{d}\lambda_2 \\ &= \frac{1}{2}\int_\Omega \omega_2 \, \mathrm{d}([u^1 \mathrm{D}_1 u^2] - [u^2 \mathrm{D}_1 u^1]) - \frac{1}{2}\int_\Omega \omega_1 \, \mathrm{d}([u^1 \mathrm{D}_2 u^2] - [u^2 \mathrm{D}_2 u^1]), \end{aligned} \tag{2.34}$$

for all $\omega = \omega_1 \, \mathrm{d}x^1 + \omega_2 \, \mathrm{d}x^2 \in \mathcal{D}^1(\Omega)$, and for all $\varphi \in \mathcal{D}^0(\Omega)$,

$$\begin{aligned}\partial(T_u)(\varphi) &= \langle Ju, \varphi \rangle_\Omega \\ &= \frac{1}{2}\int_\Omega \frac{\partial \varphi}{\partial x_2} \, \mathrm{d}([u^1 \mathrm{D}_1 u^2] - [u^2 \mathrm{D}_1 u^1]) - \frac{1}{2}\int_\Omega \frac{\partial \varphi}{\partial x_1} \, \mathrm{d}([u^1 \mathrm{D}_2 u^2] - [u^2 \mathrm{D}_2 u^1]). \end{aligned} \tag{2.35}$$

Assume now that $u \in L^p(\Omega; \mathbb{R}^m) \cap W^{1,q}(\Omega; \mathbb{R}^m)$, with $\frac{1}{p} + \frac{1}{q} = 1$; in view of Remark 2.7, the last expression reads

$$\langle Ju, \varphi \rangle_\Omega = \frac{1}{2}\int_\Omega \frac{\partial \varphi}{\partial x_2}\Big(u^1 \frac{\partial u^2}{\partial x_1} - u^2 \frac{\partial u^1}{\partial x_1}\Big) \, \mathrm{d}x - \frac{1}{2}\int_\Omega \frac{\partial \varphi}{\partial x_1}\Big(u^1 \frac{\partial u^2}{\partial x_2} - u^2 \frac{\partial u^1}{\partial x_2}\Big) \, \mathrm{d}x, \tag{2.36}$$

which coincides with the definition of the distributional determinant of ∇u. In particular, the distribution Ju extends the definition of distributional Jacobian.

Remark 2.8. In [44] the existence and well-posedness of $Ju = \partial(T_u)$ is obtained under the additional condition that the function u be approximated (strictly in BV) by a sequence of maps $v_k \in C^1(\Omega; \mathbb{R}^2)$ satisfying the following condition: There exists a constant $C > 0$, independent of k, such that

$$\int_\Omega |Jv_k| \, \mathrm{d}x \le C \qquad \forall k \in \mathbb{N}.$$

This is equivalent to require that the Jacobian total variation functional relaxed w.r.t. the strict topology of BV is finite. In turn, this is equivalent to require that the relaxed area functional of v_k is finite, and then that the mass of the Cartesian currents $\mathcal{G}_{v_k}$ with underlying maps v_k have equibounded masses.

Under this assumption, it turns out that also $Ju = \partial(T_u)$ is a Radon measure with finite total variation. In particular, whenever a sequence $\{u_k\}_{k\in\mathbb{N}} \subset BV(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$ satisfies

$$\begin{aligned} &u_k \to u \qquad \text{strictly in } BV(\Omega; \mathbb{R}^2), \\ &|Ju_k|(\Omega) \le C \qquad \forall k \in \mathbb{N}, \end{aligned} \tag{2.37}$$

for some constant $C > 0$ independent of k, then

$$Ju_k \to Ju \qquad \text{weakly star as measures.} \tag{2.38}$$

We will make use of (2.38) in the proof of the upper bound of Theorem 3.1 using the following observation.

Remark 2.9. Assume that the sequence $\{u_k\}_{k\in\mathbb{N}} \in BV(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$ satisfies the two conditions in (2.37) and in addition $\mathrm{supp}(Ju_k) \subset K \subset \Omega$ for some compact set K, then (2.38) ensures that

$$Ju_k \to Ju \qquad \text{in the flat topology.} \tag{2.39}$$

3. A new approach to topological singularities

Let Ω be a bounded and open subset of $\mathbb{R}^2$ with Lipschitz continuous boundary and let $\Omega' \subset\subset \Omega$ be an open set. We introduce

$$\mathcal{AD}(\Omega,\Omega') := \{u \in SBV^2(\Omega;\mathbb{S}^1) \,:\, \overline{S}_u \subset \overline{\Omega'}\}, \tag{3.1}$$

where S_u denotes the jump set of u. Here and below, for every bounded and open set $A \subset \mathbb{R}^2$ with Lipschitz continuous boundary, $SBV^2(A;\mathbb{S}^1)$ denotes the set of the functions $u \in BV(A;\mathbb{R}^2)$ such that $|u| = 1$ a.e. in A, $\mathrm{D}^c u \equiv 0$, and $\nabla u \in L^2(A;\mathbb{R}^{2\times 2})$, where ∇u is the density of $\mathrm{D}^a u$, i.e., $\mathrm{D}^a u := \nabla u \mathcal{L}^2$. For every $\varepsilon > 0$, let $\mathcal{F}_\varepsilon : SBV^2(\Omega;\mathbb{S}^1) \to [0,\infty]$ be the functional defined by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x + \frac{1}{\varepsilon}\mathcal{H}^1(\overline{S}_u) & \text{if } u \in \mathcal{AD}(\Omega,\Omega') \\ +\infty & \text{elsewhere in } SBV^2(\Omega;\mathbb{S}^1). \end{cases} \tag{3.2}$$

In what follows, we will adopt also localized versions of the functional $\mathcal{F}_\varepsilon$; more precisely, for any $u \in \mathcal{AD}(\Omega,\Omega')$ and for any open set $\Omega' \subset\subset A \subset\subset \Omega$, we will denote by $\mathcal{F}_\varepsilon(u;A)$ the functional in (3.2) with Ω replaced by A.

Denoting by $\mathcal{M}_b(\Omega)$ the class of Radon measures with finite total variation in Ω, we set

$$X(\Omega) := \Big\{\mu = \sum_{i=1}^{I} z^i \delta_{x^i} \in \mathcal{M}_b(\Omega) \,:\, I \in \mathbb{N},\, z^i \in \mathbb{Z}\setminus\{0\},\, x^i \in \Omega\Big\}.$$

For every $u \in \mathcal{AD}(\Omega,\Omega')$ we consider the current $T_u \in \mathcal{D}_1(\Omega)$ introduced in (2.8) and in (2.34) for the 2-dimensional case, and we denote by $Ju := \partial T_u$ its boundary (whose expression is given in (2.35)), namely the Jacobian determinant of u. We recall that in general Ju is not a measure, but a mere distribution with finite flat norm (see Theorem 2.2). However, since $u \in H^1(\Omega\setminus\overline{\Omega}';\mathbb{S}^1)$, it follows that

$$\operatorname{supp} Ju \subseteq \overline{\Omega}' \qquad \text{for every } u \in \mathcal{AD}(\Omega,\Omega'). \tag{3.3}$$

Indeed, let $\varphi \in C^\infty_{\mathrm{c}}(\Omega\setminus\overline{\Omega}')$, and write

$$\begin{aligned} \langle Ju,\varphi\rangle_\Omega &= \frac{1}{2}\int_\Omega \frac{\partial\varphi}{\partial x_2}\,\mathrm{d}([u^1\mathrm{D}_1u^2] - [u^2\mathrm{D}_1u^1]) - \frac{1}{2}\int_\Omega \frac{\partial\varphi}{\partial x_1}\,\mathrm{d}([u^1\mathrm{D}_2u^2] - [u^2\mathrm{D}_2u^1]) \\ &= \frac{1}{2}\int_{\Omega\setminus\overline{\Omega}'} \frac{\partial\varphi}{\partial x_2}\Big(u^1\frac{\partial u^2}{\partial x_1} - u^2\frac{\partial u^1}{\partial x_1}\Big)\,\mathrm{d}x - \frac{1}{2}\int_{\Omega\setminus\overline{\Omega}'} \frac{\partial\varphi}{\partial x_1}\Big(u^1\frac{\partial u^2}{\partial x_2} - u^2\frac{\partial u^1}{\partial x_2}\Big)\,\mathrm{d}x \\ &= \langle Ju,\varphi\rangle_{\Omega\setminus\overline{\Omega}'} = 0, \end{aligned}$$

where the last equality follows since $u \in H^1(\Omega\setminus\overline{\Omega}';\mathbb{S}^1)$ has null distributional Jacobian determinant in $\Omega\setminus\overline{\Omega}'$.

Our main result is the following.

Theorem 3.1. *The following Γ-convergence result holds true.*

(i) *(Compactness) Let $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ be such that*

$$\sup_{\varepsilon>0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \le C, \tag{3.4}$$

for some $C > 0$. Then there exists $\mu \in X(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$ such that, up to a subsequence, $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$).

(ii) *(Γ-liminf inequality) For every $\mu \in X(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$ and for every $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ such that $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$), it holds*

$$\pi|\mu|(\Omega) \le \liminf_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}. \tag{3.5}$$

(iii) *(Γ-limsup inequality) For every $\mu \in X(\Omega)$ with $\operatorname{supp}\mu \subseteq \overline{\Omega}'$, there exists $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ with $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$), such that*

$$\pi|\mu|(\Omega) \geq \limsup_{\varepsilon\to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,. \tag{3.6}$$

In order to prove Theorem 3.1, we will make use of Theorem 3.4 below which is proven in [8, Theorem 3.2]) (see also [25, Theorem 2.4]). To this purpose, we introduce some notation.

Let $V \subset \mathbb{R}^2$ be a bounded and open set with Lipschitz continuous boundary. For every finite family of pairwise (essentially[3]) disjoint open balls $\mathscr{B} := \{B^n\}_{n=1,\dots,N}$ (with $N \in \mathbb{N}$) we set $V(\mathscr{B}) := V \setminus \bigcup_{n=1}^N \overline{B}^n$ and we denote by $\mathcal{R}ad(\mathscr{B})$ the sum of the radii of the balls B^n, namely

$$\mathcal{R}ad(\mathscr{B}) := \sum_{n=1}^N r(B^n)\,,$$

where $r(B)$ denotes the radius of the ball B. Moreover, for every $\mu \in X(V)$ with $\mu \neq 0$ of the form

$$\mu := \sum_{n=1}^N z^n \delta_{x(B^n)} \qquad \text{with } z^n \in \mathbb{Z}\setminus\{0\}\,, \tag{3.7}$$

we set

$$\mathscr{A}(\mathscr{B},\mu,V) := \{u \in H^1(V(\mathscr{B});\mathbb{S}^1) \,:\, \deg(u,\partial B^n) = z^n \text{ for every } n = 1,\dots,N\}\,.$$

Here and below, $x(B)$ denotes the center of the ball B. For every $\mathscr{B}$ and μ as above, we set

$$F(\mathscr{B},\mu,V) := \min_{u\in\mathscr{A}(\mathscr{B},\mu,V)} \int_{V(\mathscr{B})} |\nabla u|^2 \,\mathrm{d}x\,. \tag{3.8}$$

Definition 3.2 (Merging procedure)**.** Given a finite family $\mathscr{B} = \{B_{r^i}(x^i)\}_{i=1,\dots,I}$ ($I \in \mathbb{N}$) of balls in $\mathbb{R}^2$, we define a new family $\widehat{\mathscr{B}}$ as follows. If the closures of two balls in $\mathscr{B}$ are not disjoint, then we replace the two balls with a unique ball which contains both of them and with radius less or equal to the sum of the radii of the original balls. After this, we repeat this replacement recursively, until as all the balls in the family are mutually essentially disjoint. The final family is $\widehat{\mathscr{B}}$. The procedure of passing from $\mathscr{B}$ to $\widehat{\mathscr{B}}$ is called merging procedure applied to $\mathscr{B}$. Notice that a merging procedure does not increase the sum of all the radii of the balls in the family.

The following result is proven in [25, Proposition 2.2].

Proposition 3.3. *Let $\mathscr{B}$ be a finite family of pairwise essentially disjoint balls in $\mathbb{R}^2$, and let $\mu \in X(V)$ be of the form (3.7). Then, there exists a one-parameter family of open balls $\mathscr{B}(t)$ with $t \geq 0$ such that, setting $U(t) := \bigcup_{B\in\mathscr{B}(t)} B$, the following properties hold true:*

(1) *$\mathscr{B}(0) = \mathscr{B}$;*
(2) *$U(t_1) \subset U(t_2)$ for any $0 \leq t_1 < t_2$;*
(3) *the balls in $\mathscr{B}(t)$ are pairwise (essentially) disjoint for every $t > 0$;*
(4) *for any $0 \leq t_1 < t_2$ and for any open set $U \subseteq \mathbb{R}^2$,*

$$F(\mathscr{B},\mu,U\cap(U(t_2)\setminus U(t_1))) \geq \pi \sum_{\substack{B\in\mathscr{B}(t_2)\\ B\subseteq U}} |\mu(B)| \log\frac{1+t_2}{1+t_1}\,;$$

(5) *for every $t > 0$: $\displaystyle\sum_{B\in\mathscr{B}(t)} r(B) \leq (1+t)\sum_{B\in\mathscr{B}} r(B)$, where $r(B)$ denotes the radius of B.*

For every $\mathscr{B}$ and μ as above, we set $\mathscr{C}(1) := \{B \in \mathscr{B}(1) \,:\, \overline{B} \subset V\}$ and we define

$$\widetilde{\mu} := \sum_{B\in\mathscr{C}(1)} \mu(B)\delta_{x(B)}\,. \tag{3.9}$$

We can now state the crucial result which will be the starting point of the proof of Theorem 3.1.

[3]That is, whose closures are mutually disjoint.

Theorem 3.4. *Let V be a bounded open set with Lipschitz boundary. For every $\varepsilon > 0$ let $\mathscr{B}_\varepsilon := \{B^n_\varepsilon\}_{n=1,\dots,N_\varepsilon}$ (with $N_\varepsilon \in \mathbb{N}$) be a (finite) family of pairwise (essentially) disjoint open balls with $\mathcal{R}ad(\mathscr{B}_\varepsilon) \to 0$ as $\varepsilon \to 0$ and let $\mu_\varepsilon := \sum_{n=1}^{N_\varepsilon} z^n_\varepsilon \delta_{x(B^n_\varepsilon)}$ with $z^n_\varepsilon \in \mathbb{Z}$ for every $n = 1, \dots, N_\varepsilon$. Assume that*

$$\sup_{\varepsilon>0} \frac{F(\mathscr{B}_\varepsilon, \mu_\varepsilon, V)}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|} \le C\,,$$

for some constant $C > 0$ independent of ε. Then, the following facts hold true.

