# On the Action of the Symplectic Group on the Siegel Upper Half Plane

by

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*É bom ser fiel às fragas,*
*mas é melhor ser fiel à razão e à vida inteira.*

*Miguel Torga.*

To my parents Ilda and Henrique
and my grandmother Maria Hermínia.

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# Contents

**1 Introduction** **1**
1 Notation and basic results . . . . . 1
2 The hyperbolic plane . . . . . 5

**2 The Symplectic Group and the Siegel Upper Half Plane** **9**
1 Basic definitions and results . . . . . 9
2 Symplectic linear algebra . . . . . 13
3 Other models . . . . . 18

**3 Compactifications** **23**
1 The boundaries . . . . . 23
2 The Shilov boundary . . . . . 30

**4 $\mathrm{Sp}_4\mathbb{R}$ and $\mathrm{SH}_2$** **39**
1 Normal forms . . . . . 39
2 Fixed points: classification . . . . . 48
3 The boundary in the projective model . . . . . 53
4 Fixed points: proofs . . . . . 55

**5 On Discrete Subgroups** **67**
1 General concepts . . . . . 67
2 Limit sets . . . . . 69

**Bibliography** **78**

**Vita** **80**

# Summary

The symplectic group $\mathrm{Sp}_{2n}\mathbb{R}$ is one possible generalization of the group $\mathrm{SL}_2\mathbb{R} = \mathrm{Sp}_2\mathbb{R}$ to higher dimensions. This generalization goes further, since, like the latter, they act on a symmetric homogeneous space, the Siegel upper half plane, and this action has quite a few similarities with the action of $\mathrm{SL}_2\mathbb{R}$ on the hyperbolic plane. A study of this action was done by Carl Ludwig Siegel in 1943, and published in his book "Symplectic Geometry", where not only the analytical and geometrical aspects of the action are considered, but also some applications to number theory. In this work, we present a study of this action, inspired by modern studies of Fuchsian and Kleinian groups as presented for example in Beardon's "The Geometry of Discrete Groups" [Bea].

Since the Fuchsian groups, that is, the discrete subgroups of $\mathrm{SL}_2\mathbb{R}$, serve as a motivation for our studies, we review some relevant well known results about these groups in Chapter 1.

A basic study of the group and the half plane is done in Chapter 2. In this chapter we review some algebraic properties of the symplectic group, presenting new proofs to a few results, such as the fact that any symplectic matrix has determinant 1. We also present a direct proof for the Cartan decomposition in $\mathrm{Sp}_{2n}\mathbb{R}$, which makes use of the linear algebraic structure induced in $\mathbb{C}^{2n}$ by the symplectic group. We also define hyperbolic transformations and deduce a block diagonal normal form for these matrices. We also present a few more models for the space: the Siegel disk (a model analogous to the unit disk for $\mathrm{SL}_2\mathbb{R}$), two projective models and a Lie group quotient model. Using the latter we deduce a new condition for bi-transitivity of the action of $\mathrm{Sp}_{2n}\mathbb{R}$, different form the Siegel condition.

We then compactify the Siegel upper half plane, so that we can extend the action continuously to a compact domain. This is the purpose of Chapter 3. There we deal with possible compactifications of the space. The first one (and the one that will be used if the following chapters) will be the bounded domain compactification, which consists on simply closing the Siegel disk (a bounded domain) in its environment space $\mathbb{C}^N$ under the usual topology. This compactification will yield a stratified boundary in which each stratum is an orbit for the action of the group,

and in which the stratum of smallest dimension is the Shilov boundary of the space. These results are known to the experts and in fact serve as the first step in the Satake compactification for the action of the modular group. We then discuss three other compactifications: the visual boundary, the Furstenberg boundary and the compactification induced by the Busemann functions. These will be defined using the Lie group quotient model, and the relations between the Lie algebra and the tangent space to the model at a given coset. We will deduce a connection between the compactification by Busemann functions and the Shilov boundary: our new result shows that for a special non-Riemannian choice of the metric on the Siegel upper half plane, the compactification using the corresponding Busemann functions yields the Shilov boundary.

In Chapter 4 we will do a full normal form and fixed point study for the $4 \times 4$ symplectic group. Since the $2 \times 2$ symplectic group is equal $\mathrm{SL}_2\mathbb{R}$, as we said, this is the first non studied case. In order to find the normal forms we start by observing, using the Schauder fixed point theorem, that any symplectic transformation must have a fixed point in the closure of the Siegel disk. Then by making assumptions on the location of this fixed point, we can immediately simplify the matrix (by conjugation). Then we work a bit on the simplified forms and then solve the fixed point equation $M(Z) = Z$ for each case, obtaining a full classification of number and location of fixed points. After this we summarize some results that hold for transformations of a special form: given two matrices $X, Y \in \mathrm{SL}_2\mathbb{R}$, it is possible to define a matrix $X \odot Y \in \mathrm{Sp}_4\mathbb{R}$; we present then some general results for the fixed points of transformations of this type. We also obtain quite a few symplectic transformations that are not conjugate to $X \odot Y$ in $\mathrm{Sp}_4\mathbb{R}$.

In Chapter 5 we present some results about the dynamics of the action of $\mathrm{Sp}_{2n}\mathbb{R}$, and point out the similarities and differences with the action of $\mathrm{SL}_2\mathbb{R}$ on the upper half plane. The main concern in this chapter is to define the limit set for a discrete group $\Gamma$. Turns out that it is impossible to do this as it is done in the $\mathrm{SL}_2\mathbb{R}$ case, because not all orbits accumulate at the same points. We have to restrict ourselves to the limit points that lie on the Shilov boundary in order to be able to define a limit set. Using this definition we show, as in the case of the Fuchsian groups, that the limit set of $\Gamma$ lying in the Shilov boundary is independent of the orbit $\Gamma Z$. We also study the hyperbolic transformations (as defined in Chapter 2), which very much resemble the hyperbolic transformations in $\mathrm{SL}_2\mathbb{R}$. Namely, if $<A>$ is the discrete group generated by a hyperbolic element $A$, then the limit set of $<A>$, independent of the orbit $<A>Z$, has only two points located on the Shilov boundary. However, not all their fixed points of $A$ appear as limit points (that is, accumulation points for the orbits $<A>Z$), as was the case for $\mathrm{SL}_2\mathbb{R}$. This leads us to the definition of three limit sets. We finish the chapter by deducing a complete ordering of these sets and presenting and extra property for one of these limit sets in case $\Gamma$ is Zariski dense. More precisely, for such a $\Gamma$, the set of accumulation points of any $\Gamma Z$ on the Shilov boundary is the closure of the set containing the two distinguished fixed points of every hyperbolic element of $\Gamma$, described as above.

There are a few new results in Chapters 2 and 3, as we mentioned above, namely, the normal form for hyperbolic transformations and the condition for bi-transitivity in Chapter 2 and the connection between the Busemann functions and the Shilov boundary in Chapter 3. However, most of the new results (to our knowledge) are in Chapters 4 and 5. Whenever we present a known result, we will refer to the place where this result can be found.

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# Chapter 1

# Introduction

## 1 Notation and basic results

In this work we'll use the following notation:

- For a matrix $A$, with real or complex entries, we'll denote by $A^{\mathrm{T}}$ the transpose of $A$, by $\overline{A}$ the complex conjugate of $A$, by$A^*$ the transpose conjugate of $A$. For an invertible $A$, we'll denote by $A^{-1}$ the inverse of $A$ and by $A^{-\mathrm{T}}$ the transpose of the inverse of $A$, which coincides with the inverse of its transpose.

- $I_n$ or simply $I$ – the $n \times n$ identity matrix,

- $i$ – the complex unit, $i^2 = -1$,

- $\mathrm{M}_{m,n}F$ – the space of all $m \times n$ matrices with entries in the field $F$,

- $\mathrm{M}_nF$ – the space $\mathrm{M}_{n,n}F$

- $\mathrm{Sym}_nF$ – the space of all symmetric $n \times n$ matrices with entries in $F$,

- $\mathrm{Diag}_nF$ – the space of all diagonal matrices with entries in $F$,

- $\mathrm{GL}_nF$ – the group of all invertible $n \times n$ matrices with entries in $F$,

- $\mathrm{SL}_nF$ – the group of all matrices in $\mathrm{GL}_nF$ with determinant 1,

- $\mathrm{O}_n$ – the real $n \times n$ orthogonal group,

- $\mathrm{U}_n$ – the complex $n \times n$ unitary group,

- $S^n$ – the $n$-dimensional sphere.

- For $x_1, \dots, x_n \in F$ we denote by $\mathrm{diag}(x_1, \dots, x_k)$ the diagonal matrix in $\mathrm{M}_n F$ with the elements $x_1, \dots, x_k$ on the diagonal. We use a similar notation for block diagonal matrices.

- For a matrix $A \in \mathrm{M}_{m,n} F$, and $k \leq m, n$, we represent by $A[i_1, \dots, i_k | j_1 \dots, j_k]$ the minor of the matrix $A$ obtained as the determinant of the submatrix with lines $i_1, \dots, i_k$ and columns $j_1 \dots, j_k$ taken from $A$.

- For a matrix $A \in \mathrm{M}_{m,n} F$, and $k \leq m, n$, we denote by $\wedge_k A$ the $k$-th compound matrix of $A$, a matrix with $\binom{m}{k}$ lines and $\binom{n}{k}$ columns, defined as $\wedge_k A(\alpha, \beta) = A[\alpha|\beta]$, with $\alpha, \beta$ being lists of $k$ lines and columns, $\alpha = (i_1, \dots, i_k)$, $\beta = (j_1, \dots, j_k)$, in lexicographic order. The following properties hold: $\wedge_n(A^{\mathrm{T}}) = \wedge_n(A)^{\mathrm{T}}$, $\wedge_n(\overline{A}) = \overline{\wedge_n(A)}$, $\wedge_n(AB) = \wedge_n(A) \wedge_n (B)$, and if $A$ is invertible, $\wedge_n(A^{-1}) = \wedge_n(A)^{-1}$.

- For a matrix $A$, real symmetric or complex hermitian, we write $A > 0$ if $A$ is positive definite, and $A \geq 0$ if $A$ is positive semidefinite.

- For a square matrix $A$, we denote the eigenvalues of $A$ as $\lambda_j(A)$, counted with multiplicities, and consider an order such that

$$|\lambda_1(A)| \geq \ldots \geq |\lambda_n(A)|, \quad j = 1, \dots, n.$$

  We denote the singular values of $A$ as

$$\sigma_j(A) := \sqrt{\lambda_j(AA^*)}, \quad j = 1, \dots, n.$$

  For $A \in \mathrm{GL}_n F$, we have $\sigma_j(A)^{-1} = \sigma_{n-j+1}(A^{-1})$.

- For a square matrix $A$, we'll denote by $||A||$ the norm of $A$ as an operator on the $\ell_2$ space $F^n$, $F = \mathbb{R}$ or $\mathbb{C}$. This means

$$||A|| := \max_{||x||_2 = 1} ||Ax||_2,$$

  where, for $x = (x_1, \dots, x_n)$, $||x||_p = (x_1^p + \ldots + x_n^p)^{\frac{1}{p}}$. It is simple to see that $||A|| = \sigma_1(A)$.

- Let $A$ be a normal matrix, $AA^* = A^*A$. The because $A$ is unitarily diagonalizable, $A$ has a square root. If $A$ is positive semidefinite, then there is a unique positive semidefinite square root of $A$, and if $A$ is real, this unique square root will be real.

The notation is inspired in the one used in [FH2]. The next result is well known, and proofs can be found in [HJ].

**Proposition 1.1.1 (Singular value and polar decompositions)** *Let $A$ be a matrix in $\mathrm{M}_n\mathbb{R}$. Then there exist matrices $Q_1, Q_2 \in \mathrm{O}_n$ such that*

$$A = Q_1 \Sigma(A) Q_2$$

*where $\Sigma(A) = \mathrm{diag}(\sigma_1(A), \ldots, \sigma_n(A))$ is a well defined matrix. This is called the* singular value decomposition*.*

*It comes as a consequence that for $A \in \mathrm{M}_n\mathbb{R}$, there exist matrices $P \in \mathrm{Sym}_n\mathbb{R}$, positive definite, and $Q \in \mathrm{O}_n$ such that $A = PQ$. The matrix $P$ is always uniquely defined and the matrix $Q$ is uniquely defined if $A$ is invertible. This is called the* polar decomposition*.*

If $A \in \mathrm{Sym}_n\mathbb{R}$, then the singular values are the absolute values of the eigenvalues of $A$, and if $A$ is positive definite, they are the same. If $A \in \mathrm{M}_n\mathbb{C}$, we have a similar result, with $Q_1, Q_2 \in \mathrm{U}_n$.

Now, we'll present some general considerations about metrics on matrix spaces which will be useful. As they are not specific to the study of the Symplectic group, we present them in this more general setting.

**Proposition 1.1.2** *Let $p \geq 1$ and $A, B \in \mathrm{GL}_n\mathbb{R}$. Let*

$$d_p(A,B) = \left( \sum_{j=1}^{n} |\log \sigma_j(A^{-1}B)|^p \right)^{\frac{1}{p}}$$

*and as a limit,*

$$d_\infty(A,B) = \max\{|\log \sigma_1(A^{-1}B)|,\ |\log \sigma_1(B^{-1}A)|\}.$$

*Then for each $1 \leq p \leq \infty$, $d_p$ is a metric on the homogeneous space $X = \mathrm{GL}_n\mathbb{R}/\mathrm{O}_n$. Moreover, $\mathrm{GL}_n\mathbb{R}$ acts (from the left) on $X$ as group of isometries for all these metrics.*

**Proof.** We have

$$\sigma_j(M) = \sigma_j(MQ) = \sigma_j(QM),\ M \in \mathrm{M}_n\mathbb{R},\ Q \in \mathrm{O}_n,$$

which can be easily seen using the singular value decomposition. Thus, for each $p$, $d_p(\cdot,\cdot)$ is a well defined non-negative continuous function on the space $X \times X$. To see that $A$ and $B$ belong to the same left coset of $\mathrm{O}_n$ if and only if $d_p(A,B)=0$, take two points such that $d_p(A,B)=0$. This means (even in the case $p=\infty$) that $|\log \sigma_j(A^{-1}B)|=0$ for all $j$. So $\sigma_j(A^{-1}B)=1$ for all $j$, and by the singular value decomposition, this means $A^{-1}B \in \mathrm{O}_n$ and so $A$ and $B$ represent the same coset.

We have $d_p(A,B)=d_p(B,A)$ since $\sigma_j(A^{-1}B)=\sigma_{n-j+1}(B^{-1}A)^{-1}$, and as you apply the absolute values of the logarithms and add them up, the sum will be the same. For $p=\infty$ the symmetry is obvious. The only thing left to prove is the triangle inequality. Take $A,B,C \in \mathrm{SL}_n\mathbb{R}$. Since $\sigma_1(M)=||M||,\ M \in \mathrm{M}_n\mathbb{R}$ we get

$$\sigma_1(MN) \leq \sigma_1(M)\sigma_1(N), \quad M,N \in \mathrm{M}_n\mathbb{R}.$$

Apply the above inequality to the $k$-th compound matrix $\wedge_k(MN)$ to deduce

$$\prod_{j=1}^{k} \sigma_j(MN) \leq \prod_{j=1}^{k} \sigma_j(M) \prod_{j=1}^{k} \sigma_j(N), \quad k=1,...,n.$$

Because the absolute value of the determinant is the product of all singular values, we deduce that for $k=n$ equality holds in the above inequality. As $A^{-1}C = (A^{-1}B)(B^{-1}C)$ from the above inequalities we obtain, for $k=1,...,n$,

$$\sum_{j=1}^{k} \log \sigma_j(A^{-1}C) \leq \sum_{j=1}^{k} \log \sigma_j(A^{-1}B) + \sum_{j=1}^{k} \log \sigma_j(B^{-1}C).$$

For $k=n$ the above inequality becomes an equality. As $f(t)=|t|^p$ is a convex function on $\mathbb{R}$ for $1 \leq p$, the majorization principle in [HLP] yields that for $p \geq 1$,

$$\sum_{j=1}^{n} |\log \sigma_j(A^{-1}C)|^p \leq \sum_{j=1}^{n} |\log \sigma_j(A^{-1}B)|^p + \sum_{j=1}^{n} |\log \sigma_j(B^{-1}C)|^p.$$

Thus we can conclude that

$$d_p(A,C)^p \leq d_p(A,B)^p + d_p(B,C)^p.$$

Now, consider the $\ell_p$ norm in $\mathbb{R}^2$:

$$\begin{aligned} d_p(A,C) &\leq (d_p(A,B)^p + d_p(B,C)^p)^{\frac{1}{p}} \\ &= ||(d_p(A,B), d_p(B,C))||_p \\ &\leq ||(d_p(A,B),0)||_p + ||(0,d_p(B,C))||_p \\ &= d_p(A,B) + d_p(B,C), \end{aligned}$$

which was the desired inequality. Now use the continuity of $p$ at $\infty$ to obtain the triangle inequality for $p \in [1,\infty]$.

We have similar resultes for the action of $\mathrm{GL}_n\mathbb{C}$ on $\mathrm{GL}_n\mathbb{C}/\mathrm{U}_n$. $\square$

## 2 The hyperbolic plane

In this work we are going to study the action of the symplectic group on the Siegel upper half plane. An early study of this action was done by C. L. Siegel in his 1943 book "Symplectic Geometry" [Sie], and we use some of the basic arguments in his study, hence the name of the model.

Because this action is a generalization of the action of $\mathrm{SL}_2\mathbb{R}$ over the hyperbolic half-plane, also called the Lobachevski-Bolyai plane, we will begin with a brief overview of this action and of some things pertaining to it. Most of the proofs of basic facts stated here will be given later in the more general setting of the Siegel plane. A thorough study of this action is done in [Bea]. As it is not our intention to establish this theory, we will not present proofs for the results stated; instead, we'll refer the interested reader to this book.

Given a matrix $M \in \mathrm{SL}_2\mathbb{R}$, and a point $z \in \mathrm{H}_2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, we define its action as a Möbius transformation:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad M(z) := \frac{az+b}{cz+d}.$$

These transformations indeed map the complex upper half-plane to itself, this action is transitive (hence we can call this a homogeneous space), and it is straightforward to prove that we do have all the properties pertaining to a group action. The matrices $M$ and $-M$ have the same action, so we can identify them, and consider that $\mathrm{PSL}_2\mathbb{R} = \mathrm{SL}_2\mathbb{R}/<-I_2>$ acts on H. This space has also a disk model, namely $\mathrm{D} = \{z \in C : |z| < 1\}$ and there are two complex Möbius maps connecting these models bijectively:

$$\begin{array}{ccc} \mathrm{H} & \to & \mathrm{D} \\ z & \mapsto & \frac{z-i}{z+i} \end{array} \qquad \begin{array}{ccc} \mathrm{D} & \to & \mathrm{H}. \\ z & \mapsto & i\frac{1+z}{1-z} \end{array}$$

Using these maps one can define a conjugate action of $\mathrm{SL}_2\mathbb{R}$ over D.

It can be shown that the maps defined by matrices in $\mathrm{SL}_2\mathbb{R}$ are the analytic isometries for the hyperbolic metric that can be defined on these models. One way of defining it is through the element of distance.

$$\text{On H, } ds^2 = \frac{dx^2+dy^2}{y^2}, \text{ and on D, } ds^2 = \frac{4dz^2}{(1-|z|^2)^2}.$$

The next important step is to compactify the model, so that each transformation will have a fixed point. To do so, the easiest way is to consider the compactification of D in $\mathbb{C}$, $\mathrm{Cl(D)} = \{z \in \mathbb{C} : |z| \leq 1\}$, and the corresponding compactification of H, $\mathrm{Cl(H)} = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\} \cup \{\infty\}$. The action is easily extended to the closed models, by continuity (using the expected computation rules when considering $\infty$). We then get that the action is also transitive on the boundary $\partial\mathrm{H}$ (which is $S^1$), but it is impossible to extend the metric to this boundary, so we can no longer talk about isometries. We now consider a normal form classification of the elements of $\mathrm{SL}_2\mathbb{R}$ that will be relevant to our study. It is easy to see that any matrix in $\mathrm{SL}_2\mathbb{R}$ not equal to $\pm I_2$ is conjugate (in $\mathrm{GL}_2\mathbb{R}$) to one and only one of the following:

$$\begin{aligned} g_\lambda &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \text{ if it has a real eigenvalue } \lambda \neq \pm 1 \\ \pm g_1 &= \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ if it has } \pm 1 \text{ as an eigenvalue, or} \\ g_\alpha &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \text{ if } \alpha = a + bi \text{ is an eigenvalue, } a^2 + b^2 = 1,\ b \neq 0. \end{aligned}$$

These transformations are called *hyperbolic, parabolic* and *elliptic*, depending on whether they're conjugate to a matrix of the first, second or third type, respectively. The corresponding actions as Möbius transformations are

$$z \mapsto \lambda^2 z, \quad z \mapsto z + 1 \quad \text{and} \quad z \mapsto \frac{az + b}{-bz + a}.$$

It is clear that the two first ones are a dilation and a translation, respectively. As to the third one, if we look at its conjugate action on the circle $z \mapsto (a+bi)^2 z$, $a^2+b^2 = 1$, we readily see it is a rotation.

Even though the conjugation may be in $\mathrm{GL}_2\mathbb{R}$, the above forms still give us all the right information about the number and dynamical nature of the fixed points of the transformations. This is because for elliptic and hyperbolic elements, the conjugation can be had in $\mathrm{SL}_2\mathbb{R}$ and a parabolic element will be similar to either $g_1$ or $g_1^{-1}$, up to sign.

A hyperbolic map will have two fixed points on $\partial\mathrm{H}$ (0 and $\infty$ for the normal form above), a parabolic one will have one, also on $\partial\mathrm{H}$ ($\infty$ for the normal form), and an elliptic one will have a fixed point inside H (the point $i$ for the normal form), since the number of fixed points is preserved under conjugation.

Now we consider the iterated action of each one of these maps. For this we consider the group generated by each one of the maps above. Let $z \in \mathrm{H}$ be any point. Its orbit under the action of $<g_\lambda>$ is $\{\lambda^{2n} z : n \in \mathbb{Z}\}$, which accumulates at 0 and $\infty$, infinity being an attracting point and 0 a repelling one if $|\lambda| > 1$, otherwise, the dynamical behaviors are interchanged. As for $<g_1>$, the only accumulation point for the orbit will be $\infty$. In each of these cases, we note that the accumulation points are exactly the fixed points of the transformation. Finally, in the third case, if the argument of $\alpha$ is a rational multiple of $\pi$, the orbit is finite, otherwise it is dense in the (hyperbolic) circle centered at $i$ and passing through $z$, so all points in this circle are accumulation points. We see that this is a very different case from the ones above, since the fixed point $i$ is not an accumulation point and we may have an infinite number of accumulation points.

As we now generalize these ideas, we'll now consider discrete groups.

**Definition 1.2.1** *A discrete subgroup of* $\mathrm{GL}_n\mathbb{C}$ *is a subgroup that is a discrete set for topology induced by the Euclidean norm in* $\mathrm{M}_n\mathbb{C}$*.*

In the rest of this section we'll always consider $\Gamma$ to be a discrete subgroup of $\mathrm{SL}_2\mathbb{R}$. In the examples above, $<g_\lambda>$ and $<g_1>$ were discrete subgroups, and $<g_\alpha>$, $\alpha \in \mathbb{C}$ will be discrete if and only if it is finite. Discrete subgroups of $\mathrm{SL}_2\mathbb{R}$ have important properties, for instance, if you factor $\mathrm{H}$ by the action of a discrete subgroup $\Gamma$, that is, if you identify two points $p_1, p_2 \in \mathrm{H}$ if they belong to the same $\Gamma$-orbit, then you get a Riemann surface. Moreover, all Riemann surfaces can be obtained this way.

**Definition 1.2.2** *The action of a group $G$ on a topological space $X$ is said to be properly discontinuous if for any compact set $K \subset X$*

$$K \cap g(K) = \emptyset,$$

*except for a finite number of elements $g \in G$.*

**Proposition 1.2.3** *A discrete group of* $\mathrm{SL}_2\mathbb{R}$ *has a properly discontinuous action on* $\mathrm{H}$*.*

**Lemma 1.2.4** *Let $z_1, z_2 \in \mathrm{H}$. Then if $\Gamma$ is a discrete group, the orbits $\Gamma z_1$ and $\Gamma z_2$ have the same accumulation points.*

The above lemma allows us to define the limit set for a discrete group.