(i) *If $\widetilde{\mu}_\varepsilon$ are the measures defined in (3.9) starting from the family $\mathscr{C}_\varepsilon(1) = \{B \in \mathscr{B}_\varepsilon(1) : \overline{B} \subset V\}$, then $|\widetilde{\mu}_\varepsilon|(V) \le C|\log\varepsilon|$ for all $\varepsilon > 0$, and, up to a subsequence, $\widetilde{\mu}_\varepsilon \llcorner V \overset{\text{flat}}{\to} \mu$ (as $\varepsilon \to 0$) for some $\mu \in X(V)$.*

(ii) $\pi|\mu|(V) \le \liminf_{\varepsilon\to 0} \frac{F(\mathscr{B}_\varepsilon,\mu_\varepsilon,V)}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|}$.

Proof. See [8, Theorem 3.2]. □

We are now in a position to prove Theorem 3.1. The more involved part is the compactness property. For this reason we have splitted the argument in seven steps.

Proof of Theorem 3.1. Proof of (i). We start by covering with balls the jump set of u_ε and define the measure $\widetilde{\mu}_\varepsilon$ in Step 1. In Steps 2 and 3 we suitably modify these balls in order to show - in Steps 4, 5, 6 and 7 - that the obtained measures $\pi\widetilde{\mu}_\varepsilon$ are close to Ju_ε with respect to the flat distance. Throughout the proof, the symbol C denotes an absolute positive constant independent of the parameters, which might change from line to line.

Step 1: Construction of the starting family of balls. By the energy bound (3.4), we have that

$$\mathcal{H}^1(\overline{S}_{u_\varepsilon}) \le C\varepsilon|\log\varepsilon|\,, \tag{3.10}$$

for all $\varepsilon > 0$. By the very definition of Hausdorff measure, since $\overline{S}_{u_\varepsilon}$ is compact, there exists a finite family of open balls $\mathscr{B}_\varepsilon$ (in $\mathbb{R}^2$) such that $\overline{S}_{u_\varepsilon} \subset \bigcup_{B\in\mathscr{B}_\varepsilon} B$ and

$$\mathcal{R}ad(\mathscr{B}_\varepsilon) \le C\varepsilon|\log\varepsilon|\,, \tag{3.11}$$

for some $C > 0$. The number of balls in $\mathscr{B}_\varepsilon$ depends on ε. Moreover we notice that

$$u_\varepsilon \in H^1(\Omega(\mathscr{B}_\varepsilon); \mathbb{S}^1)\,, \tag{3.12}$$

where we recall $\Omega(\mathscr{B}_\varepsilon) := \Omega \setminus (\cup_{B\in\mathscr{B}_\varepsilon}\overline{B})$. By (3.11) and recalling that $\overline{S}_\varepsilon \subset \overline{\Omega'}$, we can assume, for ε small enough, that all the balls in $\mathscr{B}_\varepsilon$ are contained in Ω. Up to applying a merging procedure (as in Definition 3.2) for the balls in $\mathscr{B}_\varepsilon$, we can assume without loss of generality that these balls are mutually (essentially) disjoint, and still satisfy (3.11). For every $\varepsilon > 0$ we set

$$\mu_\varepsilon := \sum_{B\in\mathscr{B}_\varepsilon} \deg(u_\varepsilon, \partial B)\delta_{x(B)}\,.$$

By (3.12), (3.11) and (3.4), for ε small enough it holds

$$F(\mathscr{B}_\varepsilon, \mu_\varepsilon, \Omega) \le \mathcal{F}_\varepsilon(u_\varepsilon) \le C|\log\varepsilon| \le C|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|\,, \tag{3.13}$$

where F is defined in (3.8). Using Proposition 3.3, we set

$$\mathscr{C}_\varepsilon(1) := \{B \in \mathscr{B}_\varepsilon(1) : \overline{B} \subset \Omega\} \tag{3.14}$$

and, according to (3.9), we define

$$\widetilde{\mu}_\varepsilon := \sum_{B\in\mathscr{C}_\varepsilon(1)} \deg(u_\varepsilon, \partial B)\delta_{x(B)}\,. \tag{3.15}$$

Notice that, by definition of $\mathcal{AD}(\Omega, \Omega')$, actually $\mathscr{C}_\varepsilon(1) \equiv \mathscr{B}_\varepsilon(1)$ for ε small enough (i.e., all the closures of the balls in $\mathscr{B}_\varepsilon(1)$ are contained in Ω). Therefore, by (3.13), we can apply Theorem 3.4(i) with $V = \Omega$, deducing that

$$|\widetilde{\mu}_\varepsilon|(\Omega) \le C|\log\varepsilon|, \tag{3.16}$$

and, up to a subsequence, $\widetilde{\mu}_\varepsilon \overset{\text{flat}}{\to} \mu$ (as $\varepsilon \to 0$) for some $\mu \in X(\Omega)$. In order to conclude the proof of Theorem 3.1(i) it is enough to show that

$$\|Ju_\varepsilon - \pi\widetilde{\mu}_\varepsilon\|_{\text{flat},\Omega} \to 0 \qquad \text{as } \varepsilon \to 0\,. \tag{3.17}$$

Indeed, on the one hand this shows that $Ju_\epsilon \overset{\text{flat}}{\to} \pi\mu$ (as $\varepsilon \to 0$); on the other hand, since $\operatorname{supp} Ju_\epsilon \subseteq \overline{\Omega}'$ by (3.3), we can also conclude that $\operatorname{supp}\mu \subseteq \overline{\Omega}'$.

Step 2: Fattening of the non-zero degree clusters. Let $\mathscr{C}_\varepsilon(1)$ be the family of balls in (3.14). Since $\{u_\varepsilon\}_\varepsilon \subset \mathcal{AD}(\Omega,\Omega')$ and in view of (3.11), for some open set Ω'' with $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, we have that, for ε small enough, $B \subset \Omega''$ for every $B \in \mathscr{C}_\varepsilon(1)$. We set

$$\mathscr{C}_\varepsilon^{\neq 0}(1) := \{B \in \mathscr{C}_\varepsilon(1) \,:\, \deg(u_\varepsilon, \partial B) \neq 0\}\,,$$

and

$$\mathscr{C}_\varepsilon^{0}(1) := \{B \in \mathscr{C}_\varepsilon(1) \,:\, \deg(u_\varepsilon, \partial B) = 0\}\,.$$

For every $B \in \mathscr{C}_\varepsilon^{\neq 0}(1)$ we replace the ball B by $B^{\text{mod}} := B_{\varepsilon \vee r(B)}(x(B))$ (hence increasing the radius up to ε if it is smaller). Again, since the balls in $\widetilde{\mathscr{C}_\varepsilon} := \mathscr{C}_\varepsilon^0(1) \cup \{B^{\text{mod}} \,:\, B \in \mathscr{C}_\varepsilon^{\neq 0}\}$ are not necessarily mutually disjoint, we pass to a merging procedure described in Definition 3.2 in order to obtain the new family $\mathscr{C}_\varepsilon$ of pairwise (essentially) disjoint balls.

By (3.11) and (3.16), we still have

$$\mathcal{R}ad(\mathscr{C}_\varepsilon) \leq C\varepsilon|\log\varepsilon|\,. \tag{3.18}$$

Setting

$$\mathscr{C}_\varepsilon^{\neq 0} := \{B \in \mathscr{C}_\varepsilon \,:\, \deg(u_\varepsilon, \partial B) \neq 0\}\,,$$

by construction and by (3.16), we have that

$$\sharp\mathscr{C}_\varepsilon^{\neq 0} \leq \sharp\mathscr{C}_\varepsilon^{\neq 0}(1) \leq C|\log\varepsilon|\,.$$

Once again, due to (3.18) and the fact that the original balls in $\mathscr{C}_\varepsilon(1)$ have centers in Ω', we can assume that the balls in $\mathscr{C}_\varepsilon$ are all contained in Ω''.

Step 3: Construction of suitable clusters. In this step we construct a family $\mathscr{I}_\varepsilon$ of pairwise disjoint balls with

$$\sum_{B\in\mathscr{I}_\varepsilon} r(B) \leq 5\mathcal{R}ad(\mathscr{C}_\varepsilon) \leq \overline{C}\varepsilon|\log\varepsilon|, \tag{3.19}$$

(with $\overline{C} > 0$ a fixed constant) and

$$\bigcup_{B\in\mathscr{C}_\varepsilon} B \subset \bigcup_{B\in\mathscr{I}_\varepsilon} B\,, \tag{3.20}$$

such that for any $B \in \mathscr{I}_\varepsilon$ at least one of the following conditions is satisfied

$$\text{(i) } r(B) \geq \frac{\varepsilon}{2} \qquad\qquad \text{(ii) } \int_{\partial B} |\nabla u_\varepsilon|\,\mathrm{d}\mathcal{H}^1 \leq C|\log\varepsilon|^{\frac{1}{2}}\,, \tag{3.21}$$

for some universal constant $C > 0$ (independent of ε).

We start by classifying the balls in $\mathscr{C}_\varepsilon$ by setting

$$\mathscr{C}_\varepsilon^{<} := \left\{B \in \mathscr{C}_\varepsilon \,:\, r(B) < \frac{\varepsilon}{2}\right\} \quad \text{and} \quad \mathscr{C}_\varepsilon^{>} := \left\{B \in \mathscr{C}_\varepsilon \,:\, r(B) \geq \frac{\varepsilon}{2}\right\}.$$

Notice that, if $\mathscr{C}_\varepsilon^{<} = \varnothing$, then, we can set $\mathscr{I}_\varepsilon := \mathscr{C}_\varepsilon$ and, by (3.18), the claim immediately follows.

If this is not the case, then we adopt the iterative procedure described below. For every $k = 0, 1, \ldots$, we will construct a pair of family of balls $(\mathscr{C}_\varepsilon(k); \mathscr{I}_\varepsilon(k))$ such that the balls in

$$\mathscr{S}_\varepsilon(k) := \mathscr{C}_\varepsilon(k) \cup \mathscr{I}_\varepsilon(k)$$

are (essentially) pairwise disjoint. We classify the balls in $\mathscr{C}_\varepsilon(k)$ into two subclasses

$$\mathscr{C}_\varepsilon^{<}(k) := \left\{B \in \mathscr{C}_\varepsilon(k) \,:\, r(B) < \frac{\varepsilon}{2}\right\} \quad \text{and} \quad \mathscr{C}_\varepsilon^{>}(k) := \left\{B \in \mathscr{C}_\varepsilon(k) \,:\, r(B) \geq \frac{\varepsilon}{2}\right\},$$

and the balls in $\mathscr{I}_\varepsilon(k)$ into two further subclasses

$$\mathscr{I}_\varepsilon^<(k) := \left\{B \in \mathscr{I}_\varepsilon(k) \,:\, r(B) < \frac{\varepsilon}{2}\right\} \quad \text{and} \quad \mathscr{I}_\varepsilon^>(k) := \left\{B \in \mathscr{I}_\varepsilon(k) \,:\, r(B) \geq \frac{\varepsilon}{2}\right\}.$$

We initialize such families by setting $\mathscr{C}_\varepsilon(0) := \mathscr{C}_\varepsilon$, $\mathscr{C}_\varepsilon^<(0) := \mathscr{C}_\varepsilon^<$, $\mathscr{C}_\varepsilon^>(0) := \mathscr{C}_\varepsilon^>$, and $\mathscr{I}_\varepsilon(0) = \mathscr{I}_\varepsilon^<(0) = \mathscr{I}_\varepsilon^>(0) = \varnothing$. We now define recursively $(\mathscr{C}_\varepsilon(k); \mathscr{I}_\varepsilon(k))$ so that for a finite number $K_\varepsilon \in \mathbb{N}$ we set

$$\mathscr{I}_\varepsilon := \mathscr{I}_\varepsilon(K_\varepsilon).$$

We remark that (3.20) now reads

$$\bigcup_{B\in\mathscr{C}_\varepsilon(0)} B \subset \bigcup_{B\in\mathscr{I}_\varepsilon(K_\varepsilon)} B\,.$$

At the end of the procedure we will also have

$$\mathscr{C}_\varepsilon(K_\varepsilon) = \varnothing.$$

The pair $(\mathscr{C}_\varepsilon(k); \mathscr{I}_\varepsilon(k))$ will be such that $\mathscr{C}_\varepsilon(k) \supset \mathscr{C}_\varepsilon(k+1)$ for every $k = 0, 1, \dots, K_\varepsilon - 1$; moreover, for all $k = 0, 1, \dots, K_\varepsilon$,

$$\mathscr{C}_\varepsilon(k) \cup \mathscr{I}_\varepsilon(k) \text{ is made by pairwise (essentially) disjoint balls,} \tag{3.22}$$

$$\mathscr{C}_\varepsilon(k) \subset \mathscr{C}_\varepsilon(0)\,, \tag{3.23}$$

$$\mathcal{R}ad(\mathscr{S}_\varepsilon(k)) \leq \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B}\in\mathscr{I}_\varepsilon(k)} \sum_{\substack{B\in\mathscr{C}_\varepsilon(0)\\ B\subseteq\widetilde{B}}} r(B)\,, \tag{3.24}$$

and for any $B \in \mathscr{I}_\varepsilon(k)$

$$\text{either } r(B) \geq \frac{\varepsilon}{2} \quad \text{or} \quad \int_{\partial B} |\nabla u_\varepsilon| \,\mathrm{d}\mathcal{H}^1 \leq C|\log\varepsilon|^{\frac{1}{2}}\,. \tag{3.25}$$

Eventually, we observe that (3.25) is equivalent to require

$$\int_{\partial B} |\nabla u_\varepsilon| \,\mathrm{d}\mathcal{H}^1 \leq C|\log\varepsilon|^{\frac{1}{2}} \quad \text{for any } B \in \mathscr{I}_\varepsilon^<(k)\,.$$

The pair $(\mathscr{C}_\varepsilon(0); \mathscr{I}_\varepsilon(0))$ satisfies (3.22), (3.23), (3.24) and (3.25). Let $k = 0, 1, \dots$ and assume that $(\mathscr{C}_\varepsilon(k); \mathscr{I}_\varepsilon(k))$ satisfies (3.22), (3.23), (3.24) and (3.25).