**Definition 1.2.5** *Let $\Gamma$ be a discrete subgroup of* $\mathrm{SL}_2\mathbb{R}$*. Then the limit set $\Lambda(\Gamma)$ is defined as the set of accumulation points of any orbit $\Gamma z$, with $z \in \mathrm{H}$. If $\Lambda(\Gamma)$ is finite we call $\Gamma$ elementary, and if it is infinite, we call $\Gamma$ non-elementary.*

Here are some important properties of the limit set:

**Proposition 1.2.6** *Let $\Gamma$ be an infinite discrete subgroup of* $\mathrm{SL}_2\mathbb{R}$ *and $\Lambda(\Gamma)$ its limit set. Then the following hold:*

- *We have $\Gamma \subset \partial\mathrm{H}$ and its cardinality can only be one or two, if it is finite. If it is infinite, it is a perfect set.*

- *The set* $\Lambda(\Gamma)$ *is closed and* $\Gamma$*-invariant. Moreover, if* $\Gamma$ *is non-elementary, then* $\Lambda(\Gamma)$ *is the smallest closed* $\Gamma$*-invariant set in* $\mathrm{Cl}(\mathrm{H})$*, that is, if* $X \in \mathrm{Cl}(\mathrm{H})$ *and* $X$ *is closed and* $\Gamma$*-invariant, then* $\Lambda(\Gamma) \subset X$*.*

- *If* $\Gamma$ *is non-elementary, then* $\Lambda$ *is the closure of the set of the fixed points of the hyperbolic maps in* $\Gamma$*. Moreover,* $\Omega := \mathrm{Cl}(\mathrm{H}) \setminus \Lambda(\Gamma)$ *is the largest set in* $\mathrm{Cl}(\mathrm{H})$ *where the action of* $\mathrm{SL}_2\mathbb{R}$ *is properly discontinuous.*

In the next chapters we present a generalization of these results in the Siegel upper half plane.

# Chapter 2

# The Symplectic Group and the Siegel Upper Half Plane

## 1 Basic definitions and results

Here we define the group and the action that is going to be the object of our study in this work.

**Definition 2.1.1** *Let $F$ be either the real or the complex field. The* **Symplectic Group** *is the group of all matrices $M \in \mathrm{GL}_{2n}F$ satisfying*

$$M^{\mathrm{T}} J M = J, \text{ with } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

*We denote it by* $\mathrm{Sp}_{2n}F$.

If we decompose $M \in \mathrm{M}_{2n}F$ in four $n \times n$ blocks according to $J$, and work a bit on the matrix equation, we get that

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is symplectic if and only if

$$A^{\mathrm{T}}C \text{ and } B^{\mathrm{T}}D \text{ are symmetric and } A^{\mathrm{T}}D - C^{\mathrm{T}}B = I_n. \tag{2.1}$$

We can readily see that a symplectic matrix has have determinant $\pm 1$, by taking the determinant on both sides of the defining equation—we'll see later that the determinant has to be 1. Moreover, $M^{\mathrm{T}} = JM^{-1}J^{-1}$, and since $M$ is similar to

$M^{\mathrm{T}}$, $M$ has to be similar to its inverse. Also, if $M$ is symplectic, then $M^{\mathrm{T}}$ is symplectic too:

$$MJM^{\mathrm{T}} = MJ(JM^{-1}J^{-1}) = -MM^{-1}(-J) = J, \text{ since } J^{-1} = -J.$$

We have also the following formula for the inverse of $M$:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} D^{\mathrm{T}} & -B^{\mathrm{T}} \\ -C^{\mathrm{T}} & A^{\mathrm{T}} \end{pmatrix}.$$

Finally, it is clear from equations (2.1) that $\mathrm{Sp}_2 F = \mathrm{SL}_2 F$.

We now present the generalization of the upper half plane.

**Definition 2.1.2** *The* **Siegel upper half plane** *is the set of all complex symmetric $n \times n$ matrices with positive definite imaginary part. We denote it by* $\mathrm{SH}_n$*:*

$$\mathrm{SH}_n = \{X + iY \in \mathrm{Sym}_n\mathbb{C}, X, Y \in \mathrm{Sym}_n\mathbb{R} : Y > 0\}.$$

We now present the action. For

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}\mathbb{R} \text{ and } Z \in \mathrm{SH}_n,\ M(Z) := (AZ + B)(CZ + D)^{-1}.$$

In order to see that this is indeed an action, we have to verify first that the matrix $CZ + D$ is invertible, and then that the product is in $\mathrm{SH}_n$. We'll follow the proofs in [Sie]. For $Z$ to be a point in $\mathrm{SH}_n$, it has to be symmetric, and to have a positive definite imaginary part. These two statements can be written, respectively, as follows:

$$\begin{pmatrix} Z^{\mathrm{T}} & I \end{pmatrix} J \begin{pmatrix} Z \\ I \end{pmatrix} = 0 \quad \text{and} \quad -\frac{1}{2i} \begin{pmatrix} Z^* & I \end{pmatrix} J \begin{pmatrix} Z \\ I \end{pmatrix} > 0,$$

since the first equation is just $Z^{\mathrm{T}} - Z = 0$ and the second $1/2i\ (Z - Z^*) > 0$.

Now we set $E := AZ + B$ and $F := CZ + D$, or equivalently

$$M \begin{pmatrix} Z \\ I \end{pmatrix} = \begin{pmatrix} E \\ F \end{pmatrix}.$$

We then have

$$\begin{aligned} \begin{pmatrix} E^{\mathrm{T}} & F^{\mathrm{T}} \end{pmatrix} J \begin{pmatrix} E \\ F \end{pmatrix} &= \begin{pmatrix} Z^{\mathrm{T}} & I \end{pmatrix} M^{\mathrm{T}} J M \begin{pmatrix} Z \\ I \end{pmatrix} \\ &= \begin{pmatrix} Z^{\mathrm{T}} & I \end{pmatrix} J \begin{pmatrix} Z \\ I \end{pmatrix} = 0 \end{aligned}$$

and

$$-\frac{1}{2i}\begin{pmatrix} E^* & F^* \end{pmatrix} J \begin{pmatrix} E \\ F \end{pmatrix} = -\frac{1}{2i}\begin{pmatrix} Z^* & I \end{pmatrix} M^{\mathrm{T}} J M \begin{pmatrix} Z \\ I \end{pmatrix}$$
$$= -\frac{1}{2i}\begin{pmatrix} Z^* & I \end{pmatrix} J \begin{pmatrix} Z \\ I \end{pmatrix} > 0.$$

If we now develop the left hand side of both of the equations above, we get

$$E^{\mathrm{T}}F = F^{\mathrm{T}}E \text{ and } -\frac{1}{2i}(E^*F - F^*E) > 0.$$

Now, to see that $F$ is invertible, suppose that $v$ is a solution to $Fv = 0$. Then we have $v^*F^* = 0$ and $v^*(E^*F - F^*E)v = 0$, so $v = 0$, and hence $F$ is invertible, and we can write $M(Z) = EF^{-1}$. From $E^{\mathrm{T}}F = F^{\mathrm{T}}E$ we get that $EF^{-1}$ is symmetric and from the last equality, we get

$$-\frac{1}{2i}F^*\left((F^{-1})^*E^* - EF^{-1}\right)F > 0, \quad \frac{1}{2i}(EF^{-1} - (EF^{-1})^*) > 0$$

so $\mathrm{Im}(EF^{-1}) > 0$. $\square$

We will call these maps *generalized Möbius transformations*. Here, like in the 2-dimensional upper half plane, the matrices $M$ and $-M$ have the same action. This action does indeed generalize the action of $\mathrm{SL}_2\mathbb{R}$ over $\mathrm{H}$, since $\mathrm{H} = \mathrm{SH}_1$, and $\mathrm{Sp}_2\mathbb{R} = \mathrm{SL}_2\mathbb{R}$ as we have seen. It is possible to get a closer connection between the $\mathrm{SL}_2\mathbb{R}$ action and the $\mathrm{Sp}_{2n}\mathbb{R}$ action. First, it is easy to see that we can define a 1-1 map from $\mathrm{H}^n$ to $\mathrm{SH}_n$ as follows:

$$\begin{array}{rcl} \mathrm{H}^n & \to & \mathrm{SH}_n \\ (z_1, \ldots, z_n) & \mapsto & \mathrm{diag}(z_1, \ldots, z_n) \end{array}$$

Since all the imaginary parts of the $z_j$'s are positive, $\mathrm{diag}(z_1, \ldots, z_n) \in \mathrm{SH}_n$. We have a corresponding map for the groups:

$$\begin{array}{rcl} \Phi : (\mathrm{SL}_2\mathbb{R})^n & \to & \mathrm{Sp}_{2n}\mathbb{R} \\ (M_1, \ldots, M_n) & \mapsto & T(M_1 \oplus \ldots \oplus M_n)T^{-1} \end{array}$$

where $T$ is a permutation matrix. To define $T$, let's denote by $E_j,\ j = 1, \ldots, n$ the $2n \times 1$ matrix with zeros in all entries except in entry $(j, 1)$ where it has a 1, and by $F_k$ the $2n \times 1$ matrix also with zeros in all entries except in entry $(n + k, 1)$ where it has a 1. Then we can define $T$ column by column:

$$T = \begin{pmatrix} E_1 & F_1 & E_2 & F_2 & \ldots & E_n & F_n \end{pmatrix}.$$

We will denote the image of $(M_1, \ldots, M_n)$ under this map by $M_1 \odot \ldots \odot M_n$. If

$$M_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \; j = 1, \ldots, n, \text{ then}$$

$$M_1 \odot \ldots \odot M_n = \left(\begin{array}{ccc|ccc} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ \hline c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_n & & & d_n \end{array}\right).$$

It is straightforward to see that the action of this matrix on the image of $\mathrm{H}^n$ can be done componentwise:

$$M_1 \odot \ldots \odot M_n(\mathrm{diag}(z_1, \ldots, z_n)) = \mathrm{diag}(M_1(z_1), \ldots, M_n(z_n)).$$

Moreover, since $\Phi$ is injective, and it is a homomorphism, $\Phi$ is a faithful representation of $(\mathrm{SL}_2\mathbb{R})^n$.

Before we start our study of the group, we will draw a parallel with the action on the two-dimensional model:

**Proposition 2.1.3** *The action of the symplectic group on the Siegel upper half plane is transitive.*

**Proof.** It is enough to prove that it is possible to find a symplectic map that sends $iI$ to any $X + iY \in \mathrm{SH}_n$, $Y > 0$. Take then the composition of the symplectic maps $Z \mapsto \sqrt{Y} Z \sqrt{Y}$ and $Z \mapsto Z + X$, associated with the symplectic matrices

$$\begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} \text{ and } \begin{pmatrix} I & X \\ 0 & I \end{pmatrix},$$

respectively. That transformation maps $iI$ to $X + iY$. $\square$

As another parallel, we refer that every holomorphic bijective map from $\mathrm{SH}_n$ onto itself can be represented as a symplectic map (see [Sie]). We'll present a result about bi-transitivity in the last section of this chapter.

## 2 Symplectic linear algebra

The matrix $J$ used to define the symplectic group can be used to define a skew-symmetric form in $\mathrm{M}_{2n,1}\mathbb{R}$: for vectors $u, v \in \mathrm{M}_{2n,1}\mathbb{R}$, define the form as

$$(u, v) := u^{\mathrm{T}} J v.$$

We will call it the symplectic form. With this definition, we can say that a matrix $M$ is symplectic if and only if $(Mu, Mv) = (u, v)$ for all $u, v \in \mathrm{M}_{2n,1}\mathbb{R}$.

Recall now the matrices $E_j$ and $F_k$ defined above. It is clear that $(E_j, F_k : j, k = 1, \ldots, n)$ is a basis for $\mathrm{M}_{2n,1}\mathbb{R}$. According to the following definition, this will be a symplectic basis.

**Definition 2.2.1** *A symplectic basis of* $\mathrm{M}_{2n,1}\mathbb{R}$ *is a basis such that the matrix of the symplectic form with respect to this basis is* $J$*. In other words, it it a basis* $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ *such that*

$$(e_j, f_k) = -(f_k, e_j) = \delta_{jk}, \quad (e_j, e_k) = (f_j, f_k) = 0 \text{ for all } j, k = 1, \ldots, n.$$

With this definition we can easily see that a matrix is symplectic if and only if it is a change of basis matrix from one symplectic basis to another one.

**Definition 2.2.2** *A subspace of* $\mathrm{M}_{2n,1}\mathbb{R}$ *is called Lagrangean if it has dimension* $n$ *and for all* $u, v$ *in the subspace,* $(u, v) = 0$.

As examples of Lagrangean spaces we have $<E_j : j = 1, \ldots, n>$ and $<F_k : k = 1, \ldots, n>$.

Now we proceed to find a normal form for symplectic matrices, a form that generalizes the concept of hyperbolic transformation in $\mathrm{SL}_2\mathbb{R}$, as defined in page 6. We start with a simple but important result. As this is probably the first basic result with a non-trivial proof, we present here two possible proofs.

**Proposition 2.2.3** *A symplectic matrix* $M$ *has determinant 1.*

**Proof**. We'll use here a quick proof (if not a straightforward one) taken from [Sie]. Write, as before,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } \begin{pmatrix} E \\ F \end{pmatrix} = M \begin{pmatrix} iI_n \\ I_n \end{pmatrix} = \begin{pmatrix} Ai + B \\ Ci + D \end{pmatrix}.$$

As we have seen, $\det(Ci + D) \neq 0$, since $iI_n \in \mathrm{SH}_n$. It is also easy to ascertain, from the equations (2.1) that $F^{\mathrm{T}} A - E^{\mathrm{T}} C = I_n$ and $F^{\mathrm{T}} B - E^{\mathrm{T}} D = -iI_n$. Now it is easy to check that

$$\begin{pmatrix} F^{\mathrm{T}} & -E^{\mathrm{T}} \\ 0 & I \end{pmatrix} M \begin{pmatrix} I_n & iI_n \\ C & I_n \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ C & F \end{pmatrix}.$$

By computing the determinant on both sides of the above inequality and recalling that $\det(F) \neq 0$ we have the result. $\square$

Sketches to other proofs can be found in [FH2]. Notice that a real orthogonal matrix can have determinant 1 or $-1$; a symplectic matrix, however, can only have determinant 1, even though the defining equation for both groups is formally similar—the first one involves a positive definite symmetric form, the second one a non-degenerate skew-symmetric form.

Now let $M$ be a symplectic matrix. According to the last proposition, and to the already known result that a symplectic matrix has to be similar to its inverse, we can list the eigenvalues of $M$, with multiplicities, in the following way:

$$(\lambda_1, \ldots, \lambda_n, \lambda_n^{-1}, \ldots, \lambda_1^{-1}), \tag{2.2}$$

and we consider the above list to be ordered decreasingly according to the absolute values of the eigenvalues as follows:

$$|\lambda_1| \geq \ldots \geq |\lambda_n| \geq 1 \geq |\lambda_n^{-1}| \geq \ldots \geq |\lambda_1^{-1}|.$$

**Definition 2.2.4** *Let $M \in \mathrm{Sp}_{2n}\mathbb{R}$. We will call the matrix $M$ (and the corresponding induced transformation)* hyperbolic *if $M$ has no eigenvalue on the complex unit circle, that is, if $|\lambda_j| \neq 1$ for all $j = 1, \ldots, n$,*

We will now present a normal form for hyperbolic matrices $M$. We know from linear algebra that we can conjugate $M$ by a matrix $Q \in \mathrm{GL}_{2n}\mathbb{R}$ in order to bring it to block diagonal form $\mathrm{diag}(A, D)$, where $A$ comprises all the blocks pertaining to the first $n$ eigenvalues in the list, and $D$ the ones pertaining to the last $n$. Consider this matrix now as the matrix of a linear map with respect to some basis $B$, $Q$ being the change of basis matrix. Let $B = (e_1, \ldots, e_n, f_1, \ldots, f_n)$.

**Proposition 2.2.5** *The spaces $<e_j : j = 1, \ldots, n>$ and $<f_k : k = 1, \ldots, n>$ are Lagrangean.*

**Proof.** They clearly have dimension $n$. Now we'll check that the restriction of the form to each of the spaces is the zero form. Let $V$ be the first space, $B_1 = (e_j : j = 1, \ldots, n)$ a basis for it—the proof for the second one would be similar.

Consider this restriction. Then put the matrix of $M|_V$ (which is $A$ for the basis $B_1$) in Jordan normal form, considering Jordan blocks with complex eigenvalues if necessary. This means we consider that $M$ is acting on the complex span of $B_1$. Now we prove that with respect to the new basis $C$ (with complex vectors if necessary), we have $v^{\mathrm{T}} J u = O$ for all $v, u \in \mathbb{C}^{2n}$. This will prove that the bilinear form restricted to $V$ is zero since this is the real span of $B_1$, contained in the complex span. We have, since $M$ is symplectic, $(Mu, Mv) = (u, v)$. Now we consider three cases.

1. Suppose the vectors $u$ and $v$ are eigenvectors for two eigenvalues $a$ and $b$. Notice that, because of the definition of $V$, neither $a$, $b$ or $ab$ are equal to 1. Then

$$(u, v) = (Mu, Mv) = ab(u, v), \text{ so } (1 - ab)(u, v) = 0,$$

and since $ab$ is not 1 because $a, b > 1$, we can conclude that $(u, v) = 0$; in particular, this is true even if $a = b$.

2. Suppose now that $u$ is and eigenvector and $v$ a generalized eigenvector (possibly associated with the same eigenvalue). In this case, if we order the generalized eigenvectors such that $Mv_0 = bv_0$, $Mv_j = v_{j-1} + bv_j$, then we can prove by induction that $(u, v_j) = 0$ in a similar way. We know $(u, v_0) = 0$ from the previous case. Now assume that for $j < n$, $(u, v_j) = 0$. Then

$$\begin{aligned} (u, v_n) = (Mu, Mv_n) &= (au, v_{n-1} + bv_n) \\ &= a(u, v_{n-1}) + ab(u, v_n) \\ &= ab(u, v_n) \end{aligned}$$

since by induction hypothesis $(u, v_{n-1}) = 0$. Again, since $ab \neq 1$, we can conclude that $(u, v_n) = 0$, and again, this is true even if $a = b$.

3. Both $u$ and $v$ are generalized eigenvectors. This is done again by induction, using the same type of arguments. □

Now, we can start to transform the basis $B$ into a symplectic basis, by altering just the last $n$ vectors. This is equivalent to saying that it is possible to find a symplectic matrix $Q'$, which will be the change of basis matrix from the canonical basis to the new symplectic basis, such that the conjugate of $M$ by $Q'$ is block diagonal and symplectic. To do this, consider the $n$ linear forms defined by $(e_j, \cdot)$, where the dot indicates the place for the argument. Now, if you consider the restriction of these forms to the space $U := <f_k : k = 1, \ldots, n>$, they're still a linearly independent set of vectors (since its action on $V$ is identically zero), and hence you can find a basis of $U$ dual to this system. These new vectors along with the original basis of $V$ will form a symplectic basis, and hence we proved the following result:

**Proposition 2.2.6** *Let $M$ be a symplectic matrix with no eigenvalue with absolute value 1. Then $M$ is symplectically similar to a matrix of the form*

$$\begin{pmatrix} A & 0 \\ 0 & A^{-\mathrm{T}} \end{pmatrix},$$

*with $A$ having all eigenvalues with absolute value greater than 1.*

Also, from the computations done in case 1. above, we can conclude the following technical result:

**Lemma 2.2.7** *Let $u$ and $v$ be eigenvectors (possibly generalized) for a symplectic matrix $A$, associated with two eigenvalues $a$ and $b$ with $ab \neq 1$. Then $(u, v) = 0$.*

Another important result we can get using these methods is the singular value decomposition in $\mathrm{Sp}_{2n}\mathbb{R}$. This is a particular case of the general Cartan decomposition for semisimple connected Lie groups, as described, for instance, in [Hel, p. 402]. We present here a direct proof in the symplectic case.

Let's denote by $\mathrm{SpO}_{2n}\mathbb{R}$ the group of symplectic orthogonal matrices.

**Proposition 2.2.8** *Let $M \in \mathrm{Sp}_{2n}\mathbb{R}$. Then there exist matrices $Q, R \in \mathrm{SpO}_{2n}\mathbb{R}$ such that $M = Q\Sigma_s(M)R$, where*

$$\Sigma_s(M) = \mathrm{diag}(\sigma_1(M), \ldots, \sigma_n(M), \sigma_{2n}(M), \sigma_{2n-1}(M), \ldots, \sigma_{n+1}(M)),$$

*and since $\sigma_j(M) = \sigma_{2n-j+1}(M)^{-1}$, the diagonal matrix $\Sigma_s(M)$ can be written as*

$$\Sigma_s(M) = S \oplus S^{-1}, \ S = \mathrm{diag}(\sigma_1(M), \ldots, \sigma_n(M)).$$

**Proof.** Using the usual singular value decomposition and conjugating $\Sigma(M)$ by a permutation matrix, we can have $MM^{\mathrm{T}} = Q\Sigma_s^2 Q^{\mathrm{T}}$ with $Q \in \mathrm{O}_{2n}$, $\Sigma_s = \Sigma_s(M)$. Let's prove we can have this decomposition with $Q$ in $\mathrm{SpO}_{2n}\mathbb{R}$. For this, we need the columns of $Q$ to be an symplectic orthonormal basis of $\mathbb{R}^{2n}$. All the diagonal entries of $\Sigma$ are positive, so the matrix $N := MM^{\mathrm{T}}$ will only have an eigenvalue with absolute value 1 if 1 is an eigenvalue. Notice that since $M^{\mathrm{T}}$ is symplectic, $N$ is still symplectic.

We know that

$$\mathbb{R}^{2n} = \bigoplus_{\lambda} V_\lambda$$

where $\lambda$ runs over all the eigenvectors of $N$ and $V_\lambda$ are the corresponding eigenspaces. Moreover, the eigenspaces are orthogonal to each other. From the lemma above, we have that if $\lambda \neq 1/\mu$ then for $v \in V_\lambda$ and $u \in V_\mu$, $(u, v) = 0$. So, if we build a symplectic orthonormal basis for each space $V_\mu \oplus V_{\mu^{-1}}$, $\mu \neq 1$, and for $V_1$ we will be done.

Thus, let $\lambda \neq 1$ be an eigenvalue, $V_\lambda$ its eigenspace, and $V_{\lambda^{-1}}$ the eigenspace for $\lambda^{-1}$, $\lambda > 1 > \lambda^{-1}$. They must have the same dimension, as you can see from equation (2.2). Now take an orthonormal basis for $V_\lambda$, say $(e_1, \ldots, e_d)$. Since $\lambda^2 \neq 1$, by the previous lemma, $(e_j, e_k) = 0$. Now take $f_j := -Je_j$ for $j = 1, \ldots, d$. This will be a symplectic orthonormal basis for $V_\lambda \oplus V_{\lambda^{-1}}$. To see that the $f_j$'s are in

$V_{\lambda^{-1}}$, notice that since $N$ is symmetric and symplectic, $NJN = J \Leftrightarrow NJ = JN^{-1}$, so

$$Nf_j = -NJe_j = -JN^{-1}e_j = -\lambda^{-1}Je_j = \lambda^{-1}f_j,\ j = 1, \ldots, d.$$

All vectors $f_j$ have norm one, since $||Jv|| = ||v||$ for any vector $v$. Now all we have to do is check that $(e_j, f_k) = \delta_{jk}$, for all $j, k$ and that all vectors are orthogonal. Using the facts that $J^2 = -I$ and $J^{\mathrm{T}} = -J$, we have, for all $j, k = 1, \ldots, d$,

$$\begin{aligned} e_j|f_k &= e_j^{\mathrm{T}}(-Je_k) = 0, & f_j|f_k &= (e_j^{\mathrm{T}}J^{\mathrm{T}})(-Je_k) = \delta_{jk}, \\ (e_j, f_k) &= e_j^{\mathrm{T}}J(-Je_k) = \delta_{jk}, & (f_j, f_k) &= (-e_j^{\mathrm{T}}J^{\mathrm{T}})J(-Je_k) = 0. \end{aligned}$$

This gives us a basis for $V_\lambda \oplus V_{\lambda^{-1}}$. Once we do this for every space $V_\mu \oplus V_{\mu^{-1}}$, $\mu \neq 1$, we have an orthonormal symplectic basis for $\bigoplus_{\lambda\neq 1} V_\lambda$. If this is the whole space, we're done. If $V_1 \neq (0)$, we still have to find an orthonormal symplectic basis for this space, and we need a different algorithm.

Notice that this space has to have even dimension, say $2d$. We'll build a set of orthonormal symplectic vectors $e_1, \ldots, e_d$ and then the other vectors of the desired basis for the space will again be given by $f_j := -Je_j$, $j = 1, \ldots, d$, since all the considerations above about these vectors will still hold.