If $\mathscr{C}_\varepsilon^<(k) = \varnothing$, then setting $k =: K_\varepsilon$ we conclude the procedure according to formula (3.36) below. Otherwise, we choose $B_r(x) \in \mathscr{C}_\varepsilon^<(k)$. We set

$$\begin{aligned}
\widehat{\mathscr{C}_\varepsilon^<}(k) &:= \mathscr{C}_\varepsilon^<(k) \setminus \{B_r(x)\}\,; & \widehat{\mathscr{C}_\varepsilon^>}(k) &:= \mathscr{C}_\varepsilon^>(k)\,;\\
\widehat{\mathscr{I}_\varepsilon^<}(k) &:= \mathscr{I}_\varepsilon^<(k)\,; & \widehat{\mathscr{I}_\varepsilon^>}(k) &:= \mathscr{I}_\varepsilon^>(k)\,;\\
\widehat{\mathscr{C}_\varepsilon}(k) &:= \widehat{\mathscr{C}_\varepsilon^<}(k) \cup \widehat{\mathscr{C}_\varepsilon^>}(k)\,; & \widehat{\mathscr{I}_\varepsilon}(k) &:= \widehat{\mathscr{I}_\varepsilon^<}(k) \cup \widehat{\mathscr{I}_\varepsilon^>}(k)\,.
\end{aligned}$$

Finally, it is convenient to introduce

$$\widehat{\mathscr{S}_\varepsilon}(k) := \widehat{\mathscr{C}_\varepsilon}(k) \cup \widehat{\mathscr{I}_\varepsilon}(k).$$

Trivially, $(\widehat{\mathscr{C}_\varepsilon}(k) \cup \{B_r(x)\}; \widehat{\mathscr{I}_\varepsilon}(k)) \equiv (\mathscr{C}_\varepsilon(k); \mathscr{I}_\varepsilon(k))$. Furthermore, in view of the inductive assumption,

$$\{B_r(x)\} \cup \widehat{\mathscr{C}_\varepsilon}(k) \cup \widehat{\mathscr{I}_\varepsilon}(k) \text{ is made by pairwise (essentially) disjoint balls,} \tag{3.26}$$

$$\widehat{\mathscr{C}_\varepsilon}(k) \subset \mathscr{C}_\varepsilon(0)\,, \tag{3.27}$$

$$\mathcal{R}ad(\widehat{\mathscr{S}_\varepsilon}(k)) + r \leq \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B}\in\widehat{\mathscr{I}}_\varepsilon(k)\cup\{B_r(x)\}} \sum_{\substack{B\in\mathscr{C}_\varepsilon(0)\\ B\subseteq\widetilde{B}}} r(B)\,, \tag{3.28}$$

and for all $B \in \widehat{\mathscr{I}_\varepsilon}(k)$ we have that

$$\text{either } r(B) \geq \frac{\varepsilon}{2} \quad \text{or} \quad \int_{\partial B} |\nabla u_\varepsilon| \, \mathrm{d}\mathcal{H}^1 \leq C|\log \varepsilon|^{\frac{1}{2}} \,. \tag{3.29}$$

Notice that (3.24) implies (3.28). Moreover, we observe that (3.28) is equivalent to saying that

$$\mathcal{R}ad(\widehat{\mathscr{S}_\varepsilon}(k)) + r \leq \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B} \in \widehat{\mathscr{I}}_\varepsilon(k)} \sum_{\substack{B \in \mathscr{C}_\varepsilon(0) \\ B \subset \widetilde{B}}} r(B) + 4r \,,$$

in the special case that $B_r(x) \in \mathscr{C}_\varepsilon(0)$ (that is the case, for instance, if $k = 0$).

We now define

$$T(r,x) := \inf \Big\{ t \geq 2r : \mathcal{H}^1(A_{B_r(x)}(t)) \geq \frac{t-r}{2} \Big\} \,, \tag{3.30}$$

where

$$A_{B_r(x)}(t) := \{ s \in [r,t] : \partial B_s(x) \cap \overline{B} = \varnothing, \ \forall B \in \widehat{\mathscr{S}_\varepsilon}(k) \} \,,$$

for all $t \geq 2r$. Notice that, being the map $t \mapsto \mathcal{H}^1(A_{B_r(x)}(t))$ continuous, the infimum in (3.30) is actually a minimum. Moreover, it is easy to see that the circumference $\partial B_{T(r,x)}(x)$ does not intersect any ball in $\widehat{\mathscr{S}_\varepsilon}(k)$. We set

$$d(x) := \operatorname{dist}\Big(x, \bigcup_{B \in \widehat{\mathscr{I}}_\varepsilon(k)} B\Big) \,,$$

with the standard convention that $d(x) = +\infty$ if $\widehat{\mathscr{I}_\varepsilon}(k) = \varnothing$.

Case 1: $T(r,x) \leq d(x)$. In such a case $B_{T(r,x)}(x)$ does not intersect any ball in $\widehat{\mathscr{I}_\varepsilon}(k)$. We distinguish two subcases.

Subcase 1a: $T(r,x) < \frac{\varepsilon}{2}$. In such a case, the ball $B_{T(r,x)}(x)$ does not intersect any ball in $\widehat{\mathscr{C}_\varepsilon^{>}}(k)$, since balls in $\widehat{\mathscr{C}_\varepsilon^{>}}(k)$ have radius at least $\frac{\varepsilon}{2}$. On the other hand, it might contain some balls in $\widehat{\mathscr{C}_\varepsilon^{<}}(k)$ and it contains for sure $B_r(x)$.

We choose a number $R = R(r,x) \in A_{B_r(x)}(T(r,x))$ such that

$$\int_{\partial B_{R(r,x)}(x)} |\nabla u_\varepsilon| \, \mathrm{d}\mathcal{H}^1 \leq \frac{1}{\mathcal{H}^1(A_{B_r(x)}(T(r,x)))} \int_{A_{B_r(x)}(T(r,x))} \int_{\partial B_s(x)} |\nabla u_\varepsilon| \, \mathrm{d}\mathcal{H}^1 \, \mathrm{d}s. \tag{3.31}$$

Denoting by $A \subset B_{T(r,x)}(x)$ the set (in polar coordinates centered at x)

$$A := \{ (\rho, \theta) : \rho \in A_{B_r(x)}(T(r,x)) \}$$

and using that $\mathcal{H}^1(A_{B_r(x)}(T(r,x))) = \frac{T(r,x)-r}{2}$, by (3.31), we estimate

$$\begin{aligned} \int_{\partial B_{R(r,x)}(x)} |\nabla u_\varepsilon| \, \mathrm{d}\mathcal{H}^1 &\leq \frac{2}{T(r,x)-r} \int_A |\nabla u_\varepsilon| \, \mathrm{d}x \leq \frac{2|A|^{\frac{1}{2}}}{T(r,x)-r} \Big(\int_A |\nabla u_\varepsilon|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \\ &\leq C \Big(\int_A |\nabla u_\varepsilon|^2 \, \mathrm{d}x \Big)^{\frac{1}{2}} \leq C|\log \varepsilon|^{\frac{1}{2}}, \end{aligned} \tag{3.32}$$

where C is an absolute constant. Since $R(r,x) < T(r,x)$, by the very definition of $T(r,x)$, we have

$$2 \sum_{\substack{B \in \widehat{\mathscr{C}_\varepsilon}(k) \\ B \subseteq B_{R(r,x)}(x)}} r(B) \geq \mathcal{H}^1\big([r, R(r,x)] \setminus A_{B_r(x)}(R(r,x))\big) > \frac{R(r,x)-r}{2}. \tag{3.33}$$

Then, we set

$$\begin{aligned}
\mathscr{C}_\varepsilon^<(k+1) :=& \widehat{\mathscr{C}_\varepsilon^<}(k) \setminus \{B \in \widehat{\mathscr{C}_\varepsilon^<}(k) \,:\, B \subseteq B_{R(r,x)}(x)\}\,,\\
\mathscr{C}_\varepsilon^>(k+1) :=& \widehat{\mathscr{C}_\varepsilon^>}(k)\,,\\
\mathscr{I}_\varepsilon^<(k+1) :=& \widehat{\mathscr{I}_\varepsilon^<}(k) \cup \{B_{R(r,x)}(x)\}\,,\\
\mathscr{I}_\varepsilon^>(k+1) :=& \widehat{\mathscr{I}_\varepsilon^>}(k)\,,\\
\mathscr{C}_\varepsilon(k+1) :=& \mathscr{C}_\varepsilon^<(k+1) \cup \mathscr{C}_\varepsilon^>(k+1)\,,\\
\mathscr{I}_\varepsilon(k+1) :=& \mathscr{I}_\varepsilon^<(k+1) \cup \mathscr{I}_\varepsilon^>(k+1)\,,\\
\mathscr{S}_\varepsilon(k+1) :=& \mathscr{C}_\varepsilon(k+1) \cup \mathscr{I}_\varepsilon(k+1)\,.
\end{aligned}$$

Notice that the pair $(\mathscr{C}_\varepsilon(k+1); \mathscr{I}_\varepsilon(k+1))$ satisfies (3.22), by construction, (3.23), by the inductive assumption, and (3.25), by (3.32). Finally, by (3.26), (3.27), (3.28), and (3.33), using the very definition of $\mathscr{I}_\varepsilon^<(k+1)$ and of $\widehat{\mathscr{C}_\varepsilon}(k)$, we have that

$$\begin{aligned}
\mathcal{R}ad(\mathscr{S}_\varepsilon(k+1)) =& \mathcal{R}ad(\widehat{\mathscr{S}_\varepsilon}(k)) + r + R(r,x) - r - \sum_{\substack{B \in \widehat{\mathscr{C}_\varepsilon}(k)\\ B \subseteq B_{R(r,x)}(x)}} r(B)\\
\leq& \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B} \in \widehat{\mathscr{I}}_\varepsilon(k) \cup \{B_r(x)\}} \sum_{\substack{B \in \mathscr{C}_\varepsilon(0)\\ B \subseteq \widetilde{B}}} r(B) + 4 \sum_{\substack{B \in \widehat{\mathscr{C}}_\varepsilon(k)\\ B \subseteq B_{R(r,x)}(x)}} r(B)\\
=& \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B} \in \widehat{\mathscr{I}}_\varepsilon(k)} \sum_{\substack{B \in \mathscr{C}_\varepsilon(0)\\ B \subseteq \widetilde{B}}} r(B) + 4 \sum_{\substack{B \in \mathscr{C}_\varepsilon(0)\\ B \subseteq B_r(x)}} r(B) + 4 \sum_{\substack{B \in \widehat{\mathscr{C}}_\varepsilon(k)\\ B \subseteq B_{R(r,x)}(x)}} r(B)\\
\leq& \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B} \in \widehat{\mathscr{I}}_\varepsilon(k)} \sum_{\substack{B \in \mathscr{C}_\varepsilon(0)\\ B \subseteq \widetilde{B}}} r(B) + 4 \sum_{\substack{B \in \mathscr{C}_\varepsilon(0)\\ B \subseteq B_{R(r,x)}(x)}} r(B)\\
\leq& \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B} \in \mathscr{I}_\varepsilon(k+1)} \sum_{\substack{B \in \mathscr{C}_\varepsilon(0)\\ B \subseteq \widetilde{B}}} r(B)\,,
\end{aligned}$$

which implies (3.24).

Subcase 1b: $T(r,x) \geq \frac{\varepsilon}{2}$. In this case, we define $R(r,x) := T(r,x)$ and we set

$$\begin{aligned}
\mathscr{C}_\varepsilon^<(k+1) :=& \mathscr{C}_\varepsilon^<(k) \setminus \{B \in \mathscr{C}_\varepsilon^<(k) \,:\, B \subset B_{R(r,x)}(x)\}\,,\\
\mathscr{C}_\varepsilon^>(k+1) :=& \mathscr{C}_\varepsilon^>(k) \setminus \{B \in \mathscr{C}_\varepsilon^>(k) \,:\, B \subset B_{R(r,x)}(x)\}\,,\\
\mathscr{I}_\varepsilon^<(k+1) :=& \mathscr{I}_\varepsilon^<(k)\,,\\
\mathscr{I}_\varepsilon^>(k+1) :=& \mathscr{I}_\varepsilon^>(k) \cup \{B_{R(r,x)}(x)\}\,,\\
\mathscr{C}_\varepsilon(k+1) :=& \mathscr{C}_\varepsilon^<(k+1) \cup \mathscr{C}_\varepsilon^>(k+1)\,,\\
\mathscr{I}_\varepsilon(k+1) :=& \mathscr{I}_\varepsilon^<(k+1) \cup \mathscr{I}_\varepsilon^>(k+1)\,.
\end{aligned}$$

By arguing as in Subcase 1a also in this case we obtain that the pair $(\mathscr{C}_\varepsilon(k+1); \mathscr{I}_\varepsilon(k+1))$ satisfies (3.22), (3.23), (3.24) and (3.25).

Case 2: $T(r,x) > d(x)$. In this case, we set

$$R(r,x) := \sup\{t \leq d(x) \,:\, \partial B_t(x) \cap B = \varnothing \text{ for all } B \in \widehat{\mathscr{S}_\varepsilon}(k)\}$$

and introduce

$$\widetilde{\mathscr{S}_\varepsilon}(k) := \widetilde{\mathscr{C}_\varepsilon^<}(k) \cup \widetilde{\mathscr{C}_\varepsilon^>}(k) \cup \widetilde{\mathscr{I}_\varepsilon^<}(k) \cup \widetilde{\mathscr{I}_\varepsilon^>}(k)\,,$$

where

$$
\begin{aligned}
&\widetilde{\mathscr{C}}_\varepsilon^<(k) := \widehat{\mathscr{C}}_\varepsilon^<(k) \setminus \{B \in \widehat{\mathscr{C}}_\varepsilon^<(k) \,:\, B \subset B_{R(r,x)}(x)\}\,,\\
&\widetilde{\mathscr{C}}_\varepsilon^>(k) := \widehat{\mathscr{C}}_\varepsilon^>(k) \setminus \{B \in \widehat{\mathscr{C}}_\varepsilon^>(k) \,:\, B \subset B_{R(r,x)}(x)\}\,,\\
&\widetilde{\mathscr{I}}_\varepsilon^<(k) := \widehat{\mathscr{I}}_\varepsilon^<(k)\,,\\
&\widetilde{\mathscr{I}}_\varepsilon^>(k) := \widehat{\mathscr{I}}_\varepsilon^>(k)\,.
\end{aligned}
\tag{3.34}
$$

Notice that $\partial B_{R(r,x)}(x)$ does not intersect other balls in $\widetilde{\mathscr{S}}_\varepsilon(k)$, but $B_{R(r,x)}(x)$ might contain balls in $\widehat{\mathscr{C}}_\varepsilon(k)$ and it can be tangent to some ball in $\widetilde{\mathscr{I}}_\varepsilon(k)$ (e.g., if $R(r,x) = d(x)$). Furthermore, we notice that, by the very definition of $R(r,x)$, $B_{R(r,x)}(x)$ is surely tangent to some ball in $\widehat{\mathscr{S}}_\varepsilon(k)$. Then, we pass through a merging procedure (Definition 3.2), including in a unique ball $B_{R'}(x')$ all the balls in $\widetilde{\mathscr{S}}_\varepsilon(k)$ whose closures intersect $\partial B_{R(r,x)}(x)$ (and possibly others). We emphasize that, by definition of $R(r,x)$, after the merging procedure $B_{R'}(x')$ may contain some other balls in $\widetilde{\mathscr{I}}_\varepsilon(k)$. Then, we redefine the sets in (3.34) by setting

$$
\begin{aligned}
&\widetilde{\mathscr{C}}_\varepsilon^<(k) := \widehat{\mathscr{C}}_\varepsilon^<(k) \setminus \{B \in \widehat{\mathscr{C}}_\varepsilon^<(k) \,:\, B \subseteq B_{R'}(x')\}\,,\\
&\widetilde{\mathscr{C}}_\varepsilon^>(k) := \widehat{\mathscr{C}}_\varepsilon^>(k) \setminus \{B \in \widehat{\mathscr{C}}_\varepsilon^>(k) \,:\, B \subseteq B_{R'}(x')\}\,,\\
&\widetilde{\mathscr{I}}_\varepsilon^<(k) := \widehat{\mathscr{I}}_\varepsilon^<(k) \setminus \{B \in \widehat{\mathscr{I}}_\varepsilon^<(k) \,:\, B \subseteq B_{R'}(x')\}\,,\\
&\widetilde{\mathscr{I}}_\varepsilon^>(k) := \widehat{\mathscr{I}}_\varepsilon^>(k) \setminus \{B \in \widehat{\mathscr{I}}_\varepsilon^>(k) \,:\, B \subseteq B_{R'}(x')\}\,;
\end{aligned}
$$

moreover, we set

$$
\widetilde{\mathscr{C}}_\varepsilon(k) := \widetilde{\mathscr{C}}_\varepsilon^<(k) \cup \widetilde{\mathscr{C}}_\varepsilon^>(k)\,, \qquad \widetilde{\mathscr{I}}_\varepsilon(k) := \widetilde{\mathscr{I}}_\varepsilon^<(k) \cup \widetilde{\mathscr{I}}_\varepsilon^>(k)\,, \qquad \widetilde{\mathscr{S}}_\varepsilon(k) := \widetilde{\mathscr{C}}_\varepsilon(k) \cup \widetilde{\mathscr{I}}_\varepsilon(k)\,.
$$