To build these first $d$ vectors, start by taking a vector $e_1$ of norm one in $V_1$. Then take $e_2$ to be any vector of norm one in $<e_1>^\perp \cap \ker(e_1, \cdot)$. Both these spaces have dimension $2d-1$, so they must intersect. Then take $e_3$ in $<e_1, e_2>^\perp \cap (\ker(e_1, \cdot) \cap \ker(e_2, \cdot))$, still of norm 1. The intersection is nonzero because both spaces have dimension $2d-2$—if the dimension of the second one were $2d-1$, then $\ker(e_2, \cdot) = \ker(e_2, \cdot)$ and the vectors would be linearly dependent.

You then iterate the procedure, and as you reach $d-1$, you need $e_d$ to be in

$$<e_1, \ldots, e_{d-1}>^\perp \bigcap \left( \bigcap_{j=1}^{d-1} \ker(e_j, \cdot) \right).$$

The intersection is still nonzero, since the first space has dimension $d+1$ and the second will have dimension at least $d+1$. This is because each kernel has dimension $2d-1$ (it's a hyperplane), and as you go along and intersect the space $\bigcap_{j=1}^{l} \ker(e_j, \cdot)$ with $\ker(e_{l+1}, \cdot)$, you can only bring the dimension down by one. It can be seen that you actually do, since otherwise $e_{l+1} \in <e_1, \ldots, e_l>$, so the said dimension is actually $d+1$—to see this, just notice that $\ker(e_j, \cdot) = <Je_j>^\perp$.

This gives us $d$ orthonormal vectors such that $(e_j, e_k) = 0$, for all $j, k = 1, \ldots, d$, by construction. As we add the $f_j$'s as defined above, we have a symplectic orthonormal basis for $V_1$, and as we add it to the basis for $\bigoplus_{\lambda\neq 1} V_\lambda$, we have a symplectic orthonormal basis for the space.

This proves that you can have $MM^{\mathrm{T}} = Q\Sigma_s^2 Q^{\mathrm{T}}$ with $Q \in \mathrm{SpO}_{2n}\mathbb{R}$. Now define $R := \Sigma_s^{-1} Q^{\mathrm{T}} M$, and we'll have

$$RR^{\mathrm{T}} = \Sigma_s^{-1} Q^{\mathrm{T}} M M^{\mathrm{T}} Q \Sigma_s^{-1} = I_2 \text{ and } Q\Sigma_s R = Q\Sigma_s \Sigma_s^{-1} Q^{\mathrm{T}} M = M,$$

which is what we wanted. $\square$

An immediate result we can draw from this one is the polar decomposition for symplectic matrices.

**Proposition 2.2.9** *Let $M$ be a symplectic matrix. Then there exist a symplectic positive definite symmetric matrix $A$ and a symplectic orthogonal matrix $Q$, both uniquely defined, such that $M = AQ$.*

Given the singular value decomposition above,

$$M = Q_1 \Sigma_s(M) R = (Q_1 \Sigma_s(M) Q_1^{\mathrm{T}})(Q_1 R),$$

and this gives a polar decomposition. The uniqueness of both matrices is true for the polar decomposition in $\mathrm{GL}_{2n}\mathbb{R}$, when $M$ is invertible, so we have the result. $\square$

In order to characterize the subgroup $\mathrm{Sp}_{2n}\mathbb{R} \cap \mathrm{O}_{2n}$, we can solve the equation $M^{-1} = M^{\mathrm{T}}$ using the formula we have for the inverse of a symplectic matrix, and get that the orthogonal symplectic matrices are the matrices of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tag{2.3}$$

with $A^{\mathrm{T}} A + B^{\mathrm{T}} B = I_n$ and $A^{\mathrm{T}} B$ symmetric.

# 3 Other models

**The disk**. We define $\mathrm{SD}_n$ as the set

$$\{Z \in \mathrm{Sym}_n\mathbb{C} : I - Z\overline{Z} > 0\}.$$

This is a generalization of the unit disk, since the condition $I - Z\overline{Z} > 0$ can be rewritten as $||Z|| < 1$, this norm being the operator norm, regarding $Z$ as an operator on an $\ell_2$ space. There are two complex symplectic maps connecting these two models, namely

$$\begin{array}{cccc} \Phi_1 : & \mathrm{SH}_n & \to & \mathrm{SD}_n \\ & Z & \mapsto & (Z - iI_n)(Z + iI_n)^{-1} \end{array}$$

and

$$\begin{array}{rcccc} \Phi_1^{-1} & : & \mathrm{SD}_n & \to & \mathrm{SH}_n. \\ & & Z & \mapsto & i(I_n+Z)(I_n-Z)^{-1} \end{array}$$

These are generalizations of the maps between H and D, presented in the Introduction. These maps can be expressed as complex symplectic transformations, associated with the matrices

$$\begin{pmatrix} I_n & -iI_n \\ I_n & iI_n \end{pmatrix} \text{ and } \begin{pmatrix} iI_n & iI_n \\ -I_n & I_n \end{pmatrix}$$

respectively. Thus, the action on this model is defined by conjugation, the acting group is a conjugate of $\mathrm{Sp}_{2n}\mathbb{R}$ within $\mathrm{Sp}_{2n}\mathbb{C}$: the group of all complex symplectic matrices such that

$$M^* \operatorname{diag}(-I_n, I_n) M = \operatorname{diag}(-I_n, I_n),$$

and the action is still by generalized Möbius transformations (see [Sie] and [Hel]).

**The projective models**. We start by considering the Grassmannian $G_{2n,n}\mathbb{C}$ which is the variety of all n-dimensional subspaces of $\mathbb{C}^{2n}$. Take all the complex matrices of type $2n \times n$ with full rank, that is, rank $n$, denoted $\mathrm{M}_{2n,n;n}\mathbb{C}$ and view the columns of each of these matrices as a basis of a subspace of $\mathbb{C}^{2n}$. Now, consider the action of $\mathrm{GL}_n\mathbb{C}$ by right multiplication on this set—this action preserves the said subspace, by changing the basis. Then the Grassmannian is

$$G_{2n,n} = \mathrm{M}_{2n,n;n}\mathbb{C}/\mathrm{GL}_n\mathbb{C}.$$

The model for our space will now be the set of all classes that admit as a representative a matrix of the type

$$\begin{pmatrix} Z \\ I_n \end{pmatrix} \text{ with } Z \in \mathrm{Sym}_n\mathbb{C},\ \ \mathrm{Im}(Z) > 0.$$

We'll denote this set by $\mathrm{SPH}_n$. We use square brackets to represent the class of a matrix. The group action is now left matrix multiplication by a representative of the class:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} Z \\ I_n \end{bmatrix} = \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix} = \begin{bmatrix} (AZ+B)(CZ+D)^{-1} \\ I_n \end{bmatrix}.$$

It's trivial to see that the action is well defined. The map connecting $\mathrm{SH}_n$ to $\mathrm{SPH}_n$ is clearly

$$\begin{array}{rcccc} \Phi_2 & : & \mathrm{SH}_n & \to & \mathrm{SPH}_n \\ & & Z & \mapsto & \begin{bmatrix} Z \\ I_n \end{bmatrix} \end{array}$$

which is a 1-1 map, as it is easy to check.

This model and the action are studied in a more general setting in [SZ]

It is also useful to consider another projective model related to this one. Take the set $\wedge_n \mathrm{SPH}_n := \{\wedge_n W : [W] \in \mathrm{SPH}_n\}$ with the identification $v = u$ if and only if there exists a nonzero complex number $z$ such that $v = uz$. This is a subset of the projective space $\mathbb{C}P^{N-1}$, $N = \binom{2n}{n}$. The action is defined as left multiplication by $\wedge_n M$: for $M \in \mathrm{Sp}_{2n}\mathbb{R}$ and $v \in \wedge_n \mathrm{SPH}_n$, the action is $[v] \mapsto [\wedge_n M v]$. Notice that if $V$ and $V'$ are two representatives of the same class in $G_{2n,n}$, then $V' = VU$, for some $U \in \mathrm{GL}_n\mathbb{C}$. Then $\wedge_n V' = (\wedge_n V).\det U$, since $\wedge_n U = \det U$. This allows us to write $[\wedge_n V] = \wedge_n [V]$, and we have a well defined map from $\mathrm{SPH}_n$ to $\wedge_n \mathrm{SPH}_n$ given by $[V] \mapsto [\wedge_n V]$.

We now see that this map gives a 1-1 correspondence between these last two models. A class in $\mathrm{SPH}_n$ is determined by the span of the columns of any of its representatives, so if $[V] \neq [W]$, $W, V \in \mathrm{M}_{2n,n}\mathbb{C}$, then the column spans of $V$ and $W$ are not the same, and in this case it is well known that $<\wedge_n V> \neq <\wedge_n W>$, and $[\wedge_n V] \neq [\wedge_n W]$ in $\mathbb{C}P^{N-1}$.

**The Lie group quotient**. Finally, we have as model a quotient of the Lie group $\mathrm{Sp}_{2n}\mathbb{R}$. To describe this model, we need to know the subgroup of $\mathrm{Sp}_{2n}\mathbb{R}$ that stabilizes an element in $\mathrm{SH}_n$. Let's consider the element $iI_n \in \mathrm{SH}_n$. Then it is easy to see by direct computation that the stabilizer subgroup is exactly the subgroup of symplectic orthogonal matrices (see equation (2.3) above). As we have said, we'll denote this subgroup by $\mathrm{SpO}_{2n}\mathbb{R}$ or simply $K$. The model is now the set of all equivalence classes $\mathrm{Sp}_{2n}\mathbb{R}/K$ and the action is again left matrix multiplication by a representative, which is a well defined action. To find the class corresponding to the element $X + iY$, we recall that this element must be the image of $iI$ under the map defined by the matrix

$$M_{X+iY} := \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \begin{pmatrix} \sqrt{Y} & 0 \\ 0 & \sqrt{Y^{-1}} \end{pmatrix} = \begin{pmatrix} \sqrt{Y} & X\sqrt{Y^{-1}} \\ 0 & \sqrt{Y^{-1}} \end{pmatrix}, \tag{2.4}$$

as we have seen in the proof of the transitivity, proposition 2.1.3. As it is natural to make the point $iI \in \mathrm{SH}_n$ correspond to the class of the identity matrix, and we expect the action to be left matrix multiplication, it is also natural to define the map between $\mathrm{SH}_n$ and $\mathrm{Sp}_{2n}\mathbb{R}/\mathrm{SpO}_{2n}\mathbb{R}$ by

$$\begin{array}{rccc} \Phi_3 & : & \mathrm{SH}_n & \rightarrow & \mathrm{Sp}_{2n}\mathbb{R}/K \\ & & Z & \mapsto & M_Z K \end{array}$$

To see that this is a 1-1 map, suppose that the image of $X_1 + iY_1$, is the same equivalence class. Then $M_{X+iY}M^{-1}_{X_1+iY_1}$ would be a symplectic orthogonal block upper triangular matrix, and, according to formula (2.3), it would have to be the identity. Hence from block $(2,2)$ we get $Y = Y_1$ and then from block $(1,2)$ we get $X = X_1$. This correspondence is also studied in [Fre]. From the symplectic polar decomposition we can also see that there is a unique symplectic positive definite symmetric representative in each class.

We'll denote by $\mathrm{sp}_{2n}\mathbb{R}$ the Lie algebra of $\mathrm{Sp}_{2n}\mathbb{R}$. This is the set of all matrices $M$ in $\mathrm{M}_{2n}\mathbb{R}$ that satisfy the equation $M^{\mathrm{T}}J + JM = 0$. We have

$$M = \begin{pmatrix} A & B \\ C & -A^{\mathrm{T}} \end{pmatrix}, \text{ with } B \text{ and } C \text{ symmetric.} \tag{2.5}$$

We see that the matrix $M$ must have trace zero.

This concludes our presentation of models. As we will see, each one of them will prove to be useful, depending on what kind of result we desire. For instance, if we restrict any of the metrics $d_p$ defined in the Introduction to the space $\mathrm{Sp}_{2n}\mathbb{R}/K$, we get from those results that $\mathrm{Sp}_{2n}\mathbb{R}$ acts as a group of isometries for any one of those metrics, since its action is the restriction of the action of $\mathrm{GL}_n\mathbb{R}$ (left multiplication). This is very easily verified using the quotient space model. Notice also that given a symplectic matrix $A$, $A^{\mathrm{T}}$ is also symplectic, and hence $AA^{\mathrm{T}}$ is symplectic. This means that $\sigma_j(A) = \sigma_{n-j+1}(A)^{-1}$ for $1 \leq j \leq n$, and if $A,\ B \in \mathrm{Sp}_{2n}\mathbb{R}$,

$$d_p(A,B)^p = \sum_{j=1}^{2n} |\log \sigma_j(A^{-1}B)|^p = 2\sum_{j=1}^{n} \log \sigma_j(A^{-1}B)^p. \tag{2.6}$$

In [Sie], it is proved that symplectic transformations are isometries for the Siegel metric in $\mathrm{SH}_n$. This metric corresponds to $d_2$ in $\mathrm{Sp}_{2n}\mathbb{R}/K$ as we will see. It can be defined on $\mathrm{SH}_n$ using the distance element at the point $Z = X + iY$, as defined in [Sie, p. 17]:

$$ds^2 = \mathrm{tr}(Y^{-1}dZY^{-1}d\overline{Z}).$$

The distance induced in $\mathrm{SH}_n$ by this Riemannian metric can be found in [Sie, p. 19], and is

$$d_S(Z_1,Z_2) = \left(\sum_{k=1}^{n} \log^2 \frac{1+\sqrt{r_k}}{1-\sqrt{r_k}}\right)^{\frac{1}{2}}, \quad Z_1, Z_2 \in \mathrm{SH}_n,$$

where the $r_k$'s are the eigenvalues of the cross-ratio

$$R(Z_1,Z_2) := (Z_1 - Z_2)(Z_1 - \overline{Z_2})^{-1}(\overline{Z_1} - \overline{Z_2})(\overline{Z_1} - Z_2)^{-1}.$$

Siegel proves that there is a symplectic map that maps $(A, B)$ to $(C, D)$ if and only if $R(A, B)$ and $R(C, D)$ have the same eigenvalues. Here we give a condition in the Lie group model.

**Proposition 2.3.1** *Let $(A, B)$, $(C, D)$ be two pairs of points in $\mathrm{Sp}_{2n}\mathbb{R}/K$. Then there exists a symplectic transformation mapping $A$ to $C$ and $B$ to $D$ if and only if*

$$\sigma_j(A^{-1}B) = \sigma_j(C^{-1}D) \text{ for all } j,$$

*or equivalently, if $d_p(A, B) = d_p(C, D)$ for all $1 \leq p \leq \infty$.*

**Proof.** If $TA \in CK, TB \in DK$ then for some $Q_1, Q_2 \in K$, $A^{-1}B = Q_1C^{-1}DQ_2$ and the singular value equality holds.

Suppose now that we have this equality. Clearly, the pairs $(A, B), (C, D)$ can be mapped to the pairs $(I_{2n}, A^{-1}B), (I_{2n}, C^{-1}D)$ respectively. Because we have a singular value decomposition with orthogonal matrices taken from $K$ (proposition 2.2.8), there exist $Q_1, Q_2 \in K$ such that $Q_1A^{-1}BQ_2 = C^{-1}D$. Hence the pairs $(Q_1K, Q_1A^{-1}BK)$ and $(K, C^{-1}DK)$ are the same. □

**Corollary 2.3.2** *The Siegel metric in $\mathrm{SH}_n$ corresponds to $d_2$ in $\mathrm{Sp}_{2n}\mathbb{R}/K$.*

**Proof.** First, take two points $X,\ Y \in \mathrm{SH}_n$. By the previous result it is possible to simultaneously bring their corresponding cosets in $\mathrm{Sp}_{2n}\mathbb{R}/K$ to

$$I_{2n}K \text{ and } (D \oplus D^{-1})K,\ D = \mathrm{diag}(\sqrt{y_1}, \ldots, \sqrt{y_n}),$$

where $\sqrt{y_j} = \sigma_j(M_X^{-1}M_Y) \geq 1,\ 1 \leq j \leq n$ (see equation (2.4)). These points correspond to $iI_n$ and $i\,\mathrm{diag}(y_1, \ldots, y_n) = iD^2$ in $\mathrm{SH}_n$. It is now a matter of carrying the computations to see that indeed

$$d_S^2(iI_n, iD^2) = d_2^2(I_{2n}, D \oplus D^{-1}) = \sum_{j=1}^{n} \log^2 y_j.$$

This proves the desired result. □

We now proceed to compactify the space in order to be able to do a dynamical study similar to the one done on the hyperbolic plane.

# Chapter 3

# Compactifications

## 1 The boundaries

In this chapter we will be mainly concerned with the compactification of the Siegel upper half plane. In the two-dimensional model, this could be done by taking the closure of the circle model of the space, since this was a bounded set. We'll do the same thing here. Thus, let

$$\mathrm{Cl}(\mathrm{SD}_n) = \{Z \in \mathrm{M}_n\mathbb{C} : I - Z\overline{Z} \geq 0\}.$$

Our first remark is that this space has a stratification of the boundary. The strata are

$$\partial_k \mathrm{SD}_n = \{Z \in \partial \mathrm{SD}_n : \mathrm{rank}(I - Z\overline{Z}) = n - k\}.$$

This can be written in terms of singular values as:

$$\partial_k \mathrm{SD}_n = \{Z \in \partial \mathrm{SD}_n : \sigma_1(Z) = \ldots = \sigma_k(Z) = 1 > \sigma_{k+1}(Z)\}$$

for $k \leq n - 1$, and

$$\partial_n \mathrm{SD}_n = \{Z \in \partial \mathrm{SD}_n : \sigma_1(Z) = \cdots = \sigma_n(Z) = 1\}.$$

Observe that

$$\mathrm{USym}_n := \partial_n \mathrm{SD}_n = \mathrm{U}_n \cap \mathrm{Sym}_n\mathbb{C}.$$

The group acting on $\mathrm{SD}_n$ is a conjugate of $\mathrm{Sp}_{2n}\mathbb{R}$ in $\mathrm{Sp}_{2n}\mathbb{C}$ as we have seen. We now extend this action by continuity to the boundary, still by Möbius transformations. We can also take the closure of the Siegel upper half plane,

$$\mathrm{Cl}(\mathrm{SH}_n) = \{Z \in \mathrm{Sym}_n\mathbb{C} : \mathrm{Im}(Z) \geq 0\},$$

and then try to match it to $\partial\mathrm{SD}_n$ using the extensions of the maps $\Phi_1$ and $\Phi_1^{-1}$ defined in the last chapter, but we'll notice that the map $\Phi_1^{-1}$ is not defined on the set

$$\{Z \in \mathrm{SD}_n : \det(Z - I) = 0\}.$$

On the hyperbolic plane, this set was reduced to one point, namely the complex number 1. This set is clearly contained in $\partial\mathrm{SD}_n$ and we will call it informally the *infinite boundary.* We will call its complement in $\partial\mathrm{SD}_n$ the *finite boundary.* We note that the finite boundary of $\mathrm{SD}_n$ contains a part of every stratum.

**Lemma 3.1.1** *The image of the finite part of the stratum $\partial_k\mathrm{SD}_n$ under the extension of $\Phi_1^{-1}$ is*

$$\mathrm{fin}(\partial_k\mathrm{SH}_n) = \{Z \in \mathrm{Sym}\mathbb{C} : \mathrm{Im}(Z) \geq 0 \ \textit{and} \ \mathrm{rank}(\mathrm{Im}\, Z) = n - k\}.$$

**Proof.** Let's compute the imaginary part of $\Phi_1^{-1}(Z)$, with $Z$ in the finite boundary of $\mathrm{SD}_n$.

$$\begin{aligned}
&\mathrm{Im}(\Phi_1^{-1}(Z)) = \\
&\quad = \frac{1}{2i}\left(\Phi_1^{-1}(Z) - \overline{\Phi_1^{-1}(Z)}\right) \\
&\quad = \frac{1}{2i}\left(i(I+Z)(I-Z)^{-1} + i(I+\overline{Z})(I-\overline{Z})^{-1}\right) \\
&\quad = \frac{1}{2}(I-Z)^{-1}\left((I+Z)(I-\overline{Z}) + (I-Z)(I+\overline{Z})\right)(I-\overline{Z})^{-1} \\
&\quad = (I-Z)^{-1}(I - Z\overline{Z})((I-Z)^{-1})^{*}.
\end{aligned}$$

So we see that this imaginary part is always similar to $I - Z\overline{Z}$, and we have the result. □

Even though $\mathrm{Cl}(\mathrm{SH}_n)$ only contains the points in the finite part of the boundary, we'll always consider that $\partial\mathrm{SH}_n$ will have the same structure as $\partial\mathrm{SD}_n$, that is, that $\partial\mathrm{SH}_n$ also contains the infinite part of the boundary.

The next result concerns the boundary in the projective model $\mathrm{SPH}_n$.

**Lemma 3.1.2** *The finite boundary in $\mathrm{SPH}_n$ is the set of all equivalence classes that admit a representative of the type*

$$\begin{pmatrix} Z \\ I \end{pmatrix} \ \textit{with } Z \textit{ symmetric and } \mathrm{Im}\, Z \geq 0,$$

*and such a representative is unique. Moreover, let $Z_1, Z_2$ be points in the finite boundary of $\mathrm{SH}_n$ such that*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{bmatrix} Z_1 \\ I \end{bmatrix} = \begin{bmatrix} Z_2 \\ I \end{bmatrix}.$$

*Then $CZ_1 + D$ is invertible.*

**Proof.** These are all immediate consequences of the definitions of the action and of the quotient space. $\square$

**Proposition 3.1.3** *Each stratum of $\partial\mathrm{SD}_n$ is an orbit for the action of $\mathrm{Sp}_{2n}\mathbb{R}$.*

**Proof.** Let $Z_1$ and $Z_2$ be points in $\partial_k\mathrm{SD}_n$. Notice that we can bring these to the finite part of this stratum, by means of a rotation: it is possible to choose $\theta$ such that neither $e^{i\pi\theta}Z_1$ nor $e^{i\pi\theta}Z_2$ have 1 as an eigenvalue. This transformation corresponds to the matrix $\mathrm{diag}(e^{\frac{i\pi\theta}{2}}I_n, e^{-\frac{i\pi\theta}{2}}I_n)$. Once this is done, we can use the extension of $\Phi_1^{-1}$ to map them to the boundary of $\mathrm{SH}_n$ and prove the transitivity on $\partial_k\mathrm{SH}_n$. As we did in the proof of the transitivity of the action, assume that $Z_2 = i\,\mathrm{diag}(1,\ldots,1,0,\ldots,0)$, with $k$ zeros on the diagonal and $Z_2 = X + iY$ is any point in $\partial_k\mathrm{SH}$. Take $Q \in \mathrm{O}_n$ such that

$$Y = Q\,\mathrm{diag}(r_1,\ldots,r_{n-k},0,\ldots,0)Q^{\mathrm{T}}, \quad r_1,\ldots,r_{n-k} > 0.$$

Set $D := \mathrm{diag}(r_1,\ldots,r_{n-k},1,\ldots,1)$. Now we see that it is possible to map $Z_1$ to $Z_2$ using the composition of the maps defined by the symplectic matrices

$$\mathrm{diag}(\sqrt{D},\sqrt{D^{-1}}), \quad \mathrm{diag}(Q,Q) \text{ and } \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$$

in the order presented.

Now, let's see that if $Z \in \partial_k\mathrm{SD}_n$, then $M(Z) \in \partial_k\mathrm{SD}_n$. Take $Z \in \partial_k\mathrm{SD}_n$ and, as above, consider both $Z$ and $M(Z)$ to be in the finite part of the boundary, since the unitary diagonal transformations clearly preserve the starta $\partial_j\mathrm{SD}_n$. Now, for

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \begin{pmatrix} E \\ F \end{pmatrix} = M\begin{pmatrix} Z \\ I \end{pmatrix},$$

we have, like in the proof of the transitivity of the action,

$$\begin{aligned} -\frac{1}{2i}(E^*F - F^*E) &= -\frac{1}{2i}\begin{pmatrix} E^* & F^* \end{pmatrix} J \begin{pmatrix} E \\ F \end{pmatrix} \\ &= -\frac{1}{2i}\begin{pmatrix} Z^* & I \end{pmatrix} M^{\mathrm{T}}JM \begin{pmatrix} Z \\ I \end{pmatrix} \\ &= -\frac{1}{2i}\begin{pmatrix} Z^* & I \end{pmatrix} J \begin{pmatrix} Z \\ I \end{pmatrix}. \\ &= \mathrm{Im}(Z). \end{aligned}$$

Notice that, by the previous lemma, $F$ is invertible, so

$$F^* \operatorname{Im} M(Z) F = -\frac{1}{2i} F^* \left( (F^{-1})^* E^* - E F^{-1} \right) F = \operatorname{Im} Z,$$

so the rank is preserved. $\square$

The previous result is a particular case of the general result about boundary components of bounded symmetric domains as described in [Bai, p. 200].

**Proposition 3.1.4** $\mathrm{USym}_n = \mathrm{U}_n \cap \mathrm{Sym}_n\mathbb{C}$ *is the Shilov boundary of* $SD_n$.

**Proof.** Note that $\mathrm{SD}_n \cap \mathrm{Diag}_n\mathbb{C}$ is equal to $\mathrm{D}^n$, where $\mathrm{D} \subset \mathbb{C}$ is the unit disk. Clearly, the Shilov boundary of $\mathrm{SD}_n \cap \mathrm{Diag}_n\mathbb{C}$ is $(S^1)^n$, which is equal to $\mathrm{USym}_n \cap \mathrm{Diag}_n\mathbb{C}$. Let $f : \mathrm{SD}_n \to \mathbb{C}$ be a holomorphic function which extends to a continuous function on $\mathrm{Cl}(\mathrm{SD}_n)$. Then

$$|f(0)| \leq \max_{Z \in \mathrm{USym}_n \cap \mathrm{Diag}_n\mathbb{C}} |f(Z)| \leq \max_{Z \in \mathrm{USym}_n} |f(Z)|.$$

As $\mathrm{Sp}_{2n}\mathbb{R}$ acts transitively on $\mathrm{SD}_n$ and preserves $\mathrm{USym}_n\mathbb{C}$, for any $W \in \mathrm{SD}_n$ we have

$$|f(W)| \leq \max_{Z \in \mathrm{USym}_n} |f(Z)|.$$

Hence the Shilov boundary of $\mathrm{SD}_n$ is contained in $\mathrm{USym}_n$. As $\mathrm{Sp}_{2n}\mathbb{R}$ acts transitively on $\mathrm{USym}_n$ we deduce the result. $\square$

As we have seen, the action of $\mathrm{Sp}_{2n}\mathbb{R}$ on the Shilov boundary is transitive, so, if we take $0 \in \partial_n\mathrm{SH}_n$, we'll have a presentation of the Shilov boundary as $\mathrm{Sp}_{2n}\mathbb{R}/\operatorname{Stab}(0)$, and it is easy to see that this stabilizer is the set of all lower block triangular symplectic matrices. These have the form

$$\begin{pmatrix} A & 0 \\ C & A^{-\mathrm{T}} \end{pmatrix} \tag{3.1}$$

with $A^{\mathrm{T}}C$ symmetric.

We now present two more possible boundaries of this space. For this, we need some more information about the Lie group structure. Recall that for $A \in \mathrm{M}_m\mathbb{R}$ the Frobenius norm $||A||_F$ is given by $\sqrt{\operatorname{tr}(AA^{\mathrm{T}})}$. Now consider the Lie group quotient model, $\mathrm{Sp}_{2n}\mathbb{R}/K$, $K := \mathrm{SpO}_n$.

**Lemma 3.1.5** *Let* $Q \in K$. *Then the automorphism of* $\mathrm{sp}_{2n}\mathbb{R}$ *defined by conjugation by* $Q$ *maps Weyl chambers to Weyl chambers.*

**Proof.** The defining equation for $\mathrm{sp}_{2n}\mathbb{R}$ is $M^{\mathrm{T}}J + JM = 0$. Then for $Q \in K$ and $M \in \mathrm{sp}_{2n}\mathbb{R}$,

$$QM^{\mathrm{T}}Q^{\mathrm{T}}J + JQMQ^{\mathrm{T}} = Q(M^{\mathrm{T}}J + JM)Q^{\mathrm{T}} = 0.$$

It is now easy to check the morphism properties. To see it preserves Weyl chambers, notice that is maps the walls of Weyl chambers one to another, since the elements of these walls are the matrices $A$ such that $\lambda_j(A) = \lambda_{j+1}(A)$ for some $j$. Moreover, it preserves the inner product $A|B = \mathrm{tr}(AB^{\mathrm{T}})$, so it must preserve the angles between these walls. Hence, it maps a Weyl chamber into another Weyl chamber. $\square$

**Proposition 3.1.6** *Any coset $AK \in \mathrm{Sp}_{2n}\mathbb{R}/K$ has a unique symmetric positive definite element $B \in AK \cap \mathrm{Sym}_{2n}\mathbb{R}$ with $B > 0$. This element can be obtained as $e^{tR}$ with $R \in \mathrm{sp}_{2n}\mathbb{R}$, $||R||_F = \sqrt{2}/2$, $t = d_2(I_{2n}, A)$.*

**Proof.** The existence and uniqueness of $B$ is a consequence of the symplectic polar decomposition: if $A = Q_1\Sigma(A)Q_2$, then

$$B = Q_1\Sigma(A)Q_1^{\mathrm{T}} = A(Q_2^{\mathrm{T}}Q_1^{\mathrm{T}}) \in AK.$$

For the result about the exponential, take

$$r_j = \frac{\log \sigma_j(A)}{d_2(I_{2n}, A)}, \quad j = 1, ..., n,$$

and take $D := \mathrm{diag}(r_1, \dots, r_n)$, $R := Q_1 \,\mathrm{diag}(D, -D)Q_1^{\mathrm{T}}$, $Q \in K$. It is now easy to check all the assertions. $\square$

In our space, the *Visibility boundary* is the set of all the geodesics coming out of a reference point. In our case, if we fix $I_{2n}K$, and consider the cosets $BK$ with $d_2(I_2K, BK) = 1$, we get one geodesic per coset, so, in view of our previous result, we have one geodesic for each matrix of the type $Q(D \oplus -D)Q^{\mathrm{T}}$ as defined above, $Q \in K$.

Before we present yet another boundary, it is pertinent to make a short study of the flats of our symmetric space. These are maximal submanifolds of the symmetric space with zero sectional curvature. Their dimension is exactly the rank of the group of symmetries, which in turn is the dimension of a Cartan subalgebra of its Lie algebra. In our case, we can take as the Cartan subalgebra[1] the space $\mathcal{C}$ of all diagonal elements of the Lie algebra:

$$\mathcal{C} = \{D \oplus -D : D = \mathrm{diag}(r_1, \dots, r_n),\ r_j \in \mathbb{R}\}.$$

[1]Every time we mention *the* Cartan subalgebra, we are referring to this space, by abuse of language.

This space has clearly dimension $n$. Then $e^{\mathcal{C}}$ is going to yield a flat $e^{\mathcal{C}}K$. This is because all flats can be obtained as images (under the exponential map) of a maximal abelian subalgebra contained in the tangent space at the coset $K$ (see [Hel, p. 245]). By the previous result, all these can be expressed as $Q\mathcal{C}Q^{\mathrm{T}}$, $Q \in K$. To verify this in our present case, for $Q \in K$, set

$$\mathcal{F}(Q) := e^{Q\mathcal{C}Q^{\mathrm{T}}} = \{Qe^{D\oplus -D}Q^{\mathrm{T}} : D = \mathrm{diag}(r_1, \dots, r_n), r_j \in \mathbb{R}\}.$$

Then, given two elements $E_1, E_2 \in Q\mathcal{C}Q^{\mathrm{T}}$,

$$E_k = Q(X_k \oplus -X_k)Q^T,\ X_k = \mathrm{diag}(x_{k1}, ..., x_{kn}),$$

with $x_{kj} \in \mathbb{R},\ j = 1 \dots, n,\ k = 1, 2$, the Riemannian distance between the corresponding elements in $\mathcal{F}(Q)$ is given by

$$d_2(e^{E_1}, e^{E_2}) = \sqrt{2}||x^1 - x^2||_2,$$

for $x^k = (x_{k1}, ..., x_{kn}),\ k = 1, 2$. So we see that $\mathcal{F}(Q)$ is indeed a flat, since with the restriction of the Siegel metric $d_2$, it becomes isometric to $\mathbb{R}^n$ with the usual Riemannian metric, and since the rank of $\mathrm{Sp}_{2n}\mathbb{R}$ is $n$, the subvariety is indeed maximal.

The rank of $\mathrm{SL}_2\mathbb{R}$ is 1, so the flats in the 2-dimensional upper plane were just the geodesics.

We now describe the space of all flats of $\mathrm{Sp}_{2n}\mathbb{R}$ containing the coset $K$. A certain geodesic ray $e^{tR}$, $R \in \mathrm{sp}_{2n}\mathbb{R}$, coming from the base coset $K$ will be in one and only one flat if and only if $R$ is a *regular element* of $\mathrm{sp}_{2n}\mathbb{R}$ (see [Kai]), that is, if $R$ is similar to an element $D \oplus -D \in \mathcal{C}$ such that

$$D = (r_1, \dots, r_n) \text{ with } r_j \neq r_k, \text{ for } j \neq k.$$

This is equivalent to saying that the dimension of its commutator in $\mathrm{sp}_{2n}\mathbb{R}$ is minimal amongst the dimensions of all commutators of elements of $\mathcal{C}$. Let $\mathcal{M}_n$ be the monomial group, that is, the subgroup of $\mathrm{O}_n$ whose elements are matrices with exactly $n$ entries equal to $\pm 1$ and all others equal to 0.

**Theorem 3.1.7** *The space of all flats of* $\mathrm{Sp}_{2n}\mathbb{R}/K$ *containing the coset* $K$ *is homeomorphic to* $K/(\mathcal{M}_{2n} \cap K)$.

**Proof.** Consider a regular element $D \oplus -D \in \mathcal{C}$. This element determines the flat $\mathcal{F}(I_{2n})$ uniquely. Then

$$\begin{aligned} \mathcal{F}(Q) = \mathcal{F}(I_{2n}) &\Leftrightarrow Q(D \oplus -D)Q^T \in \mathcal{C} \\ &\Leftrightarrow Q \in \mathcal{M}_{2n}. \end{aligned}$$

We then have our result. $\square$

Each flat can be decomposed into a union of closed Weyl chambers with disjoint interiors. To do this, consider for instance the flat $e^{\mathcal{C}}$ and remove from it the geodesic rays corresponding to the non-regular elements in the Cartan subalgebra. Then once you remove those rays from the flat, what you're left with is a union of a certain number of connected components. Each one of those is the image of a Weyl chamber in the Cartan subalgebra, and if we close them, their union gives us the whole flat. Note that we use the name "Weyl chamber" both for subsets of $\mathrm{sp}_{2n}\mathbb{R}$ and for their images under the exponential map in $\mathrm{Sp}_{2n}\mathbb{R}/K$.

The *Furstenberg boundary* can be defined as $\mathrm{Sp}_{2n}\mathbb{R}/P$ where $P$ is a maximal parabolic subgroup, that is, a subgroup that fixes a geodesic ray $e^{tR}$, with a regular $R$. This boundary can also be defined (see [Kai]) as the set of all the Weyl chambers based at a point $AK$. We'll (naturally) take as our dominant Weyl chamber in $\mathrm{sp}_{2n}\mathbb{R}$ the set $W$ of all diagonal matrices of the form

$$\mathrm{diag}(r_1, \ldots, r_n, -r_1, \ldots, -r_n), \text{ with } r_1 > r_2 > \ldots > r_n > 0,$$

as in [FH2]. From proposition 3.1.6, we can see that all other Weyl chambers in the tangent space can be obtained as $QWQ^{\mathrm{T}}$, $Q \in K$, since the action of $K$ is transitive on $\mathrm{sp}_{2n}\mathbb{R}$ and each Weyl chamber is determined by any of its elements $AK$ with $A$ symmetric and $\lambda_j(A) \neq \lambda_k(A)$ for all $j \neq k$.

Now, let $\mathcal{D}_m$ be the group of all $m \times m$ diagonal matrices with diagonal entries equal to plus or minus 1.

**Theorem 3.1.8** *The Furstenberg boundary of* $\mathrm{Sp}_{2n}\mathbb{R}/K$ *can be presented as*

$$K/(\mathcal{D}_{2n} \cap K) \sim \mathrm{U}_n/\mathcal{D}_n.$$

**Proof.** Let $D \oplus -D$, be a regular element in the dominant Weyl chamber, $D = (r_1, \ldots, r_n)$, $r_1 > r_2 > \ldots > r_n > 0$. Then, for $Q \in K$,

$$\begin{aligned} QWQ^{\mathrm{T}} = W \quad &\Leftrightarrow \quad Q(D \oplus -D)Q^{\mathrm{T}} \in W \\ &\Leftrightarrow \quad Q(D \oplus -D)Q^{\mathrm{T}} = D \oplus -D \\ &\Leftrightarrow \quad Q \in \mathcal{D}_{2n} \end{aligned}$$

This gives us the first part of the result. As for the second, notice that $K$ is the subgroup of $\mathrm{Sp}_{2n}\mathbb{R}$ that fixes the coset $K$. If we now think of the disk model $\mathrm{SD}_n$, then the coset $K$ corresponds to the element $0 \in \mathrm{SD}_n$ and its stabilizer is a conjugate of $K$, namely, the subgroup of all matrices $A \oplus \overline{A}$, with $A \in \mathrm{U}_n$, clearly isomorphic to $\mathrm{U}_n$. If we also conjugate $\mathcal{D}_{2n} \cap K$, the corresponding subgroup will be $\mathcal{D}_n$, and this concludes our proof. □

We see that for $n > 1$ the Shilov boundary $\mathrm{USym}_n$ has smaller dimension the Furstenberg boundary of $\mathrm{SD}_n$.

**Proposition 3.1.9** *The Furstenberg boundary has dimension $n^2$ and the Shilov boundary has dimension $n(n+1)/2$.*

**Proof.** The dimension of $\mathrm{U}_n$ is $n^2$, it can be seen by considering its Lie algebra, the space of skew-hermitian matrices. The subspace of this Lie algebra that yields the unitary symmetric matrices is the space of all skew-hermitian and skew-symmetric matrices, of dimension n(n+1)/2. □

# 2 The Shilov boundary

The Shilov boundary of $\mathrm{SD}_n$ can be obtained by identifying the points in the Furstenberg boundary using the action of $\mathrm{O}_n$.

**Theorem 3.2.1** *The Shilov boundary $\mathrm{USym}_n$ of $\mathrm{SD}_n$ can be presented as*

$$K/\tilde{\mathrm{O}}_n \sim \mathrm{U}_n/\mathrm{O}_n$$

*where $\tilde{\mathrm{O}}_n$ is the set of all matrices of the form $\mathrm{diag}(Q,Q)$ with $Q \in \mathrm{O}_n$*

**Proof.** Let's consider the action of $K$ over $\partial_n \mathrm{SH}_n$. We'll prove that this action is transitive. Once we've proved this, we know from equation (3.1) that the stabilizer of $0 \in \partial_n \mathrm{SH}_n$ in $K$ is the set of all symplectic orthogonal matrices with the (1,2) block equal to zero[2], that is, it's exactly $\tilde{\mathrm{O}}_n$. This gives us the first presentation. To get the second, consider the action of the conjugate groups in $\mathrm{Sp}_{2n}\mathbb{C}$ on $\mathrm{SD}_n$. These are, respectively,

$$\tilde{\mathrm{U}}_n := \{(A \oplus \overline{A}) : A \in \mathrm{U}_n\} \text{ and } \tilde{\mathrm{O}}_n,$$

since the conjugate of the element

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K \text{ is } \begin{pmatrix} A+iB & 0 \\ 0 & A-iB \end{pmatrix} \in \tilde{\mathrm{U}}_n.$$

Thus, the Shilov boundary can be presented as $\tilde{\mathrm{U}}_n/\tilde{\mathrm{O}}_n \sim \mathrm{U}_n/\mathrm{O}_n$.

[2]The stabilizer of $0 \in \partial_2 \mathrm{SH}_2$ will be studied in more detail in the next chapter

To establish the transitivity of $K$ on $\partial \mathrm{SH}_n$, let's prove we can bring any point $Z \in \partial_n \mathrm{SH}_n$ to the point 0. Notice that we've seen that it is possible to bring a point in the infinite boundary to the finite boundary of $\mathrm{SD}_n$ using the action of a diagonal unitary matrix. In $\mathrm{SH}_n$ this corresponds to a matrix in $K$, so we can assume that $Z$ is in the finite boundary of $\mathrm{SH}_n$. Now, since $Z$ is real symmetric, we can diagonalize it by an orthogonal matrix, say $Q$, which corresponds to the action of $\mathrm{diag}(Q, Q) \in K$. Now we have $Z = \mathrm{diag}(z_1, \ldots, z_n)$. Let $z_j$ be any of these diagonal entries, $z_j \in \partial \mathrm{H}$. It is possible now to find an elliptic map $M_j \in \mathrm{SL}_2\mathbb{R}$ such that $M_j(z_j) = 0$. This will be a rotation around $i \in \mathrm{H}$, given by the matrix

$$\begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix}, \text{ with } \tan\theta_j = -z_j.$$

Notice that $\theta_j$ is exactly the angle symmetric to the one formed by the imaginary axis and the line connecting $i$ and $z_j$. Now

$$M_1 \odot \ldots \odot M_n \in K \text{ and } M_1 \odot \ldots \odot M_n(\mathrm{diag}(z_1, \ldots, z_n)) = 0 \in \partial_n \mathrm{SH}_n,$$

as we wished. $\square$

The results of the two previous theorems can be found in [Jo1]. Notice that, from the previous result, we can conclude that the action of $K$ is transitive on $\partial_n \mathrm{SH}_n$.

To end this chapter, we present another model for the Shilov boundary involving Busemann functions. Consider the group $\tilde{\mathrm{O}}_n$ defined in the last theorem, and take

$$\mathcal{W} = \{D \oplus -D : D = \mathrm{diag}(r_1, \ldots, r_n),\ r_1 \geq \ldots \geq r_n > 0\} \subset \mathrm{sp}_{2n}\mathbb{R}.$$

Let $\mathcal{E}_0$ be the subset of $\mathrm{sp}_{2n}\mathbb{R}$ defined by

$$\mathcal{E}_0 := \{Q(D \oplus -D)Q^{\mathrm{T}} : Q \in \tilde{\mathrm{O}}_n,\ D \in \mathcal{W}\}.$$

Notice that $\mathcal{E}_0$ is a union of Weyl chambers. The presently relevant fact about $\mathcal{E}_0$ is that for any $A \in \mathcal{E}_0$, the ray in $\mathrm{SH}_n$ corresponding to $e^{-tA}$ converges to the point $0 \in \partial_n \mathrm{SH}_n$. To see this, notice that if $A = Q(D \oplus -D)Q^{\mathrm{T}}$ with $Q = Q_1 \oplus Q_1$,

$$e^{-tA} = \begin{pmatrix} Q_1 e^{-tD} Q_1^{\mathrm{T}} & 0 \\ 0 & Q_1 e^{tD} Q_1^{\mathrm{T}} \end{pmatrix},$$

and according to equation (2.4) in Chapter 2, the sequence corresponding to this one in $\mathrm{SH}_n$ goes to 0.

This allows us to identify the set $\mathcal{E}_0$ with this limit point. Any other point in $\partial_n \mathrm{SH}_n$ can be expressed as $R(0)$, $0 \in \partial_n \mathrm{SH}_n$, $R \in K$ (see previous theorem). We see that for $A \in \mathcal{E}_0$,

$$e^{-tRAR^{\mathrm{T}}} K = Re^{-tA} R^{\mathrm{T}} K = Re^{-tA} K,$$

and so the corresponding ray in $\mathrm{SH}_2$ will converge to the point $R(0)$. So we associate the set $R\mathcal{E}_0R^{\mathrm{T}}$ with the point $R(0)$. It is easy to see that $R\mathcal{E}_0R = \mathcal{E}_0$ if and only if $R \in \tilde{\mathrm{O}}_n$, so the set of all images

$$\{R\mathcal{E}_0R^{\mathrm{T}} : R \in K\} \tag{3.2}$$

can be expressed as $K/\tilde{\mathrm{O}}_n$, so it is indeed the Shilov boundary. Now we pick a special representative in each $R\mathcal{E}_0R^{\mathrm{T}}$, namely $R(I_n \oplus -I_n)R^{\mathrm{T}}$. Notice that there is one and only one representative of this type in each set, since

$$R_1(I_n \oplus -I_n)R_1^{\mathrm{T}} = R_2(I_n \oplus -I_n)R_2^{\mathrm{T}} \Leftrightarrow R_1R_2^{\mathrm{T}} \in \tilde{\mathrm{O}}_n,$$

and moreover, this element is in the boundary of all the Weyl chambers contained in $\mathcal{E}_0$. Set

$$\mathcal{L} := \{R(I_n \oplus 0_n)R^{\mathrm{T}} : R \in K\}.$$

It's clear that

$$R_1(I_n \oplus 0_n)R_1^{\mathrm{T}} = R_2(I_n \oplus 0_n)R_2^{\mathrm{T}} \Leftrightarrow R_1(I_n \oplus -I_n)R_1^{\mathrm{T}} = R_2(I_n \oplus -I_n)R_2^{\mathrm{T}},$$

so $\mathcal{L}$ is another presentation of the set in (3.2) above.

Now, notice that, for

$$A \in \mathcal{E}_0,\ A = Q\,\mathrm{diag}(r_1, \ldots, r_n, -r_1, \ldots, -r_n)Q^{\mathrm{T}},\ r_1 \geq \ldots \geq r_n > 0,\ Q \in \tilde{\mathrm{O}}_n$$

and $R \in K$, we have

$$\begin{aligned}\lim_{t\to\infty} \frac{\wedge_n(e^{tRAR^{\mathrm{T}}})}{||\wedge_n(e^{tRAR^{\mathrm{T}}})||} &= \lim_{t\to\infty} \wedge_n(R)\frac{\wedge_n(e^{tRAR^{\mathrm{T}}})}{e^{\sum_{j=1}^n r_j}} \wedge_n(R^{\mathrm{T}}) \\ &= \wedge_n(R(I_n \oplus 0_n)R^{\mathrm{T}}),\end{aligned} \tag{3.3}$$

In this case we'll say that $e^{tRAR^{\mathrm{T}}} \to R(I_n \oplus 0_n)R^{\mathrm{T}}$. We now prove this convergence this is well defined.

**Lemma 3.2.2** *Let $F, F'$ be two distinct elements of $\mathcal{L}$. Then, if $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are the $n$ first eigenvectors of $F$ and $F'$ respectively (associated with the eigenvalue 1), then*

$$<x_1, \ldots, x_n> \neq <y_1, \ldots, y_n>$$

*or equivalently, $\wedge_n(x_1, \ldots, x_n) \neq \pm \wedge_n(y_1, \ldots, y_n)$.*

**Proof.** Take $F = Q(I_n \oplus 0_n)Q^{\mathrm{T}}$ and $F' = R(I_n \oplus 0_n)R^{\mathrm{T}}$, $Q, R \in K$. The columns of $Q$ and $R$ are exactly the eigenvectors of $F$ and $F'$ respectively—take $Q = (x_1, \ldots, x_{2n})$ and $R = (y_1, \ldots, y_{2n})$. Suppose now that the span of the first $n$ columns of $Q$ is equal to the span of the first $n$ columns of $R$, and let's prove $F = F'$. If they are the same, then the off-diagonal $n \times n$ blocks of $Q^{\mathrm{T}}R$ are zero, since for $j \leq n$ and $k > n$, $y_j^{\mathrm{T}} y_k = 0$, and given the condition on the spans, $y_j^{\mathrm{T}} x_k = 0$. In a similar way, we could see that $x_k^{\mathrm{T}} y_j = 0$. This means that $Q^{\mathrm{T}}R \in \tilde{\mathrm{O}}_n = \mathrm{Stab}(I_n \oplus 0_n)$, and we then have

$$F = Q(I_n \oplus 0_n)Q^{\mathrm{T}} = Q(Q^{\mathrm{T}}R)(I_n \oplus 0_n)(R^{\mathrm{T}}Q)Q^{\mathrm{T}} = R(I_n \oplus 0_n)R^{\mathrm{T}} = F'.$$

For the last result, notice that the referred spans are equal if and only if the vectors $\wedge_n(x_1, \ldots, x_n)$ and $\wedge_n(y_1, \ldots, y_n)$ are collinear. They have both norm one, because they can be expressed as $\wedge_n Q e_1$, $\wedge_n R e_1$ where $e_1 \in \mathrm{M}_{N,1}$, $N = \binom{2n}{n}$ is the first vector of the canonical basis, $e_1 = \wedge_n(I_n, 0_n)^{\mathrm{T}}$. As they have norm 1, they are collinear if and only if they are either equal or symmetric. This concludes our proof. $\square$

It is easy to check that $\wedge_n(F) = \wedge_n(x_1, \ldots, x_n) \wedge_n (x_1, \ldots, x_n)^{\mathrm{T}}$. Since for a real symmetric rank 1 matrix $M$, the real vector $v$ such that $M = vv^{\mathrm{T}}$ is well defined up to sign, we have

$$\begin{aligned}
&\wedge_n(R_1(I_n \oplus O_n)R_1)^{\mathrm{T}} = \wedge_n(R(I_n \oplus O_n)R^{\mathrm{T}}) \\
&\quad \Leftrightarrow R_1(I_n \oplus 0_n)R_1^{\mathrm{T}} = R(I_n \oplus 0_n)R^{\mathrm{T}} \\
&\quad \Leftrightarrow R_1(I_n \oplus -I_n)R_1^{\mathrm{T}} = R(I_n \oplus -I_n)R^{\mathrm{T}} \\
&\quad \Leftrightarrow R_1 \mathcal{E}_0 R_1^{\mathrm{T}} = R \mathcal{E}_0 R^{\mathrm{T}}.
\end{aligned}$$

So

$$\{R\mathcal{E}_0 R^{\mathrm{T}} : R \in K\},\ \mathcal{L} = \{R(I_n \oplus 0_n)R^{\mathrm{T}} : R \in K\} \text{ and } \{\wedge_n(F) : F \in \mathcal{L}\} \tag{3.4}$$

are yet three more presentations of the Shilov boundary $K/\tilde{\mathrm{O}}_n$. We will henceforth denote $\partial_n(\mathrm{Sp}_{2n}\mathbb{R}/K) := \mathcal{L}$.