By the very definition of $T(r,x)$, since $R(r,x) < T(r,x)$, and using that $B_{R(r,x)}(x)$ contains only balls in $\widehat{\mathscr{C}}_\varepsilon(k)$, we deduce that (3.33) still holds true; hence, we have

$$
R(r,x) - r < 4 \sum_{\substack{B \in \widehat{\mathscr{C}}_\varepsilon(k)\\ B \subset B_{R(r,x)}(x)}} r(B)\,;
$$

therefore, by the very definition of merging, we deduce

$$
\begin{aligned}
\mathcal{R}ad(\widetilde{\mathscr{S}_\varepsilon}(k)) + R' \leq & \mathcal{R}ad(\widetilde{\mathscr{S}_\varepsilon}(k)) + \sum_{\substack{B\in\widetilde{\mathscr{S}_\varepsilon}(k)\\ B\subset(B_{R'}(x')\setminus B_{R(r,x)}(x))}} r(B) + R(r,x)\\
\leq & \mathcal{R}ad(\widetilde{\mathscr{S}_\varepsilon}(k)) + \sum_{\substack{B\in\widetilde{\mathscr{S}_\varepsilon}(k)\\ B\subset(B_{R'}(x')\setminus B_{R(r,x)}(x))}} r(B) + 4 \sum_{\substack{B\in\widehat{\mathscr{C}_\varepsilon}(k)\\ B\subseteq B_{R(r,x)}(x)}} r(B) + r\\
\leq & \mathcal{R}ad(\widehat{\mathscr{S}_\varepsilon}(k)) + r + 4 \sum_{\substack{B\in\widehat{\mathscr{C}_\varepsilon}(k)\\ B\subseteq B_{R(r,x)}(x)}} r(B)\\
\leq & \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B}\in\widehat{\mathscr{I}_\varepsilon}(k)\cup\{B_r(x)\}} \sum_{\substack{B\in\mathscr{C}_\varepsilon(0)\\ B\subseteq\widetilde{B}}} r(B) + 4 \sum_{\substack{B\in\widehat{\mathscr{C}_\varepsilon}(k)\\ B\subseteq B_{R(r,x)}(x)}} r(B)\\
\leq & \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B}\in\widetilde{\mathscr{I}_\varepsilon}(k)} \sum_{\substack{B\in\mathscr{C}_\varepsilon(0)\\ B\subseteq\widetilde{B}}} r(B) + 4 \sum_{\substack{B\in\mathscr{C}_\varepsilon(0)\\ B\subseteq B_r(x)}} r(B)\\
& + 4 \sum_{\substack{\widetilde{B}\in\widetilde{\mathscr{I}_\varepsilon}(k)\\ \widetilde{B}\subset B_{R'}(x')}} \sum_{\substack{B\in\mathscr{C}_\varepsilon(0)\\ B\subseteq\widetilde{B}}} r(B) + 4 \sum_{\substack{B\in\widehat{\mathscr{C}_\varepsilon}(k)\\ B\subseteq B_{R(r,x)}(x)}} r(B)\\
\leq & \mathcal{R}ad(\mathscr{C}_\varepsilon(0)) + 4 \sum_{\widetilde{B}\in\widetilde{\mathscr{I}_\varepsilon}(k)\cup\{B_{R'}(x')\}} \sum_{\substack{B\in\mathscr{C}_\varepsilon(0)\\ B\subseteq\widetilde{B}}} r(B)\,,
\end{aligned}
\tag{3.35}
$$

where we have used also (3.28) and (3.26). Now, if $R' \geq \frac{\varepsilon}{2}$, we set

$$
\begin{aligned}
\mathscr{C}_\varepsilon^<(k+1) &:= \widetilde{\mathscr{C}_\varepsilon^<}(k),\\
\mathscr{C}_\varepsilon^>(k+1) &:= \widetilde{\mathscr{C}_\varepsilon^>}(k),\\
\mathscr{I}_\varepsilon^<(k+1) &:= \widetilde{\mathscr{I}_\varepsilon^<}(k),\\
\mathscr{I}_\varepsilon^>(k+1) &:= \widetilde{\mathscr{I}_\varepsilon^>}(k) \cup \{B_{R'}(x')\},\\
\mathscr{C}_\varepsilon(k+1) &:= \mathscr{C}_\varepsilon^<(k+1) \cup \mathscr{C}_\varepsilon^>(k+1),\\
\mathscr{I}_\varepsilon(k+1) &:= \mathscr{I}_\varepsilon^<(k+1) \cup \mathscr{I}_\varepsilon^>(k+1),
\end{aligned}
$$

and we conclude the iterative step. Notice that the family

$$\mathscr{S}_\varepsilon(k+1) = \mathscr{C}_\varepsilon(k+1) \cup \mathscr{I}_\varepsilon(k+1)$$

is made of pairwise disjoint balls, and by (3.35) it satisfies (3.26), (3.27), (3.28), (3.29).

Otherwise, if $R' < \frac{\varepsilon}{2}$, we redefine $B_r(x)$ by setting $B_r(x) := B_{R'}(x')$, and relabel

$$
\begin{aligned}
\widehat{\mathscr{C}_\varepsilon^<}(k) &:= \widetilde{\mathscr{C}_\varepsilon^<}(k), & \widehat{\mathscr{C}_\varepsilon^>}(k) &:= \widetilde{\mathscr{C}_\varepsilon^>}(k)\\
\widehat{\mathscr{I}_\varepsilon^<}(k) &:= \widetilde{\mathscr{I}_\varepsilon^<}(k)\,, & \widehat{\mathscr{I}_\varepsilon^>}(k) &:= \widetilde{\mathscr{I}_\varepsilon^>}(k)\,,\\
\widehat{\mathscr{C}_\varepsilon}(k) &:= \widehat{\mathscr{C}_\varepsilon^<}(k) \cup \widehat{\mathscr{C}_\varepsilon^>}(k), & \widehat{\mathscr{I}_\varepsilon}(k) &:= \widehat{\mathscr{I}_\varepsilon^<}(k) \cup \widehat{\mathscr{I}_\varepsilon^>}(k)\,,
\end{aligned}
$$

and $\widehat{\mathscr{S}_\varepsilon}(k) := \widehat{\mathscr{C}_\varepsilon}(k) \cup \widehat{\mathscr{I}_\varepsilon}(k)$; we check that (the new ball) $B_r(x)$ and $\widehat{\mathscr{S}_\varepsilon}(k)$ satisfy (3.26), (3.27), (3.28), (3.29), and we restart the whole process by defining $T(r,x)$ as in (3.30) starting from the new (just built) ball $B_r(x)$.

If we fall again in Case 2 and the new radius $R' < \frac{\varepsilon}{2}$, we iterate again the procedure. Notice that every time we fall in this situation, the number of balls in $\widehat{\mathscr{C}_\varepsilon}(k) \cup \widehat{\mathscr{I}_\varepsilon}(k)$ decreases (due to the merging procedure), and hence after a finite number $N_k > 0$ of iterations we will end up in some other case. In other words, after a finite number of steps (depending on k and ε), such an

iteration does not happen anymore and hence we necessarily fall either in case $R' \geq \frac{\varepsilon}{2}$ or in Case 1, thus defining $\widehat{\mathscr{I}_\varepsilon}(k+1)$ accordingly.

(Conclusion of the procedure) We iterate the process described above inductively. Every time we perform this iteration the number of balls in $\mathscr{C}_\varepsilon^<$ decreases, and after a finite number K_ε of processes we end up with $\mathscr{C}_\varepsilon^<(K_\varepsilon) = \varnothing$. Finally, we set

$$\mathscr{I}_\varepsilon^< := \mathscr{I}_\varepsilon^<(K_\varepsilon) \qquad \mathscr{I}_\varepsilon^> := \mathscr{I}_\varepsilon^>(K_\varepsilon) \cup \mathscr{C}_\varepsilon^>(K_\varepsilon)\,, \qquad \mathscr{I}_\varepsilon := \mathscr{I}_\varepsilon^< \cup \mathscr{I}_\varepsilon^>\,. \tag{3.36}$$

By construction (see (3.22)-(3.25)), the balls in $\mathscr{I}_\varepsilon$ are pairwise disjoint and satisfy (3.19)-(3.21). Furthermore, we observe that

$$\int_{\partial B} |\nabla u_\varepsilon|\, \mathrm{d}\mathcal{H}^1 \leq C|\log\varepsilon|^{\frac{1}{2}} \qquad \text{for every } B \in \mathscr{I}_\varepsilon^<\,, \tag{3.37}$$

and, in view of (3.19),

$$\sharp\mathscr{I}_\varepsilon^> \leq C|\log\varepsilon|\,. \tag{3.38}$$

Step 4: Estimate of $\|\nabla u_\varepsilon\|_{L^1}$ on the boundary of the balls in $\mathscr{I}_\varepsilon^>$. For every $B = B_r(x) \in \mathscr{I}_\varepsilon^>$, we set

$$R(r,\varepsilon) := 16\,\overline{C}\,|\log\varepsilon|\,r\,, \tag{3.39}$$

where $\overline{C}$ is the constant appearing in (3.19). Notice that, by (3.19), again

$$\sum_{B_r(x)\in\mathscr{I}_\varepsilon^>} R(r,\varepsilon) \leq 16\overline{C}|\log\varepsilon|\mathcal{R}ad(\mathscr{I}_\varepsilon) \leq C\varepsilon|\log\varepsilon|^2\,, \tag{3.40}$$

and hence, for ε small enough, we can assume that all the balls $B_{R(r,\varepsilon)}(x)$ are contained in Ω. Let $\widetilde{D}^1,\dots,\widetilde{D}^{M_\varepsilon}$ denote the connected components of the set $\bigcup_{B_r(x)\in\mathscr{I}_\varepsilon^>} B_{R(r,\varepsilon)}(x)$. By (3.38) we get

$$M_\varepsilon \leq C|\log\varepsilon|\,.$$

For every $m = 1,\dots,M_\varepsilon$, let $\{B_{r^{m,k}}(x^{m,k})\}_{k=1,\dots,K^m} \subseteq \mathscr{I}_\varepsilon^>$ be such that $\widetilde{D}^m := \bigcup_{k=1}^{K^m} B_{R_\varepsilon^{m,k}}(x^{m,k})$, where $R_\varepsilon^{m,k} := R(r^{m,k},\varepsilon)$ is defined by (3.39). Moreover, for every $m = 1,\dots,M_\varepsilon$, we set $\widehat{D}^m := \bigcup_{k=1}^{K^m} B_{r^{m,k}}(x^{m,k})$ and we define the function $f^m : \widetilde{D}^m \to [0,1]$ as

$$f^m(x) := \inf_{k=1,\dots,K^m} \Big\{ \frac{1}{(16\,\overline{C}\,|\log\varepsilon| - 1)r^{m,k}} \mathrm{dist}(x, B_{r^{m,k}}(x^{m,k})) \Big\}\,;$$

we notice that f^m is Lipschitz continuous, $f^m = 0$ on $\widehat{D}^m$, $f^m = 1$ on $\partial\widetilde{D}^m$, and

$$|\nabla f^m| \leq \frac{1}{(16\,\overline{C}\,|\log\varepsilon| - 1)\frac{\varepsilon}{2}} \qquad \text{on } \widetilde{D}^m \setminus \widehat{D}^m. \tag{3.41}$$

Finally, for every $m = 1,\dots,M_\varepsilon$, let us define the sets

$$T_1^m := \Big\{ t \in [0,1] \,:\, \{f^m = t\} \cap \bigcup_{B\in\mathscr{I}_\varepsilon^<} B \neq \varnothing \Big\}\,,$$
$$T_2^m := \Big\{ t \in [0,1] \,:\, \{f^m = t\} \cap \bigcup_{B\in\mathscr{I}_\varepsilon^<} B = \varnothing \Big\}\,,$$

and we show that

$$\mathcal{H}^1(T_2^m) \geq \frac{1}{2}\,. \tag{3.42}$$

Indeed, in view of (3.41)) for every $B \in \mathscr{I}_\varepsilon^<$, by $\mathrm{osc}_B(f^m) = \sup_{x\in B} f^m(x) - \inf_{x\in B} f^m(x)$,

$$\begin{aligned}\mathcal{H}^1\Big(\{t \in [0,1] \,:\, \{f^m = t\} \cap B \neq \varnothing\} \Big) \leq& \mathrm{osc}_B(f^m) \leq 2r(B)\|\nabla f^m\|_{L^\infty(B;\mathbb{R}^2)} \\ \leq& 2r(B)\frac{1}{(16\,\overline{C}\,|\log\varepsilon| - 1)\frac{\varepsilon}{2}} \leq \frac{r(B)}{2\overline{C}\varepsilon|\log\varepsilon|}\,,\end{aligned}$$

which, together with (3.19), implies

$$\mathcal{H}^1(T_1^m) \le \sum_{B\in\mathscr{I}_\varepsilon^<} \mathcal{H}^1\Big(\{t\in[0,1]\,:\,\{f^m=t\}\cap B\neq\varnothing\}\Big) \le \frac{\mathcal{R}ad(\mathscr{I}_\varepsilon^<)}{2\overline{C}\varepsilon|\log\varepsilon|} \le \frac{1}{2}\,,$$

whence (3.42) follows. For every $k=1,\dots,K^m$ we set

$$E^m_{r^{m,k}} := \Big\{x\in\widetilde{D}^m\,:\,|\nabla f^m(x)| = \frac{1}{(16\,\overline{C}\,|\log\varepsilon|-1)r^{m,k}}\Big\}\,;$$

it is easy to check that

$$E^m_{r^{m,k}} \subset B_{16\,\overline{C}\,|\log\varepsilon|\,r^{m,k}}(x^{m,k})\setminus\overline{B}_{r^{m,k}}(x^{m,k}) \subset \widetilde{D}^m\setminus\widehat{D}^m\,. \tag{3.43}$$