Now we take for each class $AK$ its unique symmetric positive definite element $A$, in the symplectic polar decomposition, identifying thus $\mathrm{Sp}_{2n}\mathbb{R}/K$ with the set of the symplectic positive definite real symmetric matrices which we'll simply denote by $S$. We now find an expression for the Busemann functions with respect to the distance $d_1$. We present the definition of the Busemann functions (we'll follow [Bal]). For $A, B, C \in S$, let

$$b_1(A, B, C) := d_1(A, C) - d_1(A, B).$$

The function $b_1(A, B, \cdot)$ is a Lipschitz function with constant 1. We denote this function by $b_{1B}(A)$. Then

$$\begin{array}{rcl} b_{1B} : S & \to & C(S) \\ A & \mapsto & b_{1B}(A) = b_1(A, B, \cdot) \end{array}$$

is an embedding. We consider $C(S)$ endowed with the topology of uniform convergence on bounded subsets. We say that a sequence $A_n$ of elements of $S$ converges at infinity if $d_1(A_n, B) \to \infty$ for some $B$ and $b_{1B}(A_n)$ converges in $C(S)$. It's easy to see that this definition is independent of the choice of $B$. Two such sequences $(A_n)$ and $(A'_n)$ are called equivalent if $\lim_{n\to\infty} b_{1B}(A_n) = \lim_{n\to\infty} b_{1B}(A'_n)$ for some (and hence for any) $B$. We now denote the set of equivalence classes by $S(\infty)$. For any $F \in S(\infty)$, there is a well defined function $f = b_1(F, B, \cdot) \in C(S)$, called the *Busemann function* at $F$ based at $B$, namely, $f = \lim_{n\to\infty} b_{1B}(A_n)$, where $A_n$ represents $F$.

As we now define these functions, we'll always consider the sequences $(A_n)$ to be converging to $F$ along a geodesic for the Riemannian metric. For $E \in \mathrm{Sym}_{2n}\mathbb{R} \cap \mathrm{sp}_{2n}\mathbb{R}$, we know that $e^{tE}K$ is such a geodesic. Let's assume that

$$E = Q\Sigma(E)Q^{\mathrm{T}},\ Q \in K,\ Q = (x_1, \ldots, x_{2n}),$$

and $(x_1, \ldots, x_{2n})$ is the family of the orthogonal eigenvectors of $E$. Notice that for $t \in \mathbb{R}$, $e^{tE} = Q\Sigma(e^{tE})Q^{\mathrm{T}}$, so $(x_1, \ldots, x_{2n})$ are also the eigenvectors for $e^{tE}$, and this is why we can choose $Q \in K$.

Recall that $d_1(e^{tE}, B) = d_1(B, e^{tE}) = 2\sum_{j=1}^{n} \log \sigma_j(B^{-1}e^{tE})$. We now establish a formula for this distance. The arguments in the following two results are due to Shmuel Friedland.

**Proposition 3.2.3** *Let $E \in \mathrm{Sym}_{2n}\mathbb{R} \cap \mathrm{sp}_{2n}\mathbb{R}$, $B \in S$ and suppose that $\lambda_1(E) > \lambda_2(E)$, or equivalently, $\lambda_1(e^{tE}) > \lambda_2(e^{tE})$ for all $t > 0$. Then*

$$\lim_{t\to\infty} \sigma_1(Be^{tE}) = t\lambda_1(E) + \frac{\log \lambda_1(Bx_1x_1^{\mathrm{T}}B^{\mathrm{T}})}{2} + W(t),$$

*where* $\lim_{t\to\infty} W(t) = 0$.

**Proof.** By definition,

$$\begin{array}{rcl} \sigma_1^2(Be^{tE}) & = & \lambda_1(Be^{2tE}B^{\mathrm{T}}) \\ & = & \lambda_1(e^{2t\lambda_1(E)I_n}Be^{2t(E-\lambda_1(E)I_2)}B^{\mathrm{T}}) \\ & = & e^{2t\lambda_1(E)}\lambda_1(Be^{2tE'}B^{\mathrm{T}}) \end{array}$$

with $E' = E - \lambda_1(E)I_n$, and so

$$\log \sigma_1(Be^{tE}) = t\lambda_1(E) + \frac{\log \lambda_1(Be^{2tE'}B^{\mathrm{T}})}{2}.$$

We have $\lambda_j(E') < 0$ for $2 \leq j \leq n$. In this case, the matrix $e^{2tE'}$ converges to a rank one matrix, namely $x_1 x_1^{\mathrm{T}}$ and thus $Be^{2tE'}B^{\mathrm{T}}$ converges to $B(x_1 x_1^{\mathrm{T}})B^{\mathrm{T}}$, and we have our result. $\square$

**Corollary 3.2.4** *Let $E \in \mathrm{Sym}_{2n}\mathbb{R} \cap \mathrm{sp}_{2n}\mathbb{R}$ and suppose now that $\lambda_n(E) > \lambda_{n+1}(E)$. Then*

$$\begin{aligned}\sum_{j=1}^{n} \log \sigma_j(Be^{tE}) &= t\sum_{j=1}^{n} \lambda_j(E) + \\ &+ \frac{\log \lambda_1\left(\wedge_n B \wedge_n (x_1, \ldots, x_n) \wedge_n (x_1, \ldots, x_n)^{\mathrm{T}} \wedge_n B^{\mathrm{T}}\right)}{2} + \\ &+ W^{(n)}(t),\end{aligned}$$

*with* $\lim_{t\to\infty} W^{(n)}(t) = 0$.

**Proof.** Apply the previous result to $\wedge_n(Be^{tE}) = \wedge_n(B) \wedge_n e^{tE}$, since under this condition,

$$\log \lambda_1(\wedge_n e^{tE}) = t^n \prod_{j=1}^{n} \lambda_j(E) > t^n \lambda_{n+1}(E) \prod_{j=1}^{n-1} \lambda_j(E) = \log \lambda_2(\wedge_n e^{tE}),$$

and hence $\lambda_1(\wedge_n e^{tE}) > \lambda_2(\wedge_n e^{tE})$. The eigenvector corresponding to $\lambda_1(\wedge_n e^{E})$ is clearly $\wedge_n(x_1, \ldots, x_n)$ and this concludes our proof. $\square$

Notice that for a matrix $E \in \mathrm{sp}_{2n}\mathbb{R}$, to say that $\lambda_n(E) > \lambda_{n+1}(E)$ is to say that 0 is not an eigenvalue of $E$ (see equation (2.5) in Chapter 2). Hence $e^{tE}$ converges to a point in the Shilov boundary as $t \to \infty$.

**Lemma 3.2.5** *Under the conditions of the previous corollary, with $E \in Q\mathcal{E}_0 Q^{\mathrm{T}}$, let*

$$\lim_{t\to\infty} \frac{\wedge_n(e^{tE})}{|| \wedge_n (e^{tE})||} = \wedge_n(F),$$

$F = Q(I_n \oplus 0_n)Q^{\mathrm{T}} \in \partial_n(\mathrm{Sp}_{2n}\mathbb{R}/K)$. *Then*

$$\lambda_1\left(\wedge_n B \wedge_n (x_1, \ldots, x_n) \wedge_n (x_1, \ldots, x_n)^{\mathrm{T}} \wedge_n B^{\mathrm{T}}\right) = \sigma_1^2(\wedge_n B \wedge_n F).$$

**Proof.** Notice that the first $n$ columns of $Q$ are exactly $x_1, \dots, x_n$. So

$$\begin{aligned}
\wedge_n(Q(I_n \oplus 0_n)Q^{\mathrm{T}}) &= \wedge_n(Q(I_n \oplus 0_n)) \wedge_n ((I_n \oplus 0_n)Q^{\mathrm{T}}) \\
&= \wedge_n(x_1, \dots, x_n, 0, \dots, 0) \wedge_n (x_1, \dots, x_n, 0, \dots, 0)^{\mathrm{T}} \\
&= \wedge_n(x_1, \dots, x_n) \wedge_n (x_1, \dots, x_n)^{\mathrm{T}}.
\end{aligned}$$

Now, since $FF^{\mathrm{T}} = F^2 = F$, we have

$$\begin{aligned}
&\lambda_1 \left(\wedge_n B \wedge_n (x_1, \dots, x_n) \wedge_n (x_1, \dots, x_n)^{\mathrm{T}} \wedge_n B^{\mathrm{T}}\right) = \\
&= \lambda_1(\wedge_n B \wedge_n F \wedge_n (B^{\mathrm{T}})) \\
&= \lambda_1(\wedge_n B \wedge_n F \wedge_n (F^{\mathrm{T}}) \wedge_n (B^{\mathrm{T}})) \\
&= \sigma_1^2(\wedge_n B \wedge_n F).
\end{aligned}$$

This concludes our proof. □

This allows us to write

$$\sum_{j=1}^{n} \log \sigma_j(Be^{tE}) = t \sum_{j=1}^{n} \lambda_j(E) + \log \sigma_1(\wedge_n B \wedge_n F) + W(t), \quad \lim_{t \to \infty} W(t) = 0.$$

For $A, B \in \mathrm{Sp}_{2n}\mathbb{R}$, we have

$$d_1(A, B) = 2 \sum_{j=1}^{n} \log \sigma_j(A^{-1}B),$$

from formula (2.6) in Chapter 2. With the previous results, we can now express the Busemann functions as follows:

$$\begin{aligned}
b_1(F, B, C) &= \lim_{e^{tE} \to F} b_1(e^{tE}, B, C) \\
&= \lim_{e^{tE} \to F} d_1(e^{tE}, C) - d_1(e^{tE}, B) \\
&= \lim_{e^{tE} \to F} d_1(C, e^{tE}) - d_1(B, e^{tE}) \qquad (3.5) \\
&= 2 \log \sigma_1(\wedge_n C^{-1} \wedge_n F) - 2 \log \sigma_1(\wedge_n B^{-1} \wedge_n F) \\
&= 2 \left(\log \sigma_1(\wedge_n C^{-1} \wedge_n F) - \log \sigma_1(\wedge_n B^{-1} \wedge_n F)\right)
\end{aligned}$$

Notice that in our previous arguments, we didn't specify which exponent matrix we took—any exponent $E \in Q\mathcal{E}_0 Q^{\mathrm{T}}$, $e^{tE} \to F$ yields the same Busemann function. We now present a 1-1 and onto correspondence between the Shilov boundary and the set of all Busemann functions that correspond to sequences that go to $\infty$ along a geodesic ray with $E$ as in the previous results. We recall here, that equation (2.3) states that the elements of $K$ are of the form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \; A^{\mathrm{T}}A + B^{\mathrm{T}}B = I_n, \; A^{\mathrm{T}}B \text{ symmetric.}$$

**Theorem 3.2.6** *Let $F \in \partial_n(\mathrm{Sp}_{2n}\mathbb{R}/K)$. Then, for a given $B \in \mathrm{Sp}_{2n}\mathbb{R}/K$, there is a 1-1 and onto correspondence between the sets*

$$\{b_1(F, B, \cdot) : F \in \partial_n(\mathrm{Sp}_{2n}\mathbb{R}/K)\} \text{ and } \partial_n(\mathrm{Sp}_{2n}\mathbb{R}/K).$$

**Proof.** We've seen that to each element $F \in \partial_n(\mathrm{Sp}_{2n}\mathbb{R}/K)$, we can associate a well defined Busemann function. Now, let's see that if $F \neq F'$, the Busemann functions we get are not the same. From formula (3.5), we can see that, for a given $B$, the functions $b_1(F, B, \cdot)$ and $b_1(F', B, \cdot)$ will be equal if and only if for all $C \in S$, $|| \wedge_n (C^{-1}F)|| = || \wedge_n (C^{-1}F')||$ for all such $C$, since the operator norm, given by the first singular value. Suppose $F = Q(I_n \oplus 0_n)Q^{\mathrm{T}}$, $F' = R(I_n \oplus 0_n)R^{\mathrm{T}}$ with $Q$ and $R$ as in the previous lemma, and take $C^{-1} = Q(2I_n \oplus \frac{1}{2}I_n)Q^{\mathrm{T}}$. Then, taking $v = \wedge_n(Q(I_n, 0_n)^{\mathrm{T}}) \in \mathrm{M}_{N,1}$, $N = \binom{2n}{n}$, we have

$$|| \wedge_n (C^{-1}F)v|| = 2^n.$$

and since $||v|| = 1$, we must have $|| \wedge (C^{-1}F)|| \geq 2^n$. However, since

$$|| \wedge_n (C^{-1}F)|| = || \wedge_n (C^{-1}) \wedge_n (F)|| \leq || \wedge_n C^{-1}||.|| \wedge_n F|| = 2^n.1 = 2^n,$$

we have $|| \wedge_n (C^{-1}F)|| = 2^n$.

Consider now $F'$. Denote by $E_{11}$ the $N \times N$ matrix with all entries equal to zero, except for entry $(1, 1)$ which is equal to 1. We have

$$\wedge_n(F') = \wedge_n(R)E_{11} \wedge_n (R^{\mathrm{T}}) = \wedge_n(y_1, \ldots, y_n) \wedge_n (y_1, \ldots, y_n)^{\mathrm{T}}.$$

This is a matrix of rank 1, and moreover, its image subspace is generated by $\wedge_n(y_1, \ldots, y_n)$. Since we assumed that $F \neq F'$, by lemma 3.2.2, this subspace is not $<\wedge_n(x_1, \ldots, x_n)>$, which is exactly the eigenspace associated with the largest eigenvalue of $\wedge_n C^{-1}$, $2^n$. Therefore, for any $v \in \mathrm{M}_{N,1}$ of norm 1, we know $\wedge_n(F')v$ will not be an eigenvector for $\wedge_n C^{-1}$, for its largest eigenvalue, $2^n$. Since $|| \wedge_n F'|| = 1$, if $w = \wedge_n F'v$, $||w|| \leq 1$ and

$$|| \wedge_n C^{-1} \wedge_n F'v|| = || \wedge_n C^{-1}w|| < 2^n||w|| \leq 2^n,$$

so $|| \wedge_n C^{-1}F'|| < 2^n$, and hence $b_1(F, B, \cdot) \neq b_1(F', B, \cdot)$ as we wished. □

We have then proved that there is a subset of the set of all Busemann functions for $d_1$ that is exactly the Shilov boundary of our space. Notice that we excluded the Busemann functions associated with points in the other strata of the boundary.

In the next chapter we do a study of the symplectic transformations that have fixed points in the Shilov boundary in the case $n = 2$.

# Chapter 4

# $\mathrm{Sp}_4\mathbb{R}$ and $\mathrm{SH}_2$

## 1 Normal forms

We are now going to present a study of the action of $\mathrm{Sp}_4\mathbb{R}$ on $\mathrm{SH}_2$, since this is the first non-trivial model of the Siegel upper half plane.

We start by locating the fixed points of a symplectic map. To do this, we'll consider the compactification as bounded space, with the corresponding stratification of the boundary. In this case the finite boundary of $\mathrm{SH}_2$ has the following stratification:

$$\mathrm{fin}(\partial_1\mathrm{SH}_2) = \{Z \in \mathrm{Sym}_2\mathbb{C} : \mathrm{rank}\,\mathrm{Im}(Z) = 1\},$$

$$\mathrm{fin}(\partial_2\mathrm{SH}_2) = \mathrm{Sym}_2\mathbb{R}.$$

The boundary in $\mathrm{SD}_2$ is $\mathrm{USym}_2$ and will have a corresponding stratification, according to the rank of $I - Z\overline{Z}$. As we extend the action of $\mathrm{Sp}_4\mathbb{R}$ to the boundary of $\mathrm{SD}_2$ by continuity, we have that every transformation must have a fixed point, by the Schauder fixed point theorem (cf. [DS]), since each map is continuous and the set in question os closed, bounded an convex.

This fixed point has to be either in one of the boundary strata or inside $\mathrm{SH}_2$. We will determine how many fixed points does the transformation have, and on which region are they located. Since the action of $\mathrm{Sp}_4\mathbb{R}$ is transitive on each stratum and also inside $\mathrm{SH}_2$, we can consider that $M$ fixes, in turn, a specific point in each one of these regions. As in some cases it will be possible to reduce the matrix $M$ to the form $X \odot Y$, $X, Y \in \mathrm{SL}_2\mathbb{R}$, by conjugation, we present a list of all normal forms that you can get for matrices in $\mathrm{SL}_2\mathbb{R}$ with the conjugation being made inside $\mathrm{SL}_2\mathbb{R}$.

**Proposition 4.1.1** *Let $X \in \mathrm{SL}_2\mathbb{R}$, $X \neq \pm I_2$. Then $X$ is similar to one and* *only one of the following normal forms within* $\mathrm{SL}_2\mathbb{R}$*:*

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix}, \ \alpha > 1,$$
$$\pm\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \ \pm\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$
$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \ b > 0.$$

**Proof.** Suppose the eigenvalues of $X$ are $a$ and $1/a$, real, and with $|a| \neq 1$. Then $X$ is similar to a matrix of the first type in $\mathrm{GL}_2\mathbb{R}$. If the conjugating matrix $T$ has positive determinant, we are done, since we can always take $T/\sqrt{\det T} \in \mathrm{SL}_2\mathbb{R}$, and this will also conjugate $X$ to the normal form. If the conjugating matrix $T$ has negative determinant, notice that the matrix $\mathrm{diag}(1, -1)$ preserves $\mathrm{diag}(a, 1/a)$ by conjugation, and hence $T\,\mathrm{diag}(1, -1)$ has positive determinant and diagonalizes $X$.

Now suppose $X$ is a Jordan block, an let's consider that the corresponding eigenvalue is 1, the case for $-1$ is similar. In this case, we know that it is possible to find a matrix in $\mathrm{SL}_2\mathbb{R}$ that maps $X$ to either one of the two normal forms presented, since they are conjugate by the matrix $\mathrm{diag}(1, -1)$. It is left to show that they are not similar to each other in $\mathrm{SL}_2\mathbb{R}$. to do so, consider the equation

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

As we write the scalar equations, we get $b = 0$ and $a = -d$, so the matrix must have negative determinant.

The elliptic case is argued similarly. □

We now establish normal forms with respect to where the fixed point is located.

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Let's assume that $M$ has a fixed point in $\partial_2\mathrm{SH}_2$, and we can assume, without loss of generality, that it is the point 0. To determine $\mathrm{Stab}(0)$, $0 \in \partial_2\mathrm{SH}_2$, the equation is $M(0) = 0$, $A0 + B = 0(C0 + D)$, which simply leads to $B = 0$. In this case $A^{\mathrm{T}}D = I_2$, so both matrices are invertible, and the matrix $C0 + D = D$ is invertible, so the equation we studied gives us the stabilizer: all matrices of the form

$$\begin{pmatrix} A & 0 \\ C & D \end{pmatrix},$$

with $A^{\mathrm{T}}D = I$ and $A^{\mathrm{T}}C$ symmetric. Now we'll divide our matrices into four types.

*Type 1.* The matrix $M$ doesn't have any eigenvalue with absolute value 1. Then we can bring the matrix to the form we already described: $\mathrm{diag}(A, A^{-\mathrm{T}})$ with $|\lambda_1(A)|,\ |\lambda_2(A)| > 1$. This form includes the case $X \odot Y$ with $X$ and $Y$ hyperbolic.

*Type 2.* Now, let's assume that $A$ has a complex eigenvalue (necessarily with absolute value 1), or that it is similar to a Jordan block associated with the eigenvalue 1 or $-1$. In this case, using again conjugation by a block diagonal matrix, we can bring $A$ to the form

$$\begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix}, \text{ and } D \text{ becomes } \begin{pmatrix} b & 1 \\ -1 & 0 \end{pmatrix},$$

where $b = \mathrm{tr}(A)$ is the sum of the eigenvalues, so $|b| \leq 2$. Then, by conjugating $M$ by a matrix of the type

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}, \tag{4.1}$$

namely with

$$X = \begin{pmatrix} 0 & \frac{c_1}{2} \\ \frac{c_1}{2} & c_3 \end{pmatrix}, \text{ with } C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix},$$

we can bring $C$ to the form

$$\begin{pmatrix} 0 & c_3 - c_2 \\ 0 & c_1 + c_4 + bc_3 \end{pmatrix}.$$

Now, from the equation $A^{\mathrm{T}}C = C^{\mathrm{T}}A$ we can see that $c_1 + c_4 + bc_3 = 0$, so the matrix $C$ has the form

$$\begin{pmatrix} 0 & c_3 - c_2 \\ 0 & 0 \end{pmatrix}.$$

If $c_2 - c_3 \neq 0$, we can, as before, conjugate $M$ by $\mathrm{diag}(c_2 - c_3, 1, 1/(c_2 - c_3), 1)$, in order to bring $C$ to the form

$$\begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix},$$

with $\delta = 1$ or, in case $C$ was symmetric to start with, 0. We end up with the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & b & 0 & 0 \\ 0 & \delta & b & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \ \delta = 0,\ 1,\ |b| \leq 2.$$

*Type 3.* Now suppose $A$ has eigenvalues 1 and $\alpha$, with $\alpha \neq \pm 1$—we're not considering the eigenvalue $-1$ because of the identification $M \equiv -M$. We can certainly diagonalize $A$, and then bring $C$ to diagonal form, by conjugating with a matrix of the type (4.1) with

$$X = \begin{pmatrix} 0 & \frac{\alpha c_3}{1-\alpha} \\ \frac{\alpha c_2}{1-\alpha} & 0 \end{pmatrix},$$

since in this case $c_2 = \alpha c_3$ because $A^{\mathrm{T}} C$ has to be symmetric. Then we can put the (2,2) entry of $C$ to zero, conjugating again by a matrix of the same type, with

$$X = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\alpha c_4}{1-\alpha^2} \end{pmatrix}.$$

Finally, we can make the (1,1) entry equal to $\pm 1$ or 0, depending on whether it was positive, negative or zero, in the original $C$, by conjugating with a diagonal matrix, as in the previous case, obtaining in the end $C = \operatorname{diag}(\delta, 0)$, with $\delta = \pm 1$ or 0. The matrix becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ \delta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \odot \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix},$$

with $\delta = 0, \pm 1$.

*Type 4.* Finally, if $A = I_2$ or similar to $\operatorname{diag}(1, -1)$ (we can consider it to be diagonalized to start with), then $C$ is respectively symmetric or skew-symmetric. In the first case, we can still bring it to diagonal form by either conjugating $M$ by $\operatorname{diag}(T, T)$, where $T$ is an orthogonal matrix diagonalizing $C$. In the second case, we can still diagonalize $C$ using a matrix of the type (4.1), with

$$X = \begin{pmatrix} 0 & \frac{c_2}{2} \\ \frac{c_2}{2} & 0 \end{pmatrix}.$$

Then we can again conjugate $M$ by a diagonal symplectic matrix, in order to bring $C$ to the form $\operatorname{diag}(\delta_1, \delta_2)$, where $\delta_1, \delta_2 = 0$ or $\pm 1$, according to the rank of $C$. We end up with the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ \delta_1 & 0 & 1 & 0 \\ 0 & \delta_2 & 0 & \pm 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \delta_1 & 1 \end{pmatrix} \odot \begin{pmatrix} \pm 1 & 0 \\ \delta_2 & \pm 1 \end{pmatrix},$$

with $\delta_1, \delta_2 = 0, \pm 1$.