Since f^m is Lipschitz continuous, by the mean value theorem, the coarea formula and (3.42), there exists $t^m\in T_2^m$ such that

$$\begin{aligned}\int_{\{f^m(x)=t^m\}}|\nabla u_\varepsilon|\,\mathrm{d}\mathcal{H}^1 &\le \frac{1}{\mathcal{H}^1(T_2^m)}\int_{T_2^m}\mathrm{d}t\int_{\{f^m(x)=t\}}|\nabla u_\varepsilon|\,\mathrm{d}\mathcal{H}^1\\ \le& 2\int_{(\widetilde{D}^m\setminus\widehat{D}^m)\setminus(\bigcup_{B\in\mathscr{I}_\varepsilon^<}B)}|\nabla u_\varepsilon||\nabla f^m|\,\mathrm{d}x \le 2\sum_{k=1}^{K^m}\frac{1}{(16\,\overline{C}\,|\log\varepsilon|-1)r^{m,k}}\int_{E^m_{r^{m,k}}}|\nabla u_\varepsilon|\,\mathrm{d}x\\ \le& 2K^m\|\nabla u_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{2\times2})} \le C|\log\varepsilon|^{\frac{3}{2}}\,,\end{aligned} \tag{3.44}$$

where the third inequality is a consequence of the fact that $(\widetilde{D}^m\setminus\widehat{D}^m)\setminus(\bigcup_{B\in\mathscr{I}_\varepsilon^<}B)\subseteq\bigcup_{k=1}^{K^m}E^m_{r^{m,k}}$, the last but one inequality follows by Hölder inequality and (3.43), and in last passage we have used (3.38) and (3.4). For every $m=1,\dots,M_\varepsilon$, we set

$$D^m := \{x\in\widetilde{D}^m\,:\,f^m(x)\le t^m\}\,,$$

so that $\widehat{D}^m\subset D^m$; by (3.44) we have that

$$\int_{\partial D^m}|\nabla u_\varepsilon|\,\mathrm{d}\mathcal{H}^1 \le C|\log\varepsilon|^{\frac{3}{2}}\,. \tag{3.45}$$

Moreover, we notice that, in view of (3.40),

$$\sum_{m=1}^{M_\varepsilon}\mathrm{diam}(D^m) \le C\varepsilon|\log\varepsilon|^2\,. \tag{3.46}$$

We remark that

$$D^m := \bigcup_{k=1}^{K^m}B_{\alpha^{m,k}r^{m,k}}(x^{m,k})\,,\quad \text{for some } \alpha^{m,k}\in(1,16\overline{C}|\log\varepsilon|)\ (\text{for all } k=1,\dots,K^m)\,; \tag{3.47}$$

by (3.40),

$$\sum_{k=1}^{K^m}\alpha^{m,k}r^{m,k} \le C\varepsilon|\log\varepsilon|^2\,. \tag{3.48}$$

Eventually, observe that, thanks to the choice of $t^m\in T_2^m$, the boundary ∂D^m does not intersect other balls in $\mathscr{I}_\varepsilon$. However, D^m might contain some other balls in $\mathscr{I}_\varepsilon^<$.

Step 5: Definition of $\widehat{\mu}_\varepsilon$. For every $\varepsilon>0$, we define $\widehat{\mu}_\varepsilon$ as

$$\widehat{\mu}_\varepsilon := \sum_{m=1}^{M_\varepsilon}\deg(u_\varepsilon,\partial D^m)\delta_{\widehat{x}^m}\,,$$

where, for every $m=1,\dots,M_\varepsilon$, $\widehat{x}^m$ is the center of an arbitrarily chosen ball among the balls $\{B_{r^{m,k}}(x^{m,k})\}_{k=1,\dots,K^m}$ contained in D^m. Notice that $\deg(u_\varepsilon,\partial D^m)=\sum_{\substack{B\in\mathscr{C}_\varepsilon\\ B\subset D^m}}\deg(u_\varepsilon,\partial B)$. As a consequence, by the very definition of $\widetilde{\mu}_\varepsilon$ in (3.15), it easily follows that

$$|\widehat{\mu}_\varepsilon|(\Omega) \le |\widetilde{\mu}_\varepsilon|(\Omega) \le C|\log\varepsilon|\,. \tag{3.49}$$

By (3.46) and Theorem 3.4(i), by setting $\overline{\varphi}_m := \frac{1}{|D^m|}\int_{D^m}\varphi dx$, we get

$$\|\widetilde{\mu}_\varepsilon-\widehat{\mu}_\varepsilon\|_{\rm flat}\le \sup_{\|\varphi\|_{C^{0,1}_{\rm c}(\Omega)}\le 1}\sum_{m=1}^{M_\varepsilon}\Big|\int_{D^m}\varphi\,{\rm d}(\widetilde{\mu}_\varepsilon-\widehat{\mu}_\varepsilon)\Big| = \sup_{\|\varphi\|_{C^{0,1}_{\rm c}(\Omega)}\le 1}\sum_{m=1}^{M_\varepsilon}\Big|\int_{D^m}\varphi-\overline{\varphi}_m\,{\rm d}(\widetilde{\mu}_\varepsilon-\widehat{\mu}_\varepsilon)\Big|$$
$$\le 2\sup_{\|\varphi\|_{C^{0,1}_{\rm c}(\Omega)}\le 1}\sum_{m=1}^{M_\varepsilon}|\widetilde{\mu}_\varepsilon|(D^m){\rm osc}_{D^m}(\varphi)\le C|\log\varepsilon|\sum_{m=1}^{M_\varepsilon}{\rm diam}(D^m)\le C\varepsilon|\log\varepsilon|^3\,,$$

and hence $\|\widetilde{\mu}_\varepsilon-\widehat{\mu}_\varepsilon\|_{\rm flat}\to 0$ as $\varepsilon\to 0$. Thus, in order to get (3.17), we are left to prove that

$$\|Ju_\varepsilon-\pi\widehat{\mu}_\varepsilon\|_{{\rm flat},\Omega}\to 0\qquad\text{as }\varepsilon\to 0\,. \tag{3.50}$$

Step 6: Definition of λ_ε and $\widehat{\lambda}_\varepsilon$. In order to prove (3.50) we introduce the measures λ_ε and $\widehat{\lambda}_\varepsilon$ such that the following identities

$$-{\rm Div}\,\lambda_\varepsilon = Ju_\varepsilon\qquad\text{and}\qquad -{\rm Div}\,\widehat{\lambda}_\varepsilon=\pi\widehat{\mu}_\varepsilon\,, \tag{3.51}$$

hold in the sense of distributions in $\mathcal{D}'(\Omega)$. We set

$$\lambda^1_\varepsilon:=\frac{1}{2}([u^1_\varepsilon{\rm D}_1u^2_\varepsilon]-[u^2_\varepsilon{\rm D}_1u^1_\varepsilon])\,,\qquad \lambda^2_\varepsilon:=\frac{1}{2}([u^1_\varepsilon{\rm D}_2u^2_\varepsilon]-[u^2_\varepsilon{\rm D}_2u^1_\varepsilon])\,,\qquad \lambda_\varepsilon:=(-\lambda^2_\varepsilon,\lambda^1_\varepsilon)\,.$$

By definition of weak Jacobian determinant (see Definition 2.3, and (2.35)), λ_ε satisfies the first equation in (3.51). Moreover, by (3.37) and (3.45),

$$\begin{aligned}&\int_{\partial B}|\lambda_\varepsilon|\,{\rm d}\mathcal{H}^1\le C|\log\varepsilon|^{\frac12}\qquad\text{for every } B\in\mathscr{I}^<_\varepsilon\\ &\int_{\partial D^m}|\lambda_\varepsilon|\,{\rm d}\mathcal{H}^1\le C|\log\varepsilon|^{\frac32}\qquad\text{for every } m=1,\dots,M_\varepsilon\,.\end{aligned} \tag{3.52}$$

Furthermore, by Hölder inequality, (3.4), and (3.10) (for the estimate of $|{\rm D}^su_\varepsilon|$), we get

$$\begin{aligned}|\lambda_\varepsilon|\Big(\bigcup_{B\in\mathscr{I}^<_\varepsilon}B\cup\bigcup_{m=1}^{M_\varepsilon}D^m\Big)\le&|{\rm D}u_\varepsilon|\Big(\bigcup_{B\in\mathscr{I}^<_\varepsilon}B\cup\bigcup_{m=1}^{M_\varepsilon}D^m\Big)\\ \le&\|\nabla u_\varepsilon\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}\Big(\sum_{B\in\mathscr{I}^<_\varepsilon}|B|+\sum_{m=1}^{M_\varepsilon}|D^m|\Big)^{\frac12}+C\varepsilon|\log\varepsilon|\\ \le&C\varepsilon|\log\varepsilon|^{\frac52}\,,\end{aligned} \tag{3.53}$$

where we have used also (3.19) and (3.46) to deduce that

$$\sum_{B\in\mathscr{I}^<_\varepsilon}|B|\le C\varepsilon^2|\log\varepsilon|^2\qquad\text{and}\qquad\sum_{m=1}^{M_\varepsilon}|D^m|\le C\varepsilon^2|\log\varepsilon|^4\,. \tag{3.54}$$

Now, for every $m=1,\dots,M_\varepsilon$, we define $v^m_\varepsilon\in W^{1,1}(\Omega;\mathbb{S}^1)$ as

$$v^m_\varepsilon(x):=e^{\imath\deg(u_\varepsilon,\partial D^m)\vartheta(x-\widehat{x}^m)},$$

where $\mathbb{R}^2$ is identified with $\mathbb{C}$ and ϑ is the angular polar coordinate defined by

$$\vartheta(x):=\begin{cases}\arctan\frac{x_2}{x_1} & \text{if } x_1>0\\ \frac{\pi}{2} & \text{if } x_1=0 \text{ and } x_2>0\\ \pi+\arctan\frac{x_2}{x_1} & \text{if } x_1<0\\ \frac{3}{2}\pi & \text{if } x_1=0 \text{ and } x_2<0\,.\end{cases} \tag{3.55}$$

We set

$$v_\varepsilon(\cdot):=\Pi_{m=1}^{M_\varepsilon}v^m_\varepsilon(\cdot)\equiv e^{\imath\sum_{m=1}^{M_\varepsilon}\deg(u_\varepsilon,\partial D^m)\vartheta(\cdot-\widehat{x}^m)}$$

and we define

$$\widehat{\lambda}^1_\varepsilon:=\frac{1}{2}(v^1_\varepsilon\nabla_1v^2_\varepsilon-v^2_\varepsilon\nabla_1v^1_\varepsilon)\,,\qquad\widehat{\lambda}^2_\varepsilon:=\frac{1}{2}(v^1_\varepsilon\nabla_2v^2_\varepsilon-v^2_\varepsilon\nabla_2v^1_\varepsilon)\,,\qquad\widehat{\lambda}_\varepsilon:=(-\widehat{\lambda}^2_\varepsilon,\widehat{\lambda}^1_\varepsilon)\,.$$

We notice that $v_\varepsilon \in W^{1,p}(\Omega;\mathbb{S}^1)$ for any $1 \le p < 2$. It is well-known that, for every $m = 1, \dots, M_\varepsilon$, the distributional determinant of v_ε^m coincides with $\pi \deg(u_\varepsilon, \partial D^m)\delta_{\widehat{x}^m} = \pi\widehat{\mu}_\varepsilon \llcorner D^m$, so that also the second equation in (3.51) holds true. Finally, we show that $\widehat{\lambda}_\varepsilon$ satisfies estimates similar to (3.52) and (3.53).

Let $m = 1, \dots, M_\varepsilon$. Recalling (3.47), we can assume without loss of generality that $\widehat{x}^m = x^{m,1}$. Then, by (3.47), (3.48), (3.46) and (3.49), we have

$$
\begin{aligned}
\int_{\partial D^m} |\widehat{\lambda}_\varepsilon| \,\mathrm{d}\mathcal{H}^1 \le \int_{\partial D^m} |\nabla v_\varepsilon| \,\mathrm{d}\mathcal{H}^1 &\le \sum_{j=1}^{M_\varepsilon} |\deg(u_\varepsilon, \partial D^j)| \int_{\partial D^m} |\nabla\vartheta(\cdot - \widehat{x}^j)| \,\mathrm{d}\mathcal{H}^1 \\
&\le |\deg(u_\varepsilon, \partial D^m)| \int_{\partial D^m} |\nabla\vartheta(\cdot - \widehat{x}^m)| \,\mathrm{d}\mathcal{H}^1 \\
&\quad + \sum_{\substack{j=1 \\ j\neq m}}^{M_\varepsilon} |\deg(u_\varepsilon, \partial D^j)| \int_{\partial D^m} |\nabla\vartheta(\cdot - \widehat{x}^j)| \,\mathrm{d}\mathcal{H}^1 \\
&\le |\deg(u_\varepsilon, \partial D^m)| \int_{\partial B_{\alpha^{m,1} r^{m,1}}(x^{m,1})} |\nabla\vartheta(\cdot - x^{m,1})| \,\mathrm{d}\mathcal{H}^1 \\
&\quad + |\deg(u_\varepsilon, \partial D^m)| \sum_{k=2}^{K^m} \int_{\partial B_{\alpha^{m,k} r^{m,k}}(x^{m,k}) \setminus B_{\alpha^{m,1} r^{m,1}}(x^{m,1})} |\nabla\vartheta(\cdot - x^{m,1})| \,\mathrm{d}\mathcal{H}^1 \\
&\quad + \sum_{\substack{j=1 \\ j\neq m}}^{M_\varepsilon} |\deg(u_\varepsilon, \partial D^j)| \frac{\mathcal{H}^1(\partial D^m)}{\varepsilon} \\
&\le 2\pi |\deg(u_\varepsilon, \partial D^m)| + 2\pi |\deg(u_\varepsilon, \partial D^m)| \sum_{k=1}^{K^m} \frac{\alpha^{m,k} r^{m,k}}{\varepsilon} \\
&\quad + C \sum_{\substack{j=1 \\ j\neq m}}^{M_\varepsilon} |\deg(u_\varepsilon, \partial D^j)| \frac{\operatorname{diam}(D^m)}{\varepsilon} \\
&\le C|\widehat{\mu}_\varepsilon|(\Omega) + C|\widehat{\mu}_\varepsilon|(\Omega)|\log\varepsilon|^2 \le C|\log\varepsilon|^3 \,. \qquad (3.56)
\end{aligned}
$$

Analogously, recalling that the balls in $\mathscr{I}_\varepsilon$ are pairwise disjoint and that the balls in $\mathscr{I}_\varepsilon^>$ have radius larger than $\frac{\varepsilon}{2}$, for every $B \in \mathscr{I}_\varepsilon^<$ we have

$$
\begin{aligned}
\int_{\partial B} |\widehat{\lambda}_\varepsilon| \,\mathrm{d}\mathcal{H}^1 \le \int_{\partial B} |\nabla v_\varepsilon| \,\mathrm{d}\mathcal{H}^1 &\le \sum_{j=1}^{M_\varepsilon} |\deg(u_\varepsilon, \partial D^j)| \int_{\partial B} |\nabla\vartheta(\cdot - \widehat{x}^j)| \,\mathrm{d}\mathcal{H}^1 \\
&\le 2\sum_{j=1}^{M_\varepsilon} |\deg(u_\varepsilon, \partial D^j)| \frac{\mathcal{H}^1(\partial B)}{\varepsilon} \le C|\log\varepsilon| \,, \qquad (3.57)
\end{aligned}
$$

where the last inequality follows by (3.19) and (3.49).

Finally, we prove that

$$
|\widehat{\lambda}_\varepsilon|\Big(\bigcup_{B\in\mathscr{I}_\varepsilon^<} B \cup \bigcup_{m=1}^{M_\varepsilon} D^m \Big) \le C\varepsilon^{\frac{2}{3}} |\log\varepsilon|^{\frac{7}{3}} \,. \qquad (3.58)
$$

By definition of v_ε we easily get, for some fixed $p \in (1,2)$, and thanks to (3.49),

$$
\begin{aligned}
\|\nabla v_\varepsilon\|_{L^p(\Omega;\mathbb{R}^{2\times 2})} &= \Big\| \sum_{m=1}^{M_\varepsilon} \deg(u_\varepsilon, \partial D^m) \nabla\vartheta(\cdot - \widehat{x}^m) \Big\|_{L^p(\Omega;\mathbb{R}^2)} \\
&\le C(p) \sum_{m=1}^{M_\varepsilon} |\deg(u_\varepsilon, \partial D^m)| = C(p)|\widehat{\mu}_\varepsilon|(\Omega) \le C(p)|\log\varepsilon| \,.
\end{aligned}
$$

Hence, by Hölder inequality and (3.54), we deduce

$$\begin{aligned}|\widehat{\lambda}_\varepsilon|\Big(\bigcup_{B\in\mathscr{J}_\varepsilon^<} B\cup\bigcup_{m=1}^{M_\varepsilon} D^m\Big)\leq&\int_{\bigcup_{B\in\mathscr{J}_\varepsilon^<} B\cup\bigcup_{m=1}^{M_\varepsilon} D^m}|\nabla v_\varepsilon|\,\mathrm{d}x\\ \leq&\|\nabla v_\varepsilon\|_{L^p(\Omega;\mathbb{R}^{2\times 2})}\Big(\sum_{B\in\mathscr{J}_\varepsilon^<}|B|+\sum_{m=1}^{M_\varepsilon}|D^m|\Big)^{\frac{p-1}{p}}\\ \leq&C\varepsilon^{\frac{2p-2}{p}}|\log\varepsilon|^{\frac{5p-4}{p}},\end{aligned}$$

whence (3.58) follows by taking $p=\frac{3}{2}$.

Step 7: Final estimate. Let $\varphi\in C^1_{\rm c}(\Omega)$ with $\|\varphi\|_{C^{0,1}(\Omega)}\leq 1$. By definition of $Ju_\varepsilon=\partial T_{u_\varepsilon}$ we have

$$\begin{aligned}\langle Ju_\varepsilon,\varphi\rangle_\Omega=&\int_\Omega\nabla\varphi\cdot\,\mathrm{d}\lambda_\varepsilon\\ =&\sum_{B\in\mathscr{J}_\varepsilon^<}\int_B\nabla\varphi\cdot\,\mathrm{d}\lambda_\varepsilon+\sum_{m=1}^{M_\varepsilon}\int_{D^m\setminus\bigcup_{B\in\mathscr{J}_\varepsilon^<}B}\nabla\varphi\cdot\,\mathrm{d}\lambda_\varepsilon\\ &+\int_{\Omega\setminus(\bigcup_{m=1}^{M_\varepsilon}D^m\cup\bigcup_{B\in\mathscr{J}_\varepsilon^<}B)}\nabla\varphi\cdot\,\mathrm{d}\lambda_\varepsilon\end{aligned}\tag{3.59}$$

and, analogously, by integrating by parts

$$\begin{aligned}\langle\pi\widehat{\mu}_\varepsilon,\varphi\rangle_\Omega=&\int_\Omega\nabla\varphi\cdot\,\mathrm{d}\widehat{\lambda}_\varepsilon\\ =&\sum_{B\in\mathscr{J}_\varepsilon^<}\int_B\nabla\varphi\cdot\,\mathrm{d}\widehat{\lambda}_\varepsilon+\sum_{m=1}^{M_\varepsilon}\int_{D^m\setminus\bigcup_{B\in\mathscr{J}_\varepsilon^<}B}\nabla\varphi\cdot\,\mathrm{d}\widehat{\lambda}_\varepsilon\\ &+\int_{\Omega\setminus(\bigcup_{m=1}^{M_\varepsilon}D^m\cup\bigcup_{B\in\mathscr{J}_\varepsilon^<}B)}\nabla\varphi\cdot\,\mathrm{d}\widehat{\lambda}_\varepsilon.\end{aligned}\tag{3.60}$$

Using that $u_\varepsilon,v_\varepsilon\in H^1(\Omega\setminus(\bigcup_{m=1}^{M_\varepsilon}D^m\cup\bigcup_{B\in\mathscr{J}_\varepsilon^<}B);\mathbb{S}^1)$, and hence $Ju_\varepsilon=Jv_\varepsilon=0$ in $\Omega\setminus(\bigcup_{m=1}^{M_\varepsilon}D^m\cup\bigcup_{B\in\mathscr{J}_\varepsilon^<}B)$, we can integrate by parts the last integrals in (3.59) and (3.60), thus obtaining

$$\begin{aligned}\Big|\langle Ju_\varepsilon-\pi\widehat{\mu}_\varepsilon,\varphi\rangle_\Omega\Big|\leq&|\lambda_\varepsilon|\big(\bigcup_{B\in\mathscr{J}_\varepsilon^<}B\cup\bigcup_{m=1}^{M_\varepsilon}D^m\big)+|\widehat{\lambda}_\varepsilon|\big(\bigcup_{B\in\mathscr{J}_\varepsilon^<}B\cup\bigcup_{m=1}^{M_\varepsilon}D^m\big)\\ &+\Big|\sum_{m=1}^{M_\varepsilon}\int_{\partial D^m}\varphi\,(\lambda_\varepsilon-\widehat{\lambda}_\varepsilon)\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^1\Big|\\ &+\Big|\sum_{\substack{B\in\mathscr{J}_\varepsilon^<\\ B\cap\bigcup_{m=1}^{M_\varepsilon}D^m=\varnothing}}\int_{\partial B}\varphi\,(\lambda_\varepsilon-\widehat{\lambda}_\varepsilon)\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^1\Big|\\ \leq&C\varepsilon|\log\varepsilon|^{\frac{5}{2}}+C\varepsilon^{\frac{2}{3}}|\log\varepsilon|^{\frac{7}{3}}\\ &+\Big|\sum_{m=1}^{M_\varepsilon}\int_{\partial D^m}\varphi\,(\lambda_\varepsilon-\widehat{\lambda}_\varepsilon)\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^1\Big|\\ &+\Big|\sum_{\substack{B\in\mathscr{J}_\varepsilon^<\\ B\cap\bigcup_{m=1}^{M_\varepsilon}D^m=\varnothing}}\int_{\partial B}\varphi\,(\lambda_\varepsilon-\widehat{\lambda}_\varepsilon)\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^1\Big|,\end{aligned}\tag{3.61}$$

where in the last inequality we have used (3.53) and (3.58).

We now estimate the remaining integrals on the right hand side of (3.61). By (3.52) and (3.56), using that $\deg(u_\varepsilon, \partial D^m) = \deg(v_\varepsilon, \partial D^m)$, for every $m = 1, \dots, M_\varepsilon$, we have that

$$\begin{aligned}\Big| \int_{\partial D^m} \varphi(\lambda_\varepsilon - \widehat{\lambda}_\varepsilon) \cdot \boldsymbol{\nu} \, \mathrm{d}\mathcal{H}^1 \Big| \leq & \mathrm{osc}_{D^m}(\varphi) \Big(\int_{\partial D^m} |\lambda_\varepsilon| \, \mathrm{d}\mathcal{H}^1 + \int_{\partial D^m} |\widehat{\lambda}_\varepsilon| \, \mathrm{d}\mathcal{H}^1 \Big) \\ \leq & \mathrm{diam}(D^m) C |\log \varepsilon|^3 \,,\end{aligned}$$

which, summing over m, and using (3.46), yields

$$\sum_{m=1}^{M_\varepsilon} \Big| \int_{\partial D^m} \varphi(\lambda_\varepsilon - \widehat{\lambda}_\varepsilon) \cdot \boldsymbol{\nu} \, \mathrm{d}\mathcal{H}^1 \Big| \leq C\varepsilon |\log \varepsilon|^5 \,. \tag{3.62}$$

Analogously, using (3.52) and (3.57) together with (3.19), we obtain

$$\sum_{B \in \mathscr{I}_\varepsilon^<} \Big| \int_{\partial B} \varphi(\lambda_\varepsilon - \widehat{\lambda}_\varepsilon) \cdot \boldsymbol{\nu} \, \mathrm{d}\mathcal{H}^1 \Big| \leq C\varepsilon |\log \varepsilon|^3 \,. \tag{3.63}$$

Therefore, by (3.61), (3.62), and (3.63), for ε small enough, we get

$$\Big| \langle J u_\varepsilon - \pi \widehat{\mu}_\varepsilon, \varphi \rangle_\Omega \Big| \leq C\varepsilon |\log \varepsilon|^5 \,,$$

for some constant C independent of φ. Taking the supremum on $\varphi \in C^1_{\mathrm{c}}(\Omega)$ such that $\|\varphi\|_{C^{0,1}} \leq 1$ we infer that $\|J u_\varepsilon - \pi \widehat{\mu}_\varepsilon\|_{\mathrm{flat},\Omega} \leq C\varepsilon |\log \varepsilon|^5$, whence (3.50) follows. This concludes the proof of (i).

Proof of (ii). We can assume without loss of generality that $\{u_\varepsilon\}_\varepsilon$ satisfies (3.4) for some $C > 0$. By (3.13) and (3.11), using (3.17) and applying Theorem 3.4(ii), we get that

$$\liminf_{\varepsilon \to 0} \frac{\mathcal{F}_\varepsilon(u_\varepsilon)}{|\log \varepsilon|} \geq \liminf_{\varepsilon \to 0} \frac{F(\mathscr{B}_\varepsilon, \mu_\varepsilon, \Omega)}{|\log \mathcal{R}ad(\mathscr{B}_\varepsilon)|} \geq \pi |\mu|(\Omega) \,,$$

i.e., the claim.

Proof of (iii). We divide the proof into three steps. In the first one we construct the sequence $\{u_\varepsilon\}_\varepsilon$ and we show that it satisfies (3.6); whereas the second and third steps are devoted to show that $\|J u_\varepsilon - \pi \mu\|_{\mathrm{flat},\Omega} \to 0$ (as $\varepsilon \to 0$).

Step 1 (Definition of u_ε and energy estimates): Let $\mu = \sum_{i=1}^I z^i \delta_{x^i}$ with $z^i \in \mathbb{Z} \setminus \{0\}$ and $x^i \in \overline{\Omega}'$ for every $i = 1, \dots, I$. By standard approximation arguments in Γ-convergence, we can assume that $x^i \in \Omega'$, and that $|z^i| = 1$, for every $i = 1, \dots, I$.

Let $R > 0$ be such that the balls $B_{2R}(x^i)$ are (essentially) pairwise disjoint and contained in Ω'. For every $\rho > 0$ we set $\Omega_\rho(\mu) := \Omega \setminus \bigcup_{i=1}^I \overline{B}_\rho(x^i)$ and we define $\Theta : \Omega_R(\mu) \to \mathbb{R}$ as $\Theta(\cdot) := \sum_{i=1}^I z^i \vartheta(\cdot - x^i)$, where ϑ is the function defined in (3.55).

For every $0 < \rho_1 < \rho_2$ and for every $x \in \mathbb{R}^2$ we set $A_{\rho_1,\rho_2}(x) := B_{\rho_2}(x) \setminus \overline{B}_{\rho_1}(x)$. Let moreover $\sigma_R \in C^\infty(A_{R,2R}(0); [0,1])$ be such that $\sigma_R \equiv 1$ in $A_{\frac{7}{4}R,2R}(0)$ and $\sigma_R \equiv 0$ in $A_{R,\frac{5}{4}R}(0)$. Analogously, for every $\varepsilon > 0$, let $\sigma_\varepsilon \in C^\infty(B_\varepsilon(0); [0,1])$ be such that $\sigma_\varepsilon \equiv 0$ in $B_{\frac{\varepsilon}{4}}(0)$ and $\sigma_\varepsilon \equiv 1$ in $A_{\frac{3}{4}\varepsilon,\varepsilon}(0)$. We can also assume that

$$|\nabla \sigma_\varepsilon(x)| \leq \frac{C}{\varepsilon} \quad \text{for every } x \in B_\varepsilon(0) \,, \qquad |\nabla \sigma_R(x)| \leq \frac{C}{R} \quad \text{for every } x \in A_{R,\frac{5}{4}R}(0) \,, \tag{3.64}$$

for some constant $C > 0$ independent of ε (and of x).