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Now suppose the fixed point is in $\partial_1 \mathrm{SH}_2$, and, without loss of generality, take it to be the point $i\,\mathrm{diag}(1,0) \in \partial_1 \mathrm{SH}_2$. Let's characterize the stabilizer of this point. Let $L := \mathrm{diag}(1,0)$, $iL \in \partial_1 \mathrm{SH}_2$. To find the stabilizer of $iL$, we take

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$$

and use a similar notation for $B$, $C$ and $D$. If $CiL + D$ is invertible, we have

$$\begin{aligned} (AiL + B)(CiL + D)^{-1} = iL \quad &\Leftrightarrow \quad ALi + B = iL(iCL + D) \\ &\Leftrightarrow \quad B + ALi = -LCL + LDi, \end{aligned}$$

which can be written as:

$$\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} = \begin{pmatrix} -c_1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} a_1 & 0 \\ a_3 & 0 \end{pmatrix} = \begin{pmatrix} d_1 & d_2 \\ 0 & 0 \end{pmatrix}.$$

So the matrix $M$ is a symplectic matrix that looks like this:

$$\begin{pmatrix} a_1 & a_2 & -c_1 & 0 \\ 0 & a_4 & 0 & 0 \\ c_1 & c_2 & a_1 & 0 \\ c_3 & c_4 & d_3 & d_4 \end{pmatrix}. \tag{4.2}$$

From the equation $M^{\mathrm{T}} J M = J$, you get that

$$\begin{aligned} &a_1^2 + c_1^2 = 1,\ a_4 d_4 = 1, \\ &c_3 = d_4(a_1 c_2 - c_1 a_2) \text{ and} \\ &d_3 = -d_4(c_1 c_2 + a_1 a_2). \end{aligned}$$

Notice that with such $C$ and $D$, the matrix $CiL + D$ is indeed invertible, since $\det(CiL + D) = (a_1 + ic_1)d_4 \neq 0$, since $d_4$ is invertible and $a_1^2 + c_1^2 = 1$.

The characteristic polynomial of the matrix above is

$$(x - a_4)(x - d_4)((x - a_1)^2 + c_1^2),$$

which means that the eigenvalues of $M$ must be $a_4, d_4 = a_4^{-1}, a_1 + ic_1, a_1 - ic_1$, $a_1^2 + c_1^2 = 1$. Denote by $\lambda$ the eigenvalue $a_1 + ic_1$,and assume for the moment that $a_4 \neq \pm 1$. As we have seen before, if $v \in V_{a_4} \oplus V_{d_4}$ and $u \in V_\lambda \oplus V_{\overline{\lambda}}$, we'll have $(v, u) = 0$, provided that $a_4\lambda \neq 1$, and to have the equality we must have $a_4 = 1 = \lambda = 1$ or $a_4 = 1 = \lambda = -1$. Otherwise, it is possible to find a symplectic basis formed by two eigenvectors for $a_4$ and $d_4$ respectively, and by the eigenvectors in $V_\lambda \oplus V_{\overline{\lambda}}$ such that the matrix of the restriction of $M$ to the eigenspace is $M[1,3|1,3]$. This is because, given any vector $v$ in either one of these spaces, any other non-collinear vector $u$ in the same space will provide $(v, u) \neq 0$, since otherwise the form $(v, \cdot)$ would be degenerate. We may have to renormalize the vectors, so that the value of the form is 1 for the corresponding pairs of vectors.

If $a_4 = \pm 1$ (assume it is 1), then the matrix may or may not have a Jordan block associated with the eigenvalue. At any rate, the above argument will hold, replacing $V_1$ by the 2-dimensional space associated with the Jordan block if this is the case. Notice, however, that $A$ can similar to two possible Jordan blocks within $\mathrm{SL}_2\mathbb{R}$, according to proposition 4.1.1: the diagonal entries of the block must be equal to 1, but the nonzero non-diagonal entry can be either 1 or $-1$.

Thus, we have proved that, unless $a_4 = a_1 + ic_1 = a_1 = \pm 1$, we can bring $M$ to the form

$$\begin{pmatrix} a_1 & 0 & -c_1 & 0 \\ 0 & a_4 & 0 & 0 \\ c_1 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_4^{-1} \end{pmatrix} = \begin{pmatrix} a_1 & -c_1 \\ c_1 & a_1 \end{pmatrix} \odot \begin{pmatrix} a_1 & 0 \\ 0 & a_4^{-1} \end{pmatrix},$$

or, up to sign,

$$\begin{pmatrix} a_1 & 0 & -c_1 & 0 \\ 0 & 1 & 0 & 0 \\ c_1 & 0 & a_1 & 0 \\ 0 & \delta & 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & -c_1 \\ c_1 & a_1 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix},$$

with $\delta = 0, \pm 1$; otherwise, the matrix is, again up to sign,

$$\begin{pmatrix} 1 & a_2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c_2 & 1 & 0 \\ c_2 & c_4 & -a_2 & 1 \end{pmatrix}.$$

which is a matrix of Type 2 or Type 4 above, depending on $a_2$ being zero or not.

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Suppose finally that $M$ has a fixed point inside $\mathrm{SH}_2$, and by conjugation, suppose it is the point $iI_2$. This means $M \in K$, and $M$ has the known form

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}, A^{\mathrm{T}}B \text{ symmetric and } A^{\mathrm{T}}A + B^{\mathrm{T}}B = I_2.$$

If we consider the action on the Siegel disk $\mathrm{SD}_2$, the action of this matrix corresponds to the action of $M' = U \oplus \overline{U}$, $U$ unitary, $U = A + iB$, fixing the point $0 \in \mathrm{SD}_2$. The group that fixes this point is $\tilde{\mathrm{U}}_2 = \{T \oplus \overline{T} : T \in \mathrm{U}_n\}$. Since any unitary matrix is unitarily diagonalizable, take $V \in \mathrm{U}_2$ such that $VUV^* = D = \mathrm{diag}(\lambda, \mu)$, $|\lambda| = |\mu| = 1$. Then $V \oplus \overline{V} \in \tilde{\mathrm{U}}_2$ and

$$(V \oplus \overline{V})(U \oplus \overline{U})(V^* \oplus V^{\mathrm{T}}) = D \oplus \overline{D},$$

and this means that $M$ can be brought to the form

$$\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ -b_1 & 0 & a_1 & 0 \\ 0 & -b_2 & 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix} \odot \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix},$$

where $\lambda = a_1 + ib_1$, $\mu = a_2 + ib_2$, $|\lambda| = |\mu| = 1$. This means every element in $K$ can be brought to the form $X \odot Y$, where $X, Y \in \mathrm{SL}_2\mathbb{R}$ are elliptic or $\pm I_2$.

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We summarize all the normal forms we got for a $4 \times 4$ symplectic matrix with $M \neq \pm I_4$.

I. If $M$ has a fixed point in $\partial_2\mathrm{SH}_2$, then $M$ can be brought to the form

$$\begin{pmatrix} A & 0 \\ C & A^{-\mathrm{T}} \end{pmatrix}$$

and we have the following forms for each type:

- Type 1. If $|\lambda_1(M)| \neq 1$, and $|\lambda_2(M)| \neq 1$, $M$ is similar to

$$\begin{pmatrix} A & 0 \\ 0 & A^{-\mathrm{T}} \end{pmatrix}.$$

This includes, but is not restricted to, the case $X \odot Y$ with $X$ and $Y$ hyperbolic.

- Type 2. If $\lambda_1(A), \lambda_2(A) \in \mathbb{C} \setminus \mathbb{R}$, $|\lambda_1(A)| = |\lambda_2(A)| = 1$ or $A$ is a Jordan block associated with the eigenvalue 1, then $M$ is similar to

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & b & 0 & 0 \\ 0 & \delta & b & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \ \delta = 0,\ 1,\ |b| \leq 2.$$

- Type 3. If $\lambda_1(A) = 1$, $\lambda_2(A) = \alpha$, $|\alpha| \neq 1$, $M$ is similar up to sign to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ \delta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\alpha \end{pmatrix}, \ \delta = 0, \pm 1.$$

  In this case, the matrix can always be expressed as $X \odot Y$, $X$ parabolic or $I_2$, and $Y$ hyperbolic.

- Type 4. If $A$ is similar to $\operatorname{diag}(1, -1)$ or $I_2$, $M$ is similar up to sign to

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ \delta_1 & 0 & 1 & 0 \\ 0 & \delta_2 & 0 & \pm 1 \end{pmatrix}, \ \delta_1, \delta_2 = 0, \pm 1.$$

  Again, in this case the matrix is always similar to $X \odot Y$, $X$ and $Y$ parabolic or $\pm I_2$.

II. If $M$ is a matrix that has a fixed point in $\partial_1 \mathrm{SH}_2$, then, unless all its eigenvalues are equal to 1 or all equal to $-1$, it can be brought to the form

$$\begin{pmatrix} a_1 & 0 & -c_1 & 0 \\ 0 & a_4 & 0 & 0 \\ c_1 & 0 & a_1 & 0 \\ 0 & \delta & 0 & a_4^{-1} \end{pmatrix},$$

$a_1^2 + c_1^2 = 1, \delta = 0, \pm 1$, and $\delta$ can only be nonzero if $a_4 = \pm 1$. We have that the matrix can be written as $X \odot Y$, with $X$ elliptic and $Y$ hyperbolic, parabolic or $\pm I_2$.

If all eigenvalues of $M$ are equal to 1 or all equal to $-1$, it can be brought to either form 2 or form 4 above.

III. If $M$ has a fixed point inside $\mathrm{SH}_2$, then $M$ can be brought to the form

$$\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ -b_1 & 0 & a_1 & 0 \\ 0 & -b_2 & 0 & a_2 \end{pmatrix}, a_1^2 + b_1^2 = a_2^2 + b_2^2 = 1,$$

which is $X \odot Y$, with $X$ and $Y$ elliptic or equal to $I_2$ or $-I_2$.

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Notice that, by the Schauder fixed point theorem, every symplectic matrix must have a fixed point in $\mathrm{Cl}(\mathrm{SD}_2)$ (and hence in $\mathrm{SH}_2 \cup \partial\mathrm{SH}_2$), therefore, any symplectic matrix can be brought to one of these forms by conjugation. Thus, by looking at the forms we have, we can get the following normal forms for a symplectic matrix $M \in \mathrm{Sp}_{2n}$.

(a) If none of the eigenvalues of $M$ is on the complex unit circle ($|\lambda_j(M)| \neq 1$ for all $j = 1, \ldots, 4$), then the transformation fixes a point in $\partial_2\mathrm{SH}_2$ and can be reduced to the generic normal form in I, type 1 above, as we have seen.

(b) If $M$ has one eigenvalue $\alpha$ in the complex unit circle, $\alpha \neq \pm 1$, and its other eigenvalue is real, the transformation must have a fixed point in $\partial_1\mathrm{SH}_2$ and can be brought to the corresponding form described above in II, and it must be similar to $X \odot Y$, with $X$ hyperbolic, parabolic or $\pm I_2$ and $Y$ elliptic.

(c) If $M$ has two complex non-real eigenvalues, they must be either both on the complex unit circle or both off of it. This last case is already classified above in (a). If they're on the circle, we have the following:

   - If they're not associated with a Jordan block (this is always the case if we have four distinct ones, for instance), the transformation must have a fixed point inside $\mathrm{SH}_2$, in which case it can be brought to the form $X \odot Y$ as in III above, with $X$ and $Y$ elliptic. We'll soon see that the transformations in case I, type 2 (with $A$ a Jordan block and $C = 0$) must also have a fixed point inside $\mathrm{SH}_2$, so they must be similar to the form mentioned.
   - If the eigenvalues are associated with a Jordan block, we must only have two distinct eigenvalues. The transformation has a fixed point in $\partial_2\mathrm{SH}_2$, and it falls under I, type 2, with $\delta = 1$.

(d) If both eigenvalues of $M$ are $\pm 1$, the transformation must have a fixed point in $\partial_2\mathrm{SH}_2$ and can be reduced to either form I, type 2, with $A$ a Jordan block, or form I, type 4, and be written as $X \odot Y$ with $X$ and $Y$ parabolic or $\pm I_2$.

We now proceed to make a classification of the fixed points of these matrices.

# 2 Fixed points: classification

In this section we present the results regarding the classification fixed points. The proofs of the results will be given in the last section of this chapter.

Notice that in case $M = X \odot Y$, if $p$ is a fixed point for $X$ and $q$ a fixed point for $Y$, then $\mathrm{diag}(p, q) \in \mathrm{Cl}(\mathrm{H} \times \mathrm{H})$ will be a fixed point for $X \odot Y$. We'll informally refer to this kind of fixed points as *expected fixed points*, in case the transformation can be written as $X \odot Y$. Also, in case we have infinitely many fixed points, they will form a semi-algebraic variety, since the defining equations for the set of fixed points are algebraic and we have some inequalities in the definition of the space. In each case, we'll present the dimension of this variety over $\mathbb{R}$.

**Proposition 4.2.1** *Let $M$ be a symplectic transformation $M$ that fixes a point in $\partial_2\mathrm{SH}_2$, (case I) and is of* **type one**, *$M = \mathrm{diag}(A, A^{-\mathrm{T}})$, $|\lambda_1(M)|$, $|\lambda_2(M)| \neq 1$. Then the map will have:*

- *a 1-dimensional variety of fixed points, if it has only two distinct real eigenvalues, each one with a corresponding eigenspace of dimension 2,*
- *four fixed points if it has four distinct real eigenvalues,*
- *three fixed points if it has only two real distinct eigenvalues, each one associated with a $2 \times 2$ Jordan block, or*
- *two fixed points if it has two complex (conjugate) eigenvalues.*

*Moreover, all fixed points will lie in the Shilov boundary.*

Notice that in the case where $M$ has two real eigenvalues $\alpha$ and $\beta$, it is conjugate (in $\mathrm{Sp}_4\mathbb{R}$) to $X \odot Y$, $X, Y \in \mathrm{SL}_2\mathbb{R}$, $X$ with eigenvalues $\alpha, \alpha^{-1}$ and $Y$ with eigenvalues $\beta, \beta^{-1}$. These were hyperbolic maps, with two fixed points each, so the four fixed points in the case $\alpha \neq \beta$ are exactly the expected fixed points: they correspond to the four possible combinations of those two pairs of fixed points in $\partial_2\mathrm{SH}_2$. Notice also that in the case $\alpha = \beta$ we have many unexpected fixed points. This result agrees with the results in section 5 of [FH1], where the case $\alpha I_n \oplus (1/\alpha) I_n$ is discussed.

**Proposition 4.2.2** *Let $M$ be a symplectic transformation that fixes a point in $\partial_2 \mathrm{SH}_2$ (case I) and is of* **type two***:*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & b & 0 & 0 \\ 0 & \delta & b & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \delta = 1,\ 0,\ |b| \leq 2$$

*(a) If $C = 0$, it will have*

- *a 2-dimensional variety of fixed points, including points inside* $\mathrm{SH}_2$ *if $A$ is not a Jordan block,*
- *a 2-dimensional variety of fixed points lying on both strata of the boundary if $A$ is a Jordan block.*

*(b) If $C \neq 0$, it will have one fixed point on the small boundary.*

Notice that this is the only case where we can have only complex eigenvalues (all on the complex unit circle) with Jordan blocks associated to them (case $\delta = 1$).

**Proposition 4.2.3** *A symplectic transformation $M$ that fixes a point in $\partial_2 \mathrm{SH}_2$ (case I) and is of* **type three***,*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ \delta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\alpha \end{pmatrix}, \quad \delta = 0, \pm 1,$$

*will have:*

- *a 2-dimensional variety of fixed points, lying on both strata, if the eigenvalue* 1 *has a corresponding eigenspace of dimension 2 ($C = 0$), or*
- *two fixed points if there is a $2 \times 2$ Jordan block associated with the eigenvalue 1 ($C \neq 0$).*

In this case, we can always write $M = X \odot Y$, where $X$ is either $I_2$ or a Jordan block (a parabolic map) and $Y$ a hyperbolic one with eigenvalues $\alpha$ and $1/\alpha$. Here, the results agree with what was expected, and moreover, all the fixed points are in $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$.

**Proposition 4.2.4** *Let $M$ be a symplectic transformation that fixes a point in $\partial_2\mathrm{SH}_2$ (case I) and is of* **type four:**

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ \delta_1 & 0 & 1 & 0 \\ 0 & \delta_2 & 0 & \pm 1 \end{pmatrix}, \quad \delta_1, \delta_2 = 0, \pm 1.$$

(a) *If $A = \mathrm{diag}(1, -1)$, it will have*

- *a 4-dimensional variety of fixed points, both inside $\mathrm{SH}_2$, and in both strata of the boundary, if $C = 0$,*
- *a 2-dimensional variety of fixed points, located on both strata, if $C$ has rank 1, or*
- *one fixed point, if $C$ has rank 2.*

(b) *If $A = I$ (and $C \neq 0$), it will have*

- *a 2-dimensional variety of fixed points, on both strata if $C \neq \pm I$, and*
- *one fixed point if $C = \pm I_2$.*

*All the fixed points here are in $\mathrm{Cl}(\mathrm{H}\times\mathrm{H})$, except in case $A = I_2$ and $C = \mathrm{diag}(1, -1)$.*

Again, in this case, we can write $M = X \odot Y$, with $X, Y$ being $\pm I_2$ or a Jordan block. Notice however, that the previous result gives us the expected fixed point in the cases

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

where $A = I_2$ and $C = \pm I_2$, but it shows many unexpected fixed points in the case

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

which corresponds to $C = \mathrm{diag}(1, -1)$. This shows that, even though these last two transformations have the same Jordan structure, they have different dynamics, which means that, even though they are conjugate in $\mathrm{GL}_4\mathbb{R}$ they are not so in $\mathrm{Sp}_4\mathbb{R}$.

Notice also that whenever we got a finite number of fixed points, they were all in $\partial_2\mathrm{SH}_2$.

**Proposition 4.2.5** *A symplectic transformation that fixes a point in* $\partial_1 \mathrm{SH}_2$ *(case II) is either conjugate to* $X \odot Y$*, with* $X$ *elliptic, or must fall under types 2 or 4 above. In case it is similar to* $X \odot Y$*, it has the form*

$$\begin{pmatrix} a_1 & 0 & -c_1 & 0 \\ 0 & a_4 & 0 & 0 \\ c_1 & 0 & a_1 & 0 \\ 0 & \delta & 0 & a_4^{-1} \end{pmatrix} \quad \begin{array}{c} \text{with } a_1^2 + c_1^2 = 1,\ \delta = 0, \pm 1, \\ \\ \delta \neq 0 \Rightarrow a_4 = \pm 1. \end{array}$$

*The map will have*

- *two fixed points in* $\partial_1 \mathrm{SH}_2$ *if* $Y$ *is hyperbolic,*
- *one fixed point if* $Y$ *is parabolic, and*
- *a 2-dimensional variety of fixed points if* $Y = I_2$*.*

*All the fixed points will be in* $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$*.*

**Proposition 4.2.6** *A symplectic transformation not equal to* $\pm I_4$ *with a fixed point inside* $\mathrm{SH}_2$ *(case III) is always similar to* $X \odot Y$*, with* $X$ *and* $Y$ *elliptic or* $\pm I_2$*. The corresponding normal form is*

$$\begin{pmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ -b_1 & 0 & a_1 & 0 \\ 0 & -b_2 & 0 & a_2 \end{pmatrix} \quad \begin{array}{l} \alpha = a_1 + ib_1,\ \beta = a_2 + ib_2, \\ |\alpha| = |\beta| = 1. \end{array}$$

*The map will have:*

- *one fixed point if* $\alpha \neq \overline{\beta}$*, and neither of them is real,*
- *a 2-dimensional variety of fixed points if one of* $X$ *and* $Y$ *is elliptic and the other is* $\pm I_2$*,*
- *a 2-dimensional variety of fixed points if* $\alpha = \overline{\beta}$*,* $\alpha, \beta \notin \mathbb{R}$*,*
- *a 4-dimensional variety of fixed points if one of* $X$ *and* $Y$ *is* $I_2$ *and the other is* $-I_2$*.*

*All the fixed points will be in* $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$*, except in case* $\alpha = \overline{\beta}$*.*

The last case had already been studied in proposition 4.2.4, it corresponds to $A = \mathrm{diag}(1, -1)$ and $C = 0$. Here we get only the expected fixed points unless $\alpha = \overline{\beta}$, or equivalently, $\mathrm{diag}(\alpha, \overline{\alpha}) \neq \mathrm{diag}(\beta, \overline{\beta})^{-1}$. We can also conclude that in the third case here ($\alpha = \overline{\beta}$), $M$ has to be conjugate to a matrix that fixes a point in $\partial_2\mathrm{SH}$ of type 2, with $\delta = 0$, since that matrix had 4 complex non-real eigenvalues and it had a 2-dimensional variety of fixed points inside $\mathrm{SH}_2$.

As we now look back to all our cases, we can conclude the following result.

**Proposition 4.2.7** *Let $M$ be a symplectic transformation with only a finite number of fixed points. Then they must be either all in the same boundary stratum or all inside* $\mathrm{SH}_2$*.*

To finish this section, we state a result that brings together our considerations about the maps that can be expressed as $X \odot Y$.

**Theorem 4.2.8** *Let $X$ and $Y$ be any matrices in* $\mathrm{SL}_2\mathbb{R}$*. Then the map induced on* $\mathrm{SH}_2$ *by $X \odot Y$ will have infinitely many unexpected fixed points in* $\mathrm{Cl}(\mathrm{SH}_2)$ *if and only if $Y$ is similar to $X^{-1}$ in* $\mathrm{SL}_2\mathbb{R}$*. In case it isn't, we'll have only the expected fixed points, finite in number if neither $X$ nor $Y$ are $\pm I_2$.*

**Proof.** Let's consider first the case when $Y$ is similar to $X^{-1}$ in $\mathrm{SL}_2\mathbb{R}$. Then $X \odot Y$ is similar to $X \odot X^{-1}$ in $\mathrm{Sp}_4\mathbb{R}$, since if $X^{-1} = QYQ^{-1}$, we have $X \odot X^{-1} = (I_2 \odot Q)(X \odot Y)(I_2 \odot Q)^{-1}$.

If $X$ is hyperbolic, then $X \odot X^{-1}$ is of type 1, proposition 4.2.1, and can be brought to the form $aI_2 \oplus \frac{1}{a}I_2$, $a > 1$. We have infinitely many fixed points as opposed to the four expected ones.

If $X$ is parabolic, then

$$X \odot X^{-1} \text{ is similar to } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \text{ up to sign,}$$

which is our Type 4, proposition 4.2.4, with $A = I_2$ and $C = \mathrm{diag}(1, -1)$, which also has infinitely many fixed points.

If $X$ is elliptic, the result comes from proposition 4.2.6, $Y$ is similar to $X^{-1}$ in $\mathrm{SL}_2\mathbb{R}$ if $\alpha = \overline{\beta}$. Notice that the case $Y$ similar to $X$ in $\mathrm{SL}_2\mathbb{R}$ corresponds to $\alpha = \beta$, that yields only one fixed point.

If $X = I_2$ or $-I_2$, the transformation $X \odot X^{-1}$ is the identity, and it will fix every point in $\mathrm{Cl}(\mathrm{SH}_2)$, more than just $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$. In any of these cases, the unexpected fixed points lie outside $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$, of course.

If $X$ is not similar to $Y$ in $\mathrm{SL}_2\mathbb{R}$, we now list the possible cases and refer to where their study is done. In each case, we get only the expected fixed points. The cases $X \odot Y$ are presented up to sign, but since $M$ and $-M$ have the same action, the transformation is the same. Also, these are the only cases we need to consider, since $X \odot Y$ is similar to $Y \odot X$ by a symplectic permutation matrix.

| $X$ | $Y$ | $X \odot Y$ |
|---|---|---|
| hyperbolic | hyperbolic | Prop. 4.2.1, $\alpha \neq \beta$ |
| parabolic | hyperbolic | Prop. 4.2.3, $\delta = 1$ |
| elliptic | hyperbolic | Prop. 4.2.5 |
| $I_2$ | hyperbolic | Prop. 4.2.3 |
| parabolic | parabolic | Prop. 4.2.4, $A = I_2$, $C = \pm I_2$ or $A = \mathrm{diag}(1, -1)$, $\mathrm{rank}(C) = 2$ |
| elliptic | parabolic | Prop. 4.2.5 |
| $I_2$ | parabolic | Prop. 4.2.3 |
| elliptic | elliptic | Prop. 4.2.6 |
| $I_2$ | elliptic | Prop. 4.2.5 |
| $I_2$ | $-I_2$ | Props. 4.2.5 or 4.2.4 with $A = \mathrm{diag}(1, -1)$, $C = 0$. |

This concludes our proof. $\square$

# 3 The boundary in the projective model

Before we present the proofs to the results above, we present some considerations on the boundary. We'll consider the projective model $\mathrm{SPH}_2$. Recall that this is the subset of the Grassmanian $G_{2n,n}$ formed by all equivalence classes $\left[(Z, I_2)^{\mathrm{T}}\right]$, $Z \in \mathrm{SH}_2$ with

$$\begin{bmatrix} W_1 \\ V_1 \end{bmatrix} = \begin{bmatrix} W_2 \\ V_2 \end{bmatrix} \Leftrightarrow \begin{pmatrix} W_1 \\ V_1 \end{pmatrix} = \begin{pmatrix} W_2 \\ V_2 \end{pmatrix} U,$$

for some $U \in \mathrm{GL}_n\mathbb{C}$.

In $\mathrm{SD}_2$, the infinite boundary is the set of points that is isomorphic to the set $\{Z \in \partial\mathrm{SD}_2 : |Z - I| = 0\}$, which is the set where the map $Z \mapsto i(I + Z)(I - Z)^{-1}$ is not defined. If we conjugate the map $Z \mapsto -Z^{-1}$, which is defined on $\mathrm{SH}_2$, by the map $g$, which has as matrix

$$\begin{pmatrix} iI & iI \\ -I & I \end{pmatrix},$$

then we get the map $Z \mapsto -Z$, defined on $\mathrm{SD}_2$. Now, if $|Z - I| = 0$, it is not always true that $|-Z - I| \neq 0$, namely, if $Z$ is similar to $\mathrm{diag}(-1, 1)$, then this is not true. In this case, however, we can use the map $Z \mapsto iZ$, on $\mathrm{SD}_2$ to bring it to the finite boundary. To get the corresponding map on $\mathrm{SH}_2$, we notice that, for any $\alpha$ such that $|\alpha| = 1$, the mapping $Z \mapsto \alpha Z$ on $D$ corresponds to the mapping of $H$ having as matrix

$$\begin{pmatrix} aI & bI \\ -bI & aI \end{pmatrix},$$

where $a$ and $b$ are real, $\beta = a + ib$, $\beta^2 = \alpha$. So, in our case, for the mapping $Z \mapsto iZ$ of $D$, we get the mapping of $H$ defined by the matrix

$$\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}. \tag{4.3}$$

If we now apply these maps to the finite boundary of the projective model, we get the following result.

**Proposition 4.3.1** *For every point in the boundary of* $\mathrm{SPH}_2$*, there is a representative of one of the following forms:*

$$\begin{pmatrix} Z \\ I \end{pmatrix}, \quad \begin{pmatrix} I \\ -Z \end{pmatrix} \text{ or } \begin{pmatrix} I+Z \\ I-Z \end{pmatrix},$$

*with* $Z \in \mathrm{Sym}_2\mathbb{C}$*,* $\operatorname{Im} Z$ *positive semidefinite in all cases. Moreover, we can take* $Z$ *to be non-invertible in the second case and* $Z$ *real with eigenvalues* $1$ *and* $-1$ *in the third case.*

**Proof.** The part of the boundary represented by the first type of matrices corresponds to the finite boundary of $\mathrm{SH}_2$. The other two correspond to what you get when you apply to this boundary the maps $Z \mapsto Z^{-1}$ and the one associated with the matrix in (4.3). In this last case, what we get is

$$\frac{\sqrt{2}}{2}\begin{pmatrix} I+Z \\ I-Z \end{pmatrix},$$

but we can multiply on the right by a scalar matrix in order to get the desired form, $\operatorname{Im} Z$ being always positive semidefinite since we started out with such matrices before applying the transformations. Now, if either $I + Z$ or $I - Z$ were invertible, then we could multiply on the right by its inverse, and get one of the previous forms, so we can assume that $Z$ has eigenvalues 1 and $-1$. Let's write the matrix as

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

With eigenvalues 1 and $-1$, its trace has to be 0, which means that $a = -c$, and for the imaginary part to be positive semidefinite, this means that $\operatorname{Im} a = \operatorname{Im} c = 0$. Now, we must have $|\operatorname{Im} Z| \geq 0$, so $-(\operatorname{Im} b)^2 \geq 0$, and therefore $\operatorname{Im} b = 0$.

Finally, to see that we can take $Z$ to be non-invertible in the second case, notice that if it is, we can multiply on the right by $-Z^{-1}$ and get a representative of the first form. $\square$

We'll refer to points in this boundary as being of the first, second or third kind, according to the representatives above.

If you consider $\mathrm{H} \times \mathrm{H} \subset \mathrm{SH}_2$, then it's easy to see that $\partial(\mathrm{H} \times \mathrm{H})$ corresponds to the set of all above classes that can be expressed with a real diagonal $Z$.

# 4 Fixed points: proofs

We start with two technical lemmas.

**Lemma 4.4.1** *We have the following results.*

1. *If $Z$ is a complex symmetric matrix and $\mathrm{Im}(Z)$ is positive definite then $Z$ is invertible.*
2. *If a real matrix $A$ has a complex eigenvalue, and $v$ is an eigenvector associated to it, then $v$ cannot be real.*
3. *If $Z \in \mathrm{M}_2 F$, $Z \neq 0$ and not invertible, then there exist vectors $u, v \in F^2$ such that $Z = uv^{\mathrm{T}}$. These vectors are uniquely defined up to scalar multiplication. Moreover, if $Z \in \mathrm{Sym}_2\mathbb{C}$ or if $Z \in \mathrm{Sym}_2\mathbb{R}$ and is positive semidefinite (in the real case), then we can write $Z = uu^{\mathrm{T}}$, with $u \in \mathbb{C}^2$ or $u \in \mathbb{R}^2$ respectively, and the vector $u$ is unique up to sign.*

**Proof.** 1. Suppose $\mathrm{Im}(Z)$ is invertible, and $Zv = 0$ for some complex vector $v$. Then $\overline{Zv} = 0$, $v^{\mathrm{T}} Z = 0$ and thus $v^{\mathrm{T}}(Z - \overline{Z})\overline{v} = 0$, and $v = 0$.

2. If $v$ were real, with $Av = \alpha v$ and $\alpha \in \mathbb{C} \setminus \mathbb{R}$, we would have

$$\alpha v = Av = \overline{Av} = \overline{\alpha v} = \overline{\alpha} v,$$

and therefore $v = 0$, since $\alpha \neq \overline{\alpha}$.

3. The vector $u$ is defined as the generator of $Z\mathbb{C}^2$ and $v = (v_1, v_2)^{\mathrm{T}}$ is such that $(-v_2, v_1)^{\mathrm{T}}$ generates $\ker Z$. This defines the vectors uniquely up to scalar multiplication. For the second part, take

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}, \text{ for } k \in F, \; F = \mathbb{R} \text{ or } \mathbb{C}.$$

Then the vector $(\sqrt{z_1}, z_2/\sqrt{z_1})$ will do, if $z_1 \neq 0$, otherwise take $(z_2/\sqrt{z_3}, \sqrt{z_3})$, since $z_1$ and $z_3$ cannot be both zero, otherwise $Z = 0$ because of the rank. Since the vector $u$ such that $Z = uu^{\mathrm{T}}$ is defined up to scalar multiplication, if $v = ku$ is such that $Z = uu^{\mathrm{T}}$, then $vv^{\mathrm{T}} = uu^{\mathrm{T}} = k^2 vv^{\mathrm{T}}$ and $k = \pm 1$. □

**Lemma 4.4.2** *Consider a point of the type* $(I+Z, I-Z)^{\mathrm{T}}$, *with* $I+Z = uu^{\mathrm{T}}$ *and* $I-Z = vv^{\mathrm{T}}$. *Then* $u$ *and* $v$ *verify the following:*

$$u^{\mathrm{T}}v = 0 \text{ and } v^{\mathrm{T}}v = u^{\mathrm{T}}u = 2.$$

*Moreover, any pair* $u, v$ *satisfying the above conditions will yield a matrix* $Z$ *such that* $uu^{\mathrm{T}} = I+Z$ *and* $vv^{\mathrm{T}} = I-Z$.

**Proof.** We need that Notice that $Z^2$ has two eigenvectors with eigenvalue 1, therefore it is the identity. This means that $(Z-I)(Z+I) = Z^2 - I = 0$, which implies that $0 = uu^{\mathrm{T}}vv^{\mathrm{T}} = (u^{\mathrm{T}}v)uv^{\mathrm{T}}$, and since $uv^{\mathrm{T}}$ is not the zero matrix, for this would imply $u = 0$ or $v = 0$, we have $u^{\mathrm{T}}v = 0$. We will call this last equality the *perpendicularity condition.* Also, $u^{\mathrm{T}}u$ and $v^{\mathrm{T}}v$ are the traces of $I+Z$ and $I-Z$, so they are both 2.

Conversely, take $u$ and $v$ as above. then $Z := (uu^{\mathrm{T}} + vv^{\mathrm{T}})/2$ will be the desired matrix. To check this, notice that $(I+Z)u = 2u$ and $(I+Z)v = 0$, so $I+Z = uu^{\mathrm{T}}$, since $(u, v)$ is a basis of $\mathbb{R}^2$. We can similarly prove that $I-Z = vv^{\mathrm{T}}$. $\square$

— // —

Now we will study the four possible types of transformations fixing $0 \in \partial_2\mathrm{SH}_2$ as described above in page 45, number I.

**Type 1**. The matrix $M$ is of the form

$$\begin{pmatrix} A & 0 \\ 0 & A^{-\mathrm{T}} \end{pmatrix}.$$

as presented before. Using again square brackets to denote equivalence classes, the equation we get when we look for fixed points of the first kind is

$$\begin{pmatrix} A & 0 \\ 0 & A^{-\mathrm{T}} \end{pmatrix} \begin{bmatrix} Z \\ I \end{bmatrix} = \begin{bmatrix} AZ \\ A^{-\mathrm{T}} \end{bmatrix} = \begin{bmatrix} AZA^{\mathrm{T}} \\ I \end{bmatrix},$$

so we would need that $AZ = ZA^{-\mathrm{T}}$, which is impossible if $Z \neq 0$, in this case, since this would only be possible if $A$ and $A^{-\mathrm{T}}$ shared an eigenvalue, which is not the case here, since in this form $|\lambda_i(A)| > 1$, $i = 1, 2$. A similar analysis would show that for the representatives of the second kind, we get only one fixed point with $Z = 0$.

Now for points of the third kind. We know that, since both matrices $I+Z$ and $I-Z$ are symmetric and of rank 1, there exist vectors $u, v \in \mathbb{C}^2$ such that $I+Z = uu^{\mathrm{T}}$ and $I-Z = vv^{\mathrm{T}}$, $u$ and $v$ in the conditions of Lemma 4.4.2. we then must have

$$Avv^{\mathrm{T}}U = uu^{\mathrm{T}} \text{ and } A^{\mathrm{T}}uu^{\mathrm{T}}U = vv^{\mathrm{T}},$$

for some invertible $U$. Then we must have $Au = \alpha u$ and $A^{\mathrm{T}}v = \beta v$, because $u$ and $v$ are well defined up to scalar multiplication, and hence $\alpha$ and $\beta$ have to be eigenvalues of $A$.

Let us now assume, for the moment, that $A$ has two distinct real eigenvalues. Then you can diagonalize $A$, say by conjugation with a matrix $T \in \mathrm{GL}_2\mathbb{R}$,

$$T \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} T^{-1} = A,$$

which allows us to diagonalize $M$ by conjugation, using the matrix $T \oplus T^{-\mathrm{T}} \in \mathrm{Sp}_2\mathbb{R}$.

Now we take $u$ and $v$ to be eigenvectors of $A$ and $A^{\mathrm{T}}$ associated with different eigenvalues, such that $uu^{\mathrm{T}} = vv^{\mathrm{T}} = 2$. Clearly, we have $v^{\mathrm{T}}u = 0$. Notice that we cannot take two eigenvectors associated with the same eigenvalue, otherwise we would not have the perpendicularity condition. If we choose $u = (\sqrt{2}, 0)^{\mathrm{T}}$, we have

$$I + Z = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \; I - Z = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

and we get a fixed point of the third kind with $Z = \mathrm{diag}(1, -1)$. If we chose $u = (0, \sqrt{2})^{\mathrm{T}}$, we get another fixed point, with $Z = \mathrm{diag}(-1, 1)$. So we get two fixed points in this part of the boundary if $A$ has two distinct eigenvalues.

Now, let's suppose that the matrix $A$ has only one eigenvalue. Then, either it is similar to a Jordan block or it is scalar. In the first case, we can bring $A$ to Jordan form and assume that

$$A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix},$$

by conjugation, and find that the vectors $u = (\sqrt{2}, 0)^{\mathrm{T}}$ and $v = (0, \sqrt{2})^{\mathrm{T}}$ will do, since $Au = \alpha u$, $A^{\mathrm{T}}v = \alpha v$, $u^{\mathrm{T}}v = 0$ and $vv^{\mathrm{T}} = uu^{\mathrm{T}} = 2$. So, as before, we have a fixed point, but in this case, only one, as there is no choice possible for the eigenvectors.

If the matrix is scalar it must fix all the points in this part of the boundary. The equations become $a(I + Z)U = (I + Z)$ and $A(I - Z)U = (I - Z)$, and if $I + Z = uu^{\mathrm{T}}$ and $I - Z = vv^{\mathrm{T}}$, then we can choose $U$ such that $u^{\mathrm{T}}U = (1/a)u^{\mathrm{T}}$ and $v^{\mathrm{T}}U = av^{\mathrm{T}}$. This part of the boundary is 1-dimensional, since the matrices $Z$ can be written as $Q\,\mathrm{diag}(1, -1)Q^{\mathrm{T}}$, $Q \in \mathrm{O}_2$. Notice that many of these points will not be in $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$, even though the matrix can be expressed as $X \odot X$, with $X$ hyperbolic.

Now, if $A$ has two complex (conjugate) eigenvalues, we will have no fixed point. Indeed, if the matrix $A$ were to have a fixed point here, it would have to be with a real $Z$ (as we've seen in lemma 4.3.1), so we could choose $v$ as above to be real. But by lemma 4.4.1, we know that $A$ does not admit real eigenvectors. By lemma 4.4.2, this is impossible.

We've now proved proposition 4.2.1.

**Type 2.** Now, let's suppose that the matrix $A$ has an eigenvalue $\alpha$ with $|\alpha| = 1$, If $\alpha = \pm 1$, then $A$ is a Jordan block—because of the identification $M \equiv -M$, we can assume $\alpha = 1$. The form obtained is

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & b & 0 & 0 \\ 0 & \delta & b & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \ \delta = 0,\ 1.$$

Then the other eigenvalue of $A$ is $\overline{\alpha}$.

For fixed points of the first kind, $(Z, I)^{\mathrm{T}}$, we need

$$AZ(CZ + D)^{-1} = Z, \text{ or } AZ = Z(CZ + D).$$

We will have two very different behaviors, for $C = 0$ and $C \neq 0$. Let's assume for the moment that $C \neq 0$, that is, $\delta = 1$. In this case, if we look for an invertible $Z$ as solution of the above equation, we will find that $A$ and $CZ + D$ have to be similar. Taking into account that $A$ and $D$ are already similar, we conclude that, with

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}, \quad CZ = \begin{pmatrix} z_2 & z_3 \\ 0 & 0 \end{pmatrix} \tag{4.4}$$

cannot alter neither the trace nor the determinant of $D$. From the trace, we conclude that $z_2 = 0$, and from the determinant, $z_3 = 0$, so $Z$ would not be invertible. Thus, if we want a fixed point of this form, we need $Z$ to be non-invertible. Let's write $Z = vv^{\mathrm{T}}$, with $v$ being a complex vector. Then the equation becomes

$$Avv^{\mathrm{T}} = vv^{\mathrm{T}}(Cvv^{\mathrm{T}} + D).$$

By the unicity of decomposition of a $2 \times 2$ matrix in this form, up to scalars, we need

$$Av = \alpha v \text{ or } Av = \overline{\alpha} v.$$

Let's go on assuming $Av = \alpha v$, the other case is similar. Since we know how $A$ looks like from the normal form, we can write $v = s(1, \alpha)^{\mathrm{T}}$, for some complex $s$. Then $v^{\mathrm{T}}Cv = s^2\alpha$ and $v^{\mathrm{T}}D = \overline{\alpha}v^{\mathrm{T}}$, and the whole equation becomes

$$\alpha(vv^{\mathrm{T}}) = s^2\alpha(vv^{\mathrm{T}}) + \overline{\alpha}vv^{\mathrm{T}},$$

and thus $\alpha = s^2\alpha + \overline{\alpha}$. Solving for $s^2$, we get $s^2 = 2i(\operatorname{Im}\alpha)\overline{\alpha}$, and the candidate for fixed point will thus have

$$Z = 2i\operatorname{Im}\alpha \begin{pmatrix} \overline{\alpha} & 1 \\ 1 & \alpha \end{pmatrix}.$$

Notice, however, that the imaginary part of this matrix is

$$2\begin{pmatrix} \operatorname{Re}\alpha\operatorname{Im}\alpha & \operatorname{Im}\alpha \\ \operatorname{Im}\alpha & \operatorname{Re}\alpha\operatorname{Im}\alpha \end{pmatrix},$$

with determinant $(\operatorname{Im}^2\alpha)(\operatorname{Re}^2\alpha - 1) \leq 0$, so we only have a fixed point here if $\alpha$ is real, and in this case, the fixed point will be zero, the one we had already. The study for the other eigenvalue is absolutely similar, and would lead to the same conclusion.

Now for fixed points in the form $(I, -Z)^{\mathrm{T}}$. The equation we get in this case is linear: $DZ - ZA = C$, and if we write down the entries, we get, with $Z$ as in equation (4.4),

$$\begin{pmatrix} -bz_1 + 2z_2 & z_3 - z_1 \\ z_3 - z_1 & 2z_2 + bz_3 \end{pmatrix} = \begin{pmatrix} 0 & \delta \\ 0 & 0 \end{pmatrix},$$

which is clearly impossible if $\delta = 1$, for we would need $0 = z_3 - z_1 = 1$.

Before going to the third part of the boundary, let's examine the case $C = 0$. Then, the equations we get for fixed points in forms $(Z, I)^{\mathrm{T}}$ and $(I, -Z)^{\mathrm{T}}$ are very similar, they are:

$$AZ = ZD \text{ and } ZA = DZ,$$

respectively. Solving the first, we get as fixed points all matrices with

$$Z = \begin{pmatrix} z_1 & \frac{bz_1}{2} \\ \frac{bz_1}{2} & z_1 \end{pmatrix},$$

provided that $\operatorname{Im}(Z) \geq 0$. So, if we write $\operatorname{Im}(z_1) = y$, then we need $y \geq 0$ and

$$|\operatorname{Im} Z| \geq 0 \Leftrightarrow y^2 - \frac{b^2y^2}{4} \geq 0 \Leftrightarrow y = 0 \text{ or } b^2 \leq 4.$$

As $b = \alpha + \overline{\alpha}$, we always have $-2 \leq b \leq 2$, so all these matrices provide indeed fixed points. Notice also that the variety of fixed points will have many points inside $\mathrm{SH}_2$ if $b^2 < 4 \Leftrightarrow \alpha \in \mathbb{C} \setminus \mathbb{R}$. If $b = \pm 2$, then $A$ is a Jordan block associated with the eigenvalue 1 or $-1$. In this case, the variety will lie completely on the boundary, partly on the big one, partly on the small one, depending on $z_1$ being real or not. In any case, it will clearly have dimension 2, since we can choose any $z_1 \in \mathbb{C}$ with $\operatorname{Im} z_1 \geq 0$.

The analysis for the points of the form $(I, -Z)^{\mathrm{T}}$ would lead to a similar conclusion.

Now for fixed points of the third kind, $(I+Z, I-Z)^{\mathrm{T}}$. The equations we get are:

$$\begin{aligned} &A(I+Z)U = I+Z \text{ and} \\ &C(I+Z)U + D(I-Z)U = I-Z \end{aligned}$$

for some invertible $U$. Again, if we write $I+Z = uu^{\mathrm{T}}$ with $u$ real, we find that it must be an eigenvector of $A$, thus, if the eigenvalues of $A$ are complex (non-real) there is no fixed point here. If $A$ is a Jordan block, we take $u = (1,1)^{\mathrm{T}}$ (it's the only choice possible), and get the equation

$$\begin{pmatrix} \delta+1 & \delta-1 \\ -1 & 1 \end{pmatrix} U = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

which is only possible if $\delta = 0$, since the determinant of the matrix on the left is $2\delta$. In this case $U = I$ will do. These are the results in proposition 4.2.2.

**Type 3.** let's assume that $A$ has an eigenvalue 1 and another one $\alpha \neq \pm 1$, $\alpha^2 \neq 1$; $\alpha$ has to be real. The form obtained is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ \delta & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/\alpha \end{pmatrix}, \quad \delta = 0, \pm 1.$$

For fixed points of the first kind, we come again to the equation $AZ = Z(CZ+D)$. If we write down the entries, having as usual $Z$ as in equation (4.4), and $C = \mathrm{diag}(\delta, 0)$, $\delta = 0$ or 1, we get

$$\begin{pmatrix} z_1 & z_2 \\ \alpha z_2 & \alpha z_3 \end{pmatrix} = \begin{pmatrix} \delta z_1^2 + z_1 & \delta z_1 z_2 + \frac{z_2}{\alpha} \\ \delta z_1 z_2 + z_2 & \delta z_2^2 + \frac{z_3}{\alpha} \end{pmatrix}$$

If $z_2 \neq 0$, we can simplify the equations from entries (1,2) and (2,1), canceling $z_2$, obtaining $1 - \frac{1}{\alpha} = \delta z_1 = \alpha - 1$, which leads to the equation $\alpha^2 - 2\alpha + 1 = 0$, only possible if $\alpha = 1$. So, $z_2$ has to be zero, and we end up with the conditions:

$$\delta z_1^2 = 0 \text{ and } \frac{\alpha^2 - 1}{\alpha} z_3 = 0,$$

so to have a fixed point other that the one with $Z = 0$ we need $z_3 = 0$ and $\delta = 0$. In this case, we get a variety of fixed points, and the variety will have dimension 2 since we can choose any $z_1 \in \mathbb{C}$. It will lie on both strata.

For fixed points of the second kind, the equation is $DZ - ZA = 0$ and if we write down the entries, we get the conditions $0z_1 = \delta$ and $z_2 = z_3 = 0$, so we will only have fixed points if $\delta = 0$, in which case we get a variety of dimension 2 (any $z_3$ will do), lying on both strata.

For the third kind, we write again $I + Z = uu^{\mathrm{T}}$, and the equation $Auu^{\mathrm{T}}U = uu^{\mathrm{T}}$ tells us again that $u$ has to be an eigenvector for $A$, so $u = (\sqrt{2}, 0)$ or $(0, \sqrt{2})$. If we take the first choice, the second equation $Cuu^{\mathrm{T}}U + Dvv^{\mathrm{T}}U = vv^{\mathrm{T}}$ becomes

$$\begin{pmatrix} 2\delta & 0 \\ 0 & \frac{2}{\alpha} \end{pmatrix} U = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

which is only solvable if $\delta = 0$, in which case $U = \mathrm{diag}(1, \alpha)$ will work. If we choose $u = (0, \sqrt{2})$, we have the equation $\mathrm{diag}(2, 0)U = \mathrm{diag}(2, 0)$, always possible. So here we have two fixed points if $\delta = 0$ and one if $\delta \neq 0$.

Notice that the matrix in this case can always be written as $I_2 \odot Y$, with $Y$ parabolic or hyperbolic. In either of these cases, notice that the points we get are all inside $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$. The previous results yield proposition 4.2.3.

**Type 4.** The matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ \delta_1 & 0 & 1 & 0 \\ 0 & \delta_2 & 0 & \pm 1 \end{pmatrix}, \ \delta_1, \delta_2 = 0, \pm 1.$$

We consider first the case when $A = \mathrm{diag}(1, -1)$. In this case $D = A$. This time we'll only look for fixed points on the first part on the boundary that have a non-invertible $Z$, because the other ones will also have a representation as $(I, -Z)^{\mathrm{T}}$, with $Z$ invertible, of course. So, let's assume that $Z = vv^{\mathrm{T}}$, with $v$ complex. Then we get, from equation $AZ = Z(CZ + D)$,

$$\begin{aligned} Avv^{\mathrm{T}} &= vv^{\mathrm{T}}Cvv^{\mathrm{T}} + vv^{\mathrm{T}}A \\ &= v(v^{\mathrm{T}}Cvv^{\mathrm{T}} + v^{\mathrm{T}}A), \end{aligned}$$

so $v$ has to be an eigenvector of $A$, suppose it's $(s, 0)^{\mathrm{T}}$ for some complex $s$. Then the equation becomes

$$vv^{\mathrm{T}} = vv^{\mathrm{T}}Cvv^{\mathrm{T}} + vv^{\mathrm{T}} \Leftrightarrow vv^{\mathrm{T}}Cvv^{\mathrm{T}} = 0$$

and since $v^{\mathrm{T}}Cv = \delta_1 s^2$ so we must have $\delta_1 s^2 = 0$, so if $\delta_1 \neq 0$ we must have $s = 0$, otherwise, we get a 2-dimensional variety of fixed points. If we carry out similar computations for the other possible eigenvector, we would get to the a similar conclusion: if $\delta_2 = 0$ we get infinitely many fixed points with $v = (0, s)^{\mathrm{T}}$, otherwise, none. Therefore, if $\delta_1\delta_2 \neq 0$, we only get one fixed point here, the one with $Z = 0$. Otherwise, we get a 2-dimensional variety of fixed points if any one of the deltas is zero, with two components if they are both zero. It will be located on both strata.