For $\varepsilon < R$ we define the function $\vartheta_\varepsilon : \Omega \to \mathbb{R}$ as

$$\vartheta_\varepsilon(x) := \begin{cases} \sigma_\varepsilon(x - x^i) z^i \vartheta(x - x^i) & \text{if } x \in B_\varepsilon(x^i) \setminus \{x^i\} \text{ for some } i \\ z^i \vartheta(x - x^i) & \text{if } x \in A_{\varepsilon,R}(x^i) \text{ for some } i \\ (1 - \sigma_R(x - x^i)) z^i \vartheta(x - x^i) + \sigma_R(x - x^i) \Theta(x) & \text{if } x \in A_{R,2R}(x^i) \text{ for some } i \\ \Theta(x) & \text{if } x \in \Omega_{2R}(\mu) \end{cases}$$

and we define $u_\varepsilon : \Omega \to \mathbb{S}^1$ as

$$u_\varepsilon(\cdot) := e^{\imath \vartheta_\varepsilon(\cdot)} \,. \tag{3.65}$$

Then, $S_{u_\varepsilon} = \bigcup_{i=1}^I S^i_\varepsilon$ where

$$S^i_\varepsilon := \Big\{(x^i_1; x_2) \,:\, x^i_2 - \frac{3}{4}\varepsilon < x_2 < x^i_2 - \frac{\varepsilon}{4}\Big\} \qquad \text{for every } i = 1, \dots, I\,, \tag{3.66}$$

and hence

$$\mathcal{H}^1(\overline{S}_{u_\varepsilon}) = I\frac{\varepsilon}{2} = |\mu|(\Omega)\frac{\varepsilon}{2}\,. \tag{3.67}$$

In particular, the energy term

$$\frac{\mathcal{H}^1(\overline{S}_{u_\varepsilon})}{\varepsilon|\log\varepsilon|} \to 0 \qquad \text{as } \varepsilon \to 0^+\,. \tag{3.68}$$

Let us analyse the stored elastic energy. It is easy to see that

$$\frac{1}{2}\int_{\Omega_{2R}(\mu)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x \leq \frac{1}{2}\int_{\Omega_{2R}(\mu)} |\nabla\Theta|^2 \,\mathrm{d}x \leq C(I)\log\frac{\mathrm{diam}(\Omega)}{2R}\,, \tag{3.69}$$

where $C(I) > 0$ is a constant depending only on I. Similarly, by the second estimate in (3.64), we deduce

$$\begin{aligned}\frac{1}{2}\int_{A_{R,2R}(x^i)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x \leq& \int_{A_{R,2R}(x^i)} |\nabla\vartheta(x-x^i)|^2 \,\mathrm{d}x \\ &+ \int_{A_{R,2R}(x^i)} \Big|\nabla\Big(\sigma_R(x-x^i)\Big(\sum_{j\neq i} z^j\vartheta(x-x^j)\Big)\Big)\Big|^2 \,\mathrm{d}x \\ \leq& C(I,R)\,,\end{aligned} \tag{3.70}$$

where we have estimated

$$\begin{aligned}&\int_{A_{R,2R}(x^i)} \Big|\nabla\Big(\sigma_R(x-x^i)\Big(\sum_{j\neq i} z^j\vartheta(x-x^j)\Big)\Big)\Big|^2 \,\mathrm{d}x \\ &\qquad \leq C(I) + \int_{A_{R,2R}(x^i)} \Big|\sum_{j\neq i} z^j\nabla\vartheta(x-x^j)\Big|^2 \,\mathrm{d}x \leq C(I,R).\end{aligned}$$

Furthermore, by the first estimate in (3.64), for every $i = 1, \dots, I$, and for every $x \in B_R(x^i)$ it holds

$$\begin{aligned}|\nabla u_\varepsilon(x)| = |\nabla\vartheta_\varepsilon(x)| &\leq 2\pi\Big|\nabla\sigma_\varepsilon(x-x^i)\chi_{B_\varepsilon(x^i)}(x)\Big| + \Big|\nabla\vartheta(x-x^i)\chi_{A_{\frac{\varepsilon}{4},R}(x^i)}(x)\Big| \\ &\leq \frac{C}{\varepsilon}\chi_{B_\varepsilon(x^i)}(x) + \frac{1}{|x-x^i|}\chi_{A_{\frac{\varepsilon}{4},R}(x^i)}(x)\,,\end{aligned} \tag{3.71}$$

for some constant C independent of ε (and of x). Here, for any measurable set E, χ_E denotes the characteristic function of the set E. It follows that, for every $i = 1, \dots, I$,

$$\begin{aligned}\frac{1}{2}\int_{B_\varepsilon(x^i)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x &\leq C\,, \\ \frac{1}{2}\int_{A_{\varepsilon,R}(x^i)} |\nabla u_\varepsilon|^2 \,\mathrm{d}x &\leq \pi\log\frac{R}{\varepsilon}\,,\end{aligned} \tag{3.72}$$

which, together with (3.69) and (3.70), implies that

$$\limsup_{\varepsilon\to 0} \frac{1}{2|\log\varepsilon|}\int_\Omega |\nabla u_\varepsilon|^2 \,\mathrm{d}x \leq \pi I = \pi|\mu|(\Omega)\,.$$

This, together with (3.68), yields (3.6).

Now, in order to conclude the proof of (iii) of Theorem 3.1, it remains to prove that

$$\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},\Omega} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{3.73}$$

To this purpose in view of Remark 2.9, it is enough to prove the following two facts: There exists a constant $C > 0$ (independent of ε) such that

$$|Ju_\varepsilon|(\Omega) \leq C\,, \tag{3.74}$$

and

$$u_\varepsilon \overset{\text{strict}}{\rightharpoonup} \bar{u} \qquad \text{in } BV(\Omega;\mathbb{R}^2)\,, \tag{3.75}$$

for some $\bar{u} \in W^{1,1}(\Omega;\mathbb{S}^1)$ with $\mathrm{Det}(\nabla \bar{u}) = \pi\mu$.

Step 2: Proof of (3.74)**.** In order to show (3.74), thanks to Remark 2.8, it is sufficient to show that for all ε small enough there exists a sequence of maps $\{v_{\varepsilon,k}\}_{k\in\mathbb{N}} \subset C^1(\Omega;\mathbb{R}^2) \cap BV(\Omega;\mathbb{R}^2)$ such that

$$\sup_{k\in\mathbb{N}} |Jv_{\varepsilon,k}|(\Omega) \le C\,, \tag{3.76}$$

for some $C > 0$ independent of k and ε, and

$$v_{\varepsilon,k} \overset{\text{strict}}{\rightharpoonup} u_\varepsilon \quad \text{in } BV(\Omega;\mathbb{R}^2) \qquad (\text{as } k \to +\infty)\,. \tag{3.77}$$

Furthermore, by standard density arguments, it is enough to construct $\{v_{\varepsilon,k}\}_k \subset W^{1,\infty}(\Omega;\mathbb{R}^2)$. Let $0 < \varepsilon \ll R$ be fixed. For every $i = 1,\dots,I$, $k \in \mathbb{N}$, $t_1 < t_2$ we set $S^i_{\varepsilon,k,t_1,t_2} := [x^i_1 - \frac{\varepsilon}{k}, x^i_1 + \frac{\varepsilon}{k}] \times [x^i_2 + t_1, x^i_2 + t_2]$. We will modify u_ε in the set $S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},\frac{\varepsilon}{k}}$, that is a neighborhood of the jump set S^i_ε in (3.66). Precisely, for every $i = 1,\dots,I$, and for every $k \in \mathbb{N}$, let $u^i_{\varepsilon,k,\mathrm{int}} \in C^\infty(S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},\frac{\varepsilon}{k}};\mathbb{R}^2)$ be defined as

$$\begin{aligned} u^i_{\varepsilon,k,\mathrm{int}}(x) \;:=\; & \Big(\frac{1}{2} - \frac{k}{2\varepsilon}(x_1 - x^i_1)\Big) u_\varepsilon\Big(x^i_1 - \frac{\varepsilon}{k}; x_2\Big) \\ & + \Big(\frac{1}{2} + \frac{k}{2\varepsilon}(x_1 - x^i_1)\Big) u_\varepsilon\Big(x^i_1 + \frac{\varepsilon}{k}; x_2\Big). \end{aligned}$$

Moreover, for every $i = 1,\dots,I$, and for every $k \in \mathbb{N}$ we denote by $u^i_{\varepsilon,k,\mathrm{bottom}} \in C^\infty(S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},-\varepsilon};\mathbb{R}^2)$ and $u^i_{\varepsilon,k,\mathrm{up}}(S^i_{\varepsilon,k,0,\frac{\varepsilon}{k}};\mathbb{R}^2)$ the functions

$$\begin{aligned} u^i_{\varepsilon,k,\mathrm{bottom}}(x) \;&:=\; \frac{k}{\varepsilon}(x^i_2 - \varepsilon - x_2) u_\varepsilon(x) + \Big(1 - \frac{k}{\varepsilon}(x^i_2 - \varepsilon - x_2)\Big) u^i_{\varepsilon,k,\mathrm{int}}(x)\,, \\ u^i_{\varepsilon,k,\mathrm{up}}(x) \;&:=\; \frac{k}{\varepsilon}(x_2 - x^i_2) u_\varepsilon(x) + \Big(1 - \frac{k}{\varepsilon}(x_2 - x^i_2)\Big) u^i_{\varepsilon,k,\mathrm{int}}(x)\,. \end{aligned}$$

Finally we define $v_{\varepsilon,k} : \Omega \to \mathbb{R}^2$ as

$$v_{\varepsilon,k}(x) := \begin{cases} u^i_{\varepsilon,k,\mathrm{int}}(x) & \text{if } x \in S^i_{\varepsilon,k,-\varepsilon,0} \text{ for some } i \\ u^i_{\varepsilon,k,\mathrm{up}}(x) & \text{if } x \in S^i_{\varepsilon,k,0,\frac{\varepsilon}{k}} \text{ for some } i \\ u^i_{\varepsilon,k,\mathrm{bottom}}(x) & \text{if } x \in S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},-\varepsilon} \text{ for some } i \\ u_\varepsilon(x) & \text{elsewhere in } \Omega\,. \end{cases}$$

An easy check shows that the function $v_{\varepsilon,k} : \Omega \to \mathbb{R}^2$ is Lipschitz continuous in Ω. Indeed, the function u_ε is C^1 out of the jump set, $u^i_{\varepsilon,k,\mathrm{int}}$ is Lipschitz continuous in $S^i_{\varepsilon,k,0,\frac{\varepsilon}{k}}$ (and smooth in its interior), and $u^i_{\varepsilon,k,\mathrm{bottom}}$ and $u^i_{\varepsilon,k,\mathrm{up}}$ are Lipschitz as well in $S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},-\varepsilon}$ and $S^i_{\varepsilon,k,0,\frac{\varepsilon}{k}}$ (respectively).

We now check that $\{v_{\varepsilon,k}\}_{k\in\mathbb{N}}$ satisfies (3.77). To this purpose it is enough to show that

$$\lim_{k\to+\infty} \int_{S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},\frac{\varepsilon}{k}}} |\nabla v_{\varepsilon,k}|\,\mathrm{d}x = \int_{S^i_\varepsilon} |[u_\varepsilon]|\,\mathrm{d}\mathcal{H}^1,$$

for all $i = 1,\dots,I$. Furthermore, since $v_{\varepsilon,k} \to u_\varepsilon$ pointwise a.e. in Ω (as $k \to +\infty$), it is sufficient to prove that

$$\limsup_{k\to+\infty} \int_{S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},\frac{\varepsilon}{k}}} |\nabla v_{\varepsilon,k}|\,\mathrm{d}x \le \int_{S^i_\varepsilon} |[u_\varepsilon]|\,\mathrm{d}\mathcal{H}^1\,, \tag{3.78}$$

for all $i = 1,\dots,I$. We notice that

$$\begin{aligned} \frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_1}(x) &= \frac{k}{2\varepsilon} u_\varepsilon\Big(x^i_1 - \frac{\varepsilon}{k}; x_2\Big) - \frac{k}{2\varepsilon} u_\varepsilon\Big(x^i_1 + \frac{\varepsilon}{k}; x_2\Big)\,, \\ \frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_2}(x) &= \Big(\frac{1}{2} - \frac{k}{2\varepsilon}(x_1 - x^i_1)\Big) \frac{\partial u_\varepsilon}{\partial x_2}\Big(x^i_1 - \frac{\varepsilon}{k}; x_2\Big) + \Big(\frac{1}{2} + \frac{k}{2\varepsilon}(x_1 - x^i_1)\Big) \frac{\partial u_\varepsilon}{\partial x_2}\Big(x^i_1 + \frac{\varepsilon}{k}; x_2\Big)\,, \end{aligned}$$

whence, using that

$$|x_1 - x_1^i| \le \frac{\varepsilon}{k} \text{ in } S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},\frac{\varepsilon}{k}}\,, \tag{3.79}$$

we deduce

$$\begin{aligned}
\int_{S^i_{\varepsilon,k,-\varepsilon,0}} |\nabla v_{\varepsilon,k}|\,\mathrm{d}x &= \int_{S^i_{\varepsilon,k,-\varepsilon,0}} |\nabla u^i_{\varepsilon,k,\mathrm{int}}|\,\mathrm{d}x\\
&\le \int_{S^i_{\varepsilon,k,-\varepsilon,0}} \Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_1}\Big|\,\mathrm{d}x + \int_{S^i_{\varepsilon,k,-\varepsilon,0}} \Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_2}\Big|\,\mathrm{d}x\\
&\le \int_{-\varepsilon}^0 \Big|u_\varepsilon\Big(x_1^i + \frac{\varepsilon}{k}; x_2\Big) - u_\varepsilon\Big(x_1^i - \frac{\varepsilon}{k}; x_2\Big)\Big|\,\mathrm{d}x_2\\
&\quad + \int_{-\varepsilon}^0 \int_{-\frac{\varepsilon}{k}}^{\frac{\varepsilon}{k}} \Big|\frac{\partial u_\varepsilon}{\partial x_2}\Big(x_1^i - \frac{\varepsilon}{k}; x_2\Big)\Big| + \Big|\frac{\partial u_\varepsilon}{\partial x_2}\Big(x_1^i + \frac{\varepsilon}{k}; x_2\Big)\Big|\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&= \int_{S^i_\varepsilon} |[u_\varepsilon]|\,\mathrm{d}\mathcal{H}^1 + \mathrm{o}_k(1)\,,
\end{aligned} \tag{3.80}$$

where $\mathrm{o}_k(1)$ tends to 0 as $k \to +\infty$. Furthermore,

$$\begin{aligned}
\frac{\partial u^i_{\varepsilon,k,\mathrm{bottom}}}{\partial x_1}(x) &= \frac{k}{\varepsilon}(x_2^i - \varepsilon - x_2)\frac{\partial u_\varepsilon}{\partial x_1}(x) + \Big(1 - \frac{k}{\varepsilon}(x_2^i - \varepsilon - x_2)\Big)\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_1}(x)\,,\\
\frac{\partial u^i_{\varepsilon,k,\mathrm{bottom}}}{\partial x_2}(x) &= -\frac{k}{\varepsilon}u_\varepsilon(x) + \frac{k}{\varepsilon}u^i_{\varepsilon,k,\mathrm{int}}(x) + \frac{k}{\varepsilon}(x_2^i - \varepsilon - x_2)\frac{\partial u_\varepsilon}{\partial x_2}(x)\\
&\qquad + \Big(1 - \frac{k}{\varepsilon}(x_2^i - \varepsilon - x_2)\Big)\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_2}(x)\,,\\
\frac{\partial u^i_{\varepsilon,k,\mathrm{up}}}{\partial x_1}(x) &= \frac{k}{\varepsilon}(x_2 - x_2^i)\frac{\partial u_\varepsilon}{\partial x_1}(x) + \Big(1 - \frac{k}{\varepsilon}(x_2 - x_2^i)\Big)\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_1}(x)\,,\\
\frac{\partial u^i_{\varepsilon,k,\mathrm{up}}}{\partial x_2}(x) &= \frac{k}{\varepsilon}u_\varepsilon(x) - \frac{k}{\varepsilon}u^i_{\varepsilon,k,\mathrm{int}}(x) + \frac{k}{\varepsilon}(x_2 - x_2^i)\frac{\partial u_\varepsilon}{\partial x_2}(x) + \Big(1 - \frac{k}{\varepsilon}(x_2 - x_2^i)\Big)\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_2}(x)\,;
\end{aligned}$$

hence, using that $|u_\varepsilon|, |u^i_{\varepsilon,k,\mathrm{int}}| \le 1$, (3.79) and the fact that

$$|\varepsilon + x_2 - x_2^i| \le \frac{\varepsilon}{k} \text{ in } S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},-\varepsilon}\,, \quad |x_2 - x_2^i| \le \frac{\varepsilon}{k} \text{ in } S^i_{\varepsilon,k,0,\frac{\varepsilon}{k}},$$

we can estimate

$$\begin{aligned}
&\Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_1}(x)\Big| \le \frac{k}{\varepsilon}\,,\\
&\Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_2}(x)\Big| \le \Big|\nabla u_\varepsilon\Big(x_1^i - \frac{\varepsilon}{k}; x_2\Big)\Big| + \Big|\nabla u_\varepsilon\Big(x_1^i + \frac{\varepsilon}{k}; x_2\Big)\Big|\,,\\
&\Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{bottom}}}{\partial x_1}(x)\Big| \le |\nabla u_\varepsilon(x)| + \frac{k}{\varepsilon}\,,\\
&\Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{bottom}}}{\partial x_2}(x)\Big| \le 2\frac{k}{\varepsilon} + |\nabla u_\varepsilon(x)| + \Big|\nabla u_\varepsilon\Big(x_1^i - \frac{\varepsilon}{k}; x_2\Big)\Big| + \Big|\nabla u_\varepsilon\Big(x_1^i + \frac{\varepsilon}{k}; x_2\Big)\Big|\,,\\
&\Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{up}}}{\partial x_1}(x)\Big| \le |\nabla u_\varepsilon(x)| + \frac{k}{\varepsilon}\,,\\
&\Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{up}}}{\partial x_2}(x)\Big| \le 2\frac{k}{\varepsilon} + |\nabla u_\varepsilon(x)| + \Big|\nabla u_\varepsilon\Big(x_1^i - \frac{\varepsilon}{k}; x_2\Big)\Big| + \Big|\nabla u_\varepsilon\Big(x_1^i + \frac{\varepsilon}{k}; x_2\Big)\Big|\,.
\end{aligned} \tag{3.81}$$

By (3.81) and (3.72) it follows, in particular, that the integrals of $|\nabla v_{\varepsilon,k}|$ on $S^i_{\varepsilon,k,-\varepsilon,-\varepsilon-\frac{\varepsilon}{k}}$ and $S^i_{\varepsilon,k,0,\frac{\varepsilon}{k}}$ are both negligible as $k \to \infty$. This fact, together with (3.80), implies (3.78) and, in turn, (3.77).

In order to prove (3.76), we notice that, in view of (3.81) and of (3.71), it holds

$$\begin{aligned}\int_{S^i_{\varepsilon,k,-\varepsilon,0}} |Jv_{\varepsilon,k}|\,\mathrm{d}x &= \int_{S^i_{\varepsilon,k,-\varepsilon,0}} |Ju^i_{\varepsilon,k,\mathrm{int}}|\,\mathrm{d}x\\ &\le \int_{S^i_{\varepsilon,k,-\varepsilon,0}} \Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_1}(x)\Big|\Big|\frac{\partial u^i_{\varepsilon,k,\mathrm{int}}}{\partial x_2}(x)\Big|\,\mathrm{d}x\\ &\le 2\int_{-\varepsilon}^0 \Big|\nabla u_\varepsilon\Big(x^i_1-\frac{\varepsilon}{k};x_2\Big)\Big| + \Big|\nabla u_\varepsilon\Big(x^i_1+\frac{\varepsilon}{k};x_2\Big)\Big|\,\mathrm{d}x_2\\ &\le C\,,\end{aligned} \tag{3.82}$$

for some constant C independent of ε and k.

Analogously, using again (3.81) and (3.72), one can prove that there exists a universal constant $C>0$ such that

$$\begin{aligned}\int_{S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},-\varepsilon}} |Jv_{\varepsilon,k}|\,\mathrm{d}x &= \int_{S^i_{\varepsilon,k,-\varepsilon-\frac{\varepsilon}{k},-\varepsilon}} |Ju^i_{\varepsilon,k,\mathrm{bottom}}|\,\mathrm{d}x \le C\,,\\ \int_{S^i_{\varepsilon,k,0,\frac{\varepsilon}{k}}} |Jv_{\varepsilon,k}|\,\mathrm{d}x &= \int_{S^i_{\varepsilon,k,0,\frac{\varepsilon}{k}}} |Ju^i_{\varepsilon,k,\mathrm{up}}|\,\mathrm{d}x \le C\,.\end{aligned} \tag{3.83}$$

By (3.82) and (3.83), the estimate in (3.76) follows.

Step 3: Proof of (3.75)**.** It is easy to see that the functions ϑ_ε tend pointwise (as $\varepsilon\to 0$) to $\bar\vartheta$ defined as

$$\bar\vartheta(x) := \begin{cases} z^i\vartheta(x-x^i) & \text{if } x\in B_R(x^i) \text{ for some } i\\ (1-\sigma_R(x-x^i))z^i\vartheta(x-x^i)+\sigma_R(x-x^i)\Theta(x) & \text{if } x\in A_{R,2R}(x^i) \text{ for some } i\\ \Theta(x) & \text{if } x\in\Omega_{2R}(\mu).\end{cases}$$

By definition, the function $\bar u$ defined by $\bar u(x) := e^{\imath\bar\vartheta(x)}$ belongs to $W^{1,1}(\Omega;\mathbb{S}^1)$ and satisfies $\mathrm{Det}(\nabla\bar u)=\pi\mu$. Notice moreover that, since $\bar\vartheta-\vartheta_\varepsilon\neq 0$ only on $\bigcup_{i=1}^I B_\varepsilon(x^i)$ and $\bar u\in W^{1,1}(\Omega;\mathbb{S}^1)$, we can prove that $u_\varepsilon \overset{\text{strict}}{\rightharpoonup} \bar u$ in $BV(\Omega;\mathbb{R}^2)$ if we show that

$$|\mathrm{D}u_\varepsilon|(\cup_{i=1}^I \overline{B}_\varepsilon(x^i))\to 0 \qquad \text{as } \varepsilon\to 0\,. \tag{3.84}$$

But, by (3.72) and by Hölder inequality

$$\int_{B_\varepsilon(x^i)} |\nabla u_\varepsilon|\,\mathrm{d}x \le C\varepsilon\Big(\int_{B_\varepsilon(x^i)} |\nabla u_\varepsilon|^2\Big)^{\frac12}\,\mathrm{d}x \le C\varepsilon \qquad \forall i=1,\dots,I\,,$$

which together with (3.67) implies (3.84). This concludes the proof of (3.75), and in turn of (3.73) and of the whole Theorem 3.1. □

Dirichlet boundary conditions. We conclude this section dealing with prescribed boundary conditions on $\partial\Omega$. To this purpose, let $\widetilde\Omega,\widehat\Omega\subset\mathbb{R}^2$ be bounded and open sets with $\widetilde\Omega\subset\subset\Omega\subset\subset\widehat\Omega$, and let $w\in H^1(\widehat\Omega\setminus\widetilde\Omega;\mathbb{S}^1)$. Admissible functions for the problem with boundary conditions will be given by

$$\mathcal{AD}^w(\Omega) := \{u\in SBV^2(\widehat\Omega;\mathbb{S}^1) : u=w \text{ on } \widehat\Omega\setminus\overline{\Omega}\}.$$

For every $\varepsilon>0$, let $\mathcal{F}^w_\varepsilon : SBV^2(\Omega;\mathbb{S}^1)\to[0,\infty]$ be the functional defined by

$$\mathcal{F}^w_\varepsilon(u) := \begin{cases} \dfrac12\displaystyle\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{\varepsilon}\mathcal{H}^1(\overline{S}_u) & \text{if } u\in\mathcal{AD}^w(\Omega)\\ +\infty & \text{elsewhere in } SBV^2(\Omega;\mathbb{S}^1)\,.\end{cases}$$

Notice that we can have $S_u\cap\partial\Omega\neq\varnothing$, and that we take the closure $\overline{S}_u$ of S_u in $\widehat\Omega$ (in particular, the functional $\mathcal{F}_\varepsilon$ penalizes jumps of u also on $\partial\Omega$).

We show that, up to slight modifications in the proof of Theorem 3.1, we can obtain a compactness and Γ-convergence result for the functional $\mathcal{F}^w_\varepsilon$. Specifically, we will prove the following statement.

Theorem 3.5. *The following Γ-convergence result holds true.*

(i) *(Compactness) Let $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ be such that*

$$\sup_{\varepsilon>0} \frac{\mathcal{F}^w_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \le C, \tag{3.85}$$

for some $C>0$. Then there exists $\mu \in X(\overline{\Omega})$, such that , up to a subsequence, $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},U} \to 0$ (as $\varepsilon\to 0$), for any open set U with $\Omega \subset\subset U \subset\subset \widehat{\Omega}$. Moreover, it holds

$$\mu(U) = \deg(w,\partial\Omega)\,, \tag{3.86}$$

for any open set U with $\Omega \subset\subset U \subset\subset \widehat{\Omega}$.

(ii) *(Γ-liminf inequality) For every $\mu \in X(\overline{\Omega})$ and for every $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ such that $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},U} \to 0$ (as $\varepsilon\to 0$) for any open set U with $\Omega \subset\subset U \subset\subset \widehat{\Omega}$, it holds*

$$\pi|\mu|(\overline{\Omega}) \le \liminf_{\varepsilon\to 0} \frac{\mathcal{F}^w_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,.$$

(iii) *(Γ-limsup inequality) For every $\mu \in X(\overline{\Omega})$ satisfying (3.86), there exists $\{u_\varepsilon\}_\varepsilon \subset SBV^2(\Omega;\mathbb{S}^1)$ with $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},U} \to 0$ (as $\varepsilon\to 0$) for any open set U with $\Omega \subset\subset U \subset\subset \widehat{\Omega}$, such that*

$$\pi|\mu|(\overline{\Omega}) \ge \limsup_{\varepsilon\to 0} \frac{\mathcal{F}^w_\varepsilon(u_\varepsilon)}{|\log\varepsilon|}\,.$$

Proof. We divide the proof into three steps corresponding to each of the items in the statement.

Step 1: Proof of (i). Notice that, by (3.85), we can assume without loss of generality that $u_\varepsilon \in \mathcal{AD}(\widehat{\Omega},\Omega)$ for all $\varepsilon>0$ (see (3.1)). Moreover, recalling the definition of $\mathcal{F}_\varepsilon$ in (3.2), by (3.85) we have

$$\mathcal{F}_\varepsilon(u_\varepsilon;\widehat{\Omega}) \le \mathcal{F}^w_\varepsilon(u_\varepsilon) + \|w\|^2_{H^1(\widehat{\Omega}\setminus\overline{\Omega};\mathbb{R}^2)} \le C|\log\varepsilon|\,. \tag{3.87}$$

Hence, we can apply Theorem 3.1(i) to deduce that, up to a subsequence, $\|Ju_\varepsilon - \pi\widehat{\mu}\|_{\mathrm{flat},\widehat{\Omega}} \to 0$ for some $\widehat{\mu} \in X(\widehat{\Omega})$. Therefore, since, by construction, $\operatorname{supp} Ju_\varepsilon \subset \overline{\Omega}$, then $\operatorname{supp}\widehat{\mu} \subset \overline{\Omega}$, and therefore $\|Ju_\varepsilon - \pi\mu\|_{\mathrm{flat},U} \to 0$, for every open set U with $\Omega \subset\subset U \subset\subset \widehat{\Omega}$. Finally, if U is an open set with $\Omega \subset\subset U \subset\subset \widehat{\Omega}$, we choose a function $\varphi \in C^\infty_{\mathrm{c}}(\widehat{\Omega})$ which equals 1 on U, and write

$$\begin{aligned}\pi\mu(U) &= \langle\pi\mu,\varphi\rangle_{\widehat{\Omega}} = \lim_{\varepsilon\to 0}\langle Ju_\varepsilon,\varphi\rangle_{\widehat{\Omega}} = \lim_{\varepsilon\to 0}\int_{\widehat{\Omega}} \nabla\varphi\cdot\,\mathrm{d}\lambda_{u_\varepsilon} = \lim_{\varepsilon\to 0}\int_{\widehat{\Omega}\setminus\overline{U}} \nabla\varphi\cdot\,\mathrm{d}\lambda_{u_\varepsilon}\\ &= \lim_{\varepsilon\to 0}\int_{\partial U} \lambda_{u_\varepsilon}\cdot\boldsymbol{\nu}\,\mathrm{d}\mathcal{H}^1 = \lim_{\varepsilon\to 0}\pi\deg(u_\varepsilon,\partial U) = \pi\deg(w,\partial U)\,,\end{aligned}$$

where we have used (1.3) and the fact that $u_\varepsilon \in \mathcal{AD}^w(\widehat{\Omega},\Omega)$.

Step 2: Proof of (ii). We can assume without loss of generality that $\{u_\varepsilon\}_\varepsilon$ satisfies (3.85). By (3.87) and by Theorem 3.1(ii), we obtain

$$\liminf_{\varepsilon\to 0}\frac{\mathcal{F}^w_\varepsilon(u_\varepsilon)}{|\log\varepsilon|} \ge \liminf_{\varepsilon\to 0}\frac{\mathcal{F}_\varepsilon(u_\varepsilon;\widehat{\Omega})}{|\log\varepsilon|} \ge \pi|\mu|(\overline{\Omega})\,,$$

i.e., the claim.

Step 3: Proof of (iii). Since every $\mu \in X(\overline{\Omega})$ can be approximated by measures in $X(\Omega)$ with respect to the flat distance, by standard density arguments in Γ-convergence, it is enough to construct the recovery sequence only for measures $\mu \in X(\Omega)$.

Let $\widetilde{\Omega} \subset\subset \Omega' \subset\subset \Omega$ be an open set such that $\operatorname{supp}\mu \subset \Omega'$. Let moreover $\eta \in C^\infty(\Omega\setminus\Omega')$ be a cut-off function with $\eta\equiv 0$ in a neighborhood of $\partial\Omega'$ and $\eta\equiv 1$ in a neighborhood of $\partial\Omega$. Let furthermore $\{\widetilde{u}_\varepsilon\}_\varepsilon = \{e^{\imath\widetilde{\vartheta}_\varepsilon}\}_\varepsilon$ indicate the sequence provided by (3.65). Then there exists a function $\vartheta^w \in SBV^2(\widehat{\Omega}\setminus\widetilde{\Omega})$ such that $w = e^{\imath\vartheta^w}$ on $\widehat{\Omega}\setminus\widetilde{\Omega}$, $S_{\vartheta^w} = S_{\widetilde{\vartheta}_\varepsilon}$ in $\Omega\setminus\overline{\Omega'}$, and $[\vartheta^w] = [\widetilde{\vartheta}_\varepsilon]$ in $\Omega\setminus\overline{\Omega'}$. The last property can be achieved thanks to the fact that μ satisfies condition (3.86).

For every $\varepsilon > 0$ we define the function $u_\varepsilon : \widehat{\Omega} \to \mathbb{S}^1$ as $u_\varepsilon(\cdot) := e^{\imath\vartheta_\varepsilon(\cdot)}$, where the lifting ϑ_ε is defined by

$$\vartheta_\varepsilon(x) := \begin{cases} \widetilde{\vartheta}_\varepsilon(x) & \text{if } x \in \Omega' \\ (1-\eta(x))\widetilde{\vartheta}_\varepsilon(x) + \eta(x)\vartheta^w(x) & \text{if } x \in \Omega \setminus \Omega' \\ \vartheta^w(x) & \text{if } x \in \widehat{\Omega} \setminus \Omega\,. \end{cases}$$

It is easy to check that the sequence $\{u_\varepsilon\}_\varepsilon$ satisfies the desired properties. □

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