For points of the second kind, the equation is linear: $AZ - ZA = C$, and it can be written as

$$\begin{pmatrix} 0 & 2z_2 \\ -2z_2 & 0 \end{pmatrix} = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix},$$

so $z_2 = 0$ and we only have fixed points here if $C = 0$, lying on both strata and also inside, of the form $\operatorname{diag}(z_1, z_3) \in \mathrm{Cl}(\mathrm{H} \times \mathrm{H})$. The variety will be 4-dimensional.

Now for the third part of the boundary. As usual, if $I + Z = uu^{\mathrm{T}}$, the equation $Auu^{\mathrm{T}}U = uu^{\mathrm{T}}$ tells us that $u$ will have to be an eigenvector of $A$, so let's pick $u = (\sqrt{2}, 0)$. The second equation, $Cuu^{\mathrm{T}}U + Avv^{\mathrm{T}}U = vv^{\mathrm{T}}$, becomes, after some simplification,

$$\begin{pmatrix} 2\delta_1 & 0 \\ 0 & -2 \end{pmatrix} U = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

so it has a solution (namely $U = A$) if and only if $\delta_1 = 0$. In a similar fashion, we could see that the other eigenvector provides a fixed point if and only if $\delta_2 = 0$, so we get as many fixed points here as the number of zeros in $C$.

Finally, for the case $A = I_2$. Notice that, in this case, if $C = 0$, the transformation we get is the identity, so we'll always consider $C \neq 0$.

For the first part of the boundary, the equation is $Z = Z(CZ + I)$. If $Z$ is invertible, we simplify the equation, and it leads to $C = 0$. If it is not, then, as usual, consider $Z = vv^{\mathrm{T}}$, and the equation becomes

$$vv^{\mathrm{T}}Cvv^{\mathrm{T}} = 0 \Leftrightarrow (v^{\mathrm{T}}Cv)vv^{\mathrm{T}} = 0,$$

so, with $v = (v_1, v_2)$, $0 = v^{\mathrm{T}}Cv = \delta_1 v_1^2 + \delta_2 v_2^2$. If at least one of the deltas is zero, we will have a variety of solutions, all on the boundary. If they are both nonzero, the result will depend on the signs of the deltas. If they have the same sign, we get $v_1^2 = -v_2^2$, and we can't find any nonzero real solutions to this equation. If we take complex numbers, then $\operatorname{Im}(v_1^2) = -\operatorname{Im}(v_2^2)$, and this is only possible if $\operatorname{Im}(v_1^2) = 0$, since $v_1^2$ and $v_2^2$ are the diagonal entries of the matrix $Z$ that must have positive semidefinite imaginary part. Since its real parts have be symmetric, we need $v_1$ to be pure imaginary, say $v_1 = ki$, and $v_2$ real, $v_2 = k$ (or vice-versa). However, in this case, the imaginary part of $Z$ would be of the type

$$\mathrm{Im}(vv^{\mathrm{T}}) = \begin{pmatrix} 0 & k^2 \\ k^2 & 0 \end{pmatrix},$$

which is not positive semidefinite. So we get no fixed points in this case.

If the deltas have different signs, the equation $v_1^2 = v_2^2$ is solvable, taking $v_1 = v_2$ such that its square has non-negative imaginary part. This will provide a 2-dimensional variety of fixed points, all lying on the boundary. With $Z$ like this, the matrix $CZ + D$ will have determinant 1, so they do provide fixed points. Notice that the fixed points we got are not diagonal, so they're not in $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$.

For the points of the second kind, the equation is $C - Z = -Z$, which only has a solution if $C = 0$.

For the points of the third kind, take as usual $I + Z = uu^{\mathrm{T}}$ and $I - Z = vv^{\mathrm{T}}$, with $uv^{\mathrm{T}} = 0$ and $u^{\mathrm{T}}u = v^{\mathrm{T}}v = 2$. Then we need

$$uu^{\mathrm{T}} = uu^{\mathrm{T}}U \text{ and } Cuu^{\mathrm{T}} + vv^{\mathrm{T}} = vv^{\mathrm{T}}U,$$

which can be written as $Cuu^{\mathrm{T}} = v(v^{\mathrm{T}}U - v^{\mathrm{T}})$, and so we must have $Cu = \lambda v$, for some real $\lambda$. If $\delta_1 \neq 0$ and $\delta_2 = 0$, then $Cu = (\pm u_1, 0)$, and therefore $v = (v_1, 0)^{\mathrm{T}}$, and by the perpendicularity condition, $u = (0, u_2)^{\mathrm{T}}$, which is impossible. The same would happen if $\delta_1 = 0$ and $\delta_2 \neq 0$. If $C = \pm I$, then we would get $u = \pm\lambda v$ which is also impossible, because of the perpendicularity condition. If $C = \pm\,\mathrm{diag}(1, -1)$, then if $u = (u_1, u_2)^{\mathrm{T}}$, $v = Cu/\lambda = (\delta u_1, -\delta u_2)^{\mathrm{T}}/\lambda$, $\delta = \pm 1$, and the perpendicularity condition would give us $u_1^2 = u_2^2$, and from $u^{\mathrm{T}}u = 2$ we get $u_1, u_2 = \pm 1$. Therefore, we must have $u = (1, 1)^{\mathrm{T}}$ or $(1, -1)^{\mathrm{T}}$ (the symmetric vectors would yield the same matrix $I + Z$), and indeed $v := Cu$ does satisfy the perpendicularity condition (again, $-Cu$ would lead to the same matrix $I - Z$). To check, finally, that these two choices do indeed provide two fixed points, by finding a suitable $U$, notice that, since $Cu = \lambda v$, the equation above can be written

$$\lambda v(u^{\mathrm{T}} + v^{\mathrm{T}} - v^{\mathrm{T}}U) = 0,$$

so if we choose $U$ to be such that $u^{\mathrm{T}}U = u^{\mathrm{T}}$ and $v^{\mathrm{T}}U = v^{\mathrm{T}} + u^{\mathrm{T}}$, we'll have a uniquely defined invertible linear transformation, since $u$ and $v$ are a basis of $\mathbb{R}^2$.

All the fixed points we got were in $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$, except for the case $A = I_2$ and $C = \mathrm{diag}(1, -1)$. Proposition 4.2.4 sums up these results.

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Now, we present a study of the fixed points of the transformations that fix $iL$, $L = \mathrm{diag}(1, 0)$ which are similar to $X \odot Y$, $X,\ Y \in \mathrm{SL}_2\mathbb{R}$ and we can assume $X$ to be elliptic and $Y$ parabolic, hyperbolic or the identity. Notice that if they're not, then they can be brought to types 2 or 4 above.

The matrix $X \odot Y$ is

$$\begin{pmatrix} a_1 & 0 & -c_1 & 0 \\ 0 & a_4 & 0 & 0 \\ c_1 & 0 & a_1 & 0 \\ 0 & \delta & 0 & a_4^{-1} \end{pmatrix} \qquad \delta = 0, \pm 1,$$

with $a_1^2 + c_1^2 = 1$. Notice that $\delta$ can only be nonzero if $a_4 = 1$ (we are not considering the case $a_4 = -1$ because of the identification $M \equiv -M$). This corresponds to

$$\begin{pmatrix} a_1 & -c_1 \\ c_1 & a_1 \end{pmatrix} \odot \begin{pmatrix} a_4 & 0 \\ \delta & a_4^{-1} \end{pmatrix}.$$

Let's assume that $c_1 \neq 0$, otherwise we would get $B = 0$, which we have already studied. The equation for the first part of the boundary and the points inside is $AZ + B = Z(CZ + D)$:

$$\begin{pmatrix} a_1 z_1 - c_1 & a_1 z_2 \\ a_4 z_2 & a_4 z_3 \end{pmatrix} = \begin{pmatrix} c_1 z_1^2 + \delta z_2^2 + a_1 z_1 & c_1 z_1 z_2 + \delta z_2 z_3 + z_2/a_4 \\ c_1 z_1 z_2 + \delta z_2 z_3 + a_1 z_2 & c_1 z_2^2 + \delta z_3^2 + z_3/a_4 \end{pmatrix}.$$

If we subtract the equation coming from entry (2,1) from the one coming from entry (1,2) we get

$$z_2(a_1 - a_4) = z_2(\frac{1}{a_4} - a_1),$$

and if $z_2 \neq 0$ we get to the equation $2a_1 = a_4 + 1/a_4$ and thus $|a_1| \geq 1$, which is impossible, since we're taking $-1 < a_1 < 1$, so that $c_1 \neq 0$. So we must have $z_2 = 0$, and thus the equation from entry (1,1) gives us $z_1 = \pm i$, and only the solution $z_1 = i$ is acceptable, since we want $\operatorname{Im} Z \geq 0$. Equation from entry (2,2) is

$$z_3(\delta z_3 + a_4^{-1} - a_4) = 0,$$

so either $z_3 = 0$ or $\delta z_3 = a_4 - a_4^{-1}$. The second equation is impossible if $a_4 \neq 1$ (since in this case $\delta = 0$), possible ($z_3 = 0$) if $a_4 = 1$ and $\delta = \pm 1$ and undetermined if $a_1 = 1$ and $\delta = 0$. At any rate, we get a 2-dimensional variety of fixed points here ($z_3$ can be any) if $a_4 = 1$ and $\delta = 0$ and one fixed point, $\operatorname{diag}(i, 0)$, otherwise.

For the second part of the boundary, the equation is $C - DZ = -Z(A - BZ)$:

$$\begin{pmatrix} c_1 - a_1 z_1 & -a_1 z_2 \\ -z_2/a_4 & \delta - z_3/a_4 \end{pmatrix} = \begin{pmatrix} -c_1 z_1^2 - a_1 z_1 & -c_1 z_1 z_2 - a_4 z_2 \\ -c_1 z_1 z_2 - a_1 z_2 & -c_1 z_2^2 - a_4 z_3 \end{pmatrix}.$$

This time we get $z_1 = i$ from the (1,1) entry equation immediately (we're assuming $c_1 \neq 0$), and then we can easily conclude that $z_2 = 0$ from equation (1,2). As for $z_3$, the equation

$$a_4 z_3 = z_3/a_4 - \delta$$

becomes impossible if $a_4 = 1$ and $\delta = \pm 1$, undetermined if $a_4 = 1$ and $\delta = 0$ (any $z_3$ with $\operatorname{Im} z_3 \geq 0$ will do), and $z_3 = 0$ if $a_4 \neq 1$. So we have a variety if $a_4 = 1$ and $\delta = 0$, one more fixed point if $a_4 \neq 1$, and no fxed point otherwise.

For the third part of the boundary, the equations are

$$\begin{cases} A(I+Z)U + B(I-Z)U = I+Z \\ C(I+Z)U + D(I-Z)U = I-Z \end{cases} \tag{4.5}$$

Take $B' := \operatorname{diag}(0,1)$, and notice that $B'B = 0$. If we then multiply the first equation above by $B'$ on the left, and consider, as usual $I+Z = uu^{\mathrm{T}}$ and $I-Z = vv^{\mathrm{T}}$, we get

$$B'Auu^{\mathrm{T}}U = B'uu^{\mathrm{T}},$$

and hence $u^{\mathrm{T}}U = \alpha U$ for some $\alpha \neq 0$.

Now we consider the second equation in (4.5), which can be written as

$$Cuu^{\mathrm{T}}U + Dvv^{\mathrm{T}}U = vv^{\mathrm{T}}.$$

Taking into account that $u^{\mathrm{T}}U = \alpha u^{\mathrm{T}}$, and rearranging, we get

$$\alpha Cuu^{\mathrm{T}} - vv^{\mathrm{T}} = -Dvv^{\mathrm{T}}U.$$

Multiplying the above equation on the right by $v$, and using the facts that $v^{\mathrm{T}}v = 2$, $u^{\mathrm{T}}v = 0$, we get $-2v = -(v^{\mathrm{T}}Uv)Dv$, and we must have $Dv = \beta v$ for some $\beta$, so $v$ has to be an eigenvector of $D$, $v = (0, \sqrt{2})^{\mathrm{T}}$ or $v = (\sqrt{2}, 0)^{\mathrm{T}}$. We'll see that neither one will provide a fixed point.

If $v = (0, \sqrt{2})^{\mathrm{T}}$, we must have $u = (\sqrt{2}, 0)^{\mathrm{T}}$, $I+Z = \operatorname{diag}(2,0)$, $I-Z = \operatorname{diag}(0,2)$, and the second equation in (4.5) becomes

$$\begin{pmatrix} 2c_1 & 0 \\ 0 & 2/a_4 \end{pmatrix} U = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

which is impossible, with $c_1, a_4 \neq 0$.

If $v = (\sqrt{2}, 0)^{\mathrm{T}}$, the first equation in (4.5) becomes

$$\begin{pmatrix} -2c_1 & 0 \\ 0 & 2a_4 \end{pmatrix} U = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

which is again impossible with $c_1, a_4 \neq 0$. So we have no fixed points in this part of the boundary.

Notice that if the transformation was of the type $I_2 \times X$, with $X$ elliptic, the fixed points we got are in $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$. This concludes the proof of proposition 4.2.5.

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Finally, if $M$ fixes $iI_2$, then it must be in $K$. Consider the action of the conjugate of $K$, $\tilde{\mathrm{U}}_2 = \{U \oplus \overline{U} : U \in \mathrm{U}_n\}$ on the Siegel disk $\mathrm{SD}_n$. We've seen that every transformation is similar to a diagonal matrix is $\tilde{\mathrm{U}}$, so let's assume that $M = \mathrm{diag}(\alpha, \beta, \overline{\alpha}, \overline{\beta})$, $U = \mathrm{diag}(\alpha, \beta)$, $|\alpha|, |\beta| = 1$. This corresponds to $X \odot Y$ in $K$, with $X$ and $Y$ elliptic or $\pm I_2$,

$$X = \begin{pmatrix} \mathrm{Re}\,\alpha & \mathrm{Im}\,\alpha \\ -\mathrm{Im}\,\alpha & \mathrm{Re}\,\alpha \end{pmatrix}, \text{ and } Y = \begin{pmatrix} \mathrm{Re}\,\beta & \mathrm{Im}\,\beta \\ -\mathrm{Im}\,\beta & \mathrm{Re}\,\beta \end{pmatrix},$$

The fixed point equation in $\mathrm{SD}_2$ is $UZ(\overline{U})^{-1} = Z \Leftrightarrow UZ = Z\overline{U}$.

$$\begin{pmatrix} \alpha z_1 & \alpha z_2 \\ \beta z_2 & \beta z_3 \end{pmatrix} = \begin{pmatrix} \overline{\alpha} z_1 & \overline{\beta} z_2 \\ \overline{\alpha} z_2 & \overline{\beta} z_3 \end{pmatrix},$$

which yields the three equations

$$\alpha z_1 = \overline{\alpha} z_1, \quad \alpha z_2 = \overline{\beta} z_2, \quad \beta z_3 = \overline{\beta} z_3.$$

The first equation will have a non-zero solution if and only if $\alpha = \pm 1$, and we have a similar result for the third. The second will have a nonzero solution if $\alpha = \overline{\beta}$ and this means that $X$ and $Y$ above are inverse to each other. If $\alpha \neq \pm 1$, $\beta \neq \pm 1$ and $\alpha \neq \overline{\beta}$ we only have one solution, zero.

So we have nonzero solutions to one and only one equation (yielding a 2-dimensional variety of fixed points) if one of $\alpha$ and $\beta$ is $\pm 1$ and the other is not real or $\alpha = \overline{\beta}$. We have nonzero solutions to two equations if $\alpha = 1$ and $\beta = -1$ or vice-versa, in this case we get a 4-dimensional variety. The last case left is $\alpha = \beta = 1$ or $\alpha = \beta = -1$, which would yield $M = \pm I_2$.

The only case that yields non-diagonal fixed points (outside $\mathrm{Cl}(\mathrm{H} \times \mathrm{H})$) is $\alpha = \overline{\beta}$. This proves proposition 4.2.6.

# Chapter 5

# On Discrete Subgroups

## 1 General concepts

We now take $\Gamma$ to be a discrete subgroup of $\mathrm{Sp}_{2n}\mathbb{R}$. As in the 2-dimensional hyperbolic plane, we can define a Dirichlet domain, for $\Gamma$. This is done in [Sie, p. 27ff], we'll follow this construction. Let $\Gamma$ be a discrete group, $\Gamma = \{\gamma_j : j = 1, 2, \ldots\}$, and take $\gamma_1 = I_n$. Take $Z_0 \in \mathrm{SH}_n$ such that it is not fixed by any element in $\Gamma$, except the identity, and take

$$\Delta_p(Z_0) := \{Z \in \mathrm{SH}_n : d_p(Z, Z_0) \leq d_p(Z, \gamma_j(Z_0)),\ \forall j = 2, 3, \ldots\}.$$

We'll call this set the *Dirichlet domain* for $\Gamma$ centered at $Z_0$. This is a set that is closed in $\mathrm{SH}_n$. One important property of this set is that there is a fundamental set $D$ for $\Gamma$—that is, a set that contains one and only one element of each orbit of $\Gamma$—satisfying $\mathrm{Int}(\Delta_p(Z_0)) \subseteq D \subseteq \Delta_p(Z_0)$. The set $\Delta_p(Z_0)$ will be a star-shaped set, that is, a set such that if $Z \in \Delta_p(Z_0)$, then it contains the geodesic arc connecting $Z_0$ and $Z$.

Another important concept is the Poincaré exponent. To define it we recall the definition of the orbital counting function:

$$N_p(r, A, B) = \#\{\gamma \in \Gamma : d_p(A, \gamma B) < r\},\ 1 \leq p \leq \infty.$$

The Poincaré exponent can now be presented as:

$$\delta_p(\Gamma) := \limsup_{r\to\infty} \frac{\log N_p(r, A, B)}{r}, \quad A, B \in \mathrm{Sp}_{2n}\mathbb{R}/K.$$

This definition is independent of $A$ and $B$. Consider now the associated Poincaré series

$$g_{s,p}(A, B) := \sum_{\gamma\in\Gamma} e^{-s d_p(A,\gamma B)} \text{ for } s > 0$$

It is known that this series diverges if $s < \delta_p(\Gamma)$ and converges if $s > \delta_p(\Gamma)$. The Poincaré exponent can also be defined as the exponent $s_0$ such that the series converges for $s > s_0$ and diverges for $s < s_0$.

Finally, we present the Poisson kernel of our space, with respect to the Shilov boundary $\partial_n(\mathrm{Sp}_{2n}\mathbb{R}/K)$. There is a formula for this kernel in [Jo1] (and also in [Jo2]), with respect to the model $\mathrm{SH}_n$. Here we give a formula for the kernel in the model $\mathrm{Sp}_{2n}\mathbb{R}/K$. This kernel is a real function defined on $\mathrm{Sp}_{2n}\mathbb{R}/K \times \partial_n(\mathrm{Sp}_{2n}\mathbb{R}/K)$, and for this boundary we'll take the set defined in Chapter 3, equation (3.4):

$$\partial_n(\mathrm{Sp}_{2n}\mathbb{R}/K) = \{R(I_n \oplus 0_n)R^{\mathrm{T}},\ R \in K\}.$$

For $A$, $B$, $C \in \mathrm{Sp}_{2n}\mathbb{R}/K$, $F \in \partial_n \mathrm{Sp}_{2n}\mathbb{R}/K$ and $p \geq 1$, the function $P_p$ is such that:

$$\lim_{A \to F} \frac{e^{d_p(A,C)}}{e^{d_p(A,B)}} = \frac{P_p(B,F)}{P_p(C,F)}. \tag{5.1}$$

As with the Busemann functions, the distance $d_1$ will be the one that works better for the Shilov boundary, since for this distance the quotient on the left hand side of the equation above will not depend on the choice of Weyl chamber representing $F$. Notice that this expression can be written as

$$\frac{e^{d_p(A,C)}}{e^{d_p(A,B)}} = e^{d_p(A,C)-d_p(A,B)} = e^{b_1(A,B,C)},$$

and the limit

$$\frac{P_p(B,F)}{P_p(C,F)} = \lim_{A \to F} e^{b_p(A,B,C)} = e^{b_p(F,B,C)}.$$

As we have seen at the end of Chapter 3, formula (3.5), for $p = 1$, the Busemann function can be written as

$$b_1(F,B,C) = 2\left(\log \sigma_1(\wedge_n C^{-1} \wedge_n F) - \log \sigma_1(\wedge_n B^{-1} \wedge_n F)\right).$$

So, we can write the Poisson kernel for $d_1$ as

$$P_1(A,F) = \frac{1}{\sigma_1(\wedge_n A^{-1} \wedge_n F)} = \prod_{j=1}^{n} \frac{1}{\sigma_j(A^{-1}F)}$$

and then indeed

$$\begin{aligned} e^{b_1(F,B,C)} &= e^{2\log\sigma_1(\wedge_n C^{-1}\wedge_n F) - 2\log\sigma_1(\wedge_n B^{-1}\wedge_n F)} \\ &= \frac{\sigma_1(\wedge_n C^{-1} \wedge_n F)}{\sigma_1(\wedge_n B^{-1} \wedge_n F)} \\ &= \frac{P_1(B,F)}{P_1(C,F)}. \end{aligned}$$

## 2 Limit sets

We are about to define the concept of a limit set for a discrete subgroup of $\mathrm{Sp}_{2n}\mathbb{R}$. In this chapter we'll always work with the bounded domain boundary, with its stratification, just like in the study we've made of the case $n = 2$ in Chapter 4. If we now define the limit set of a discrete subgroup $\Gamma$ by taking all the accumulation points of an orbit of $\Gamma$, the first thing we notice is that this set depends on the orbit, as can be seen from the following simple example. Take

$$\Gamma = <\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \odot I_2> = <\mathrm{diag}(1/2, 1, 2, 1)>.$$

Then, if you take

$$Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in \mathrm{SH}_2,$$

its orbit will accumulate at

$$\begin{pmatrix} 0 & 0 \\ 0 & z_3 \end{pmatrix} \in \partial_1 \mathrm{SH}_2,$$

so the accumulation point does depend on the point chosen to define the orbit. However, this does not happen if we only look at the accumulation points in $\partial_n \mathrm{SH}_n$, as we sill see. This is another very important property of the Shilov boundary. To prove this, we start with a technical lemma.

**Lemma 5.2.1** *Let $A, B \in \mathrm{M}_n\mathbb{R}$ and $j = 1, ..., n$. Then we have the following inequalities:*

$$\sigma_n(A)\sigma_j(B) \le \sigma_j(AB) \le \sigma_1(A)\sigma_j(B),$$
$$\sigma_j(A)\sigma_n(B) \le \sigma_j(AB) \le \sigma_j(A)\sigma_1(B).$$

This is the multiplicative version of Weyl's inequalities, and a proof of the right-hand side of the two inequalities can be found in [HJ, p. 423]. The left-hand side is argued similarly.

**Theorem 5.2.2** *Let $\Gamma < \mathrm{Sp}_{2n}\mathbb{R}$ be a discrete subgroup, and let $Z, W \in \mathrm{SH}_n$. Then*

$$\mathrm{Cl}(\Gamma Z) \cap \partial_n \mathrm{SH}_n = \mathrm{Cl}(\Gamma W) \cap \partial_n \mathrm{SH}_n.$$

*Moreover, if $(\gamma_k)$ is a sequence of elements in $\Gamma$ such that $\gamma_k Z \to Z_0 \in \partial_n \mathrm{SH}_n$, then $\gamma_k W \to Z_0 \in \partial_n \mathrm{SH}_n$.*

**Proof.** Clearly the last result implies the one about the sets. In order to prove it, notice that, because of the transitivity of the action of $\mathrm{Sp}_{2n}\mathbb{R}$ on $\partial_n \mathrm{SH}_n$, we can assume that $\gamma_k Z \to 0 \in \partial_n \mathrm{SD}_n$. Let $W \in \mathrm{SH}_n$ be any point, and write $\gamma_k Z = X_k + iY_k$, $\gamma_k W = U_k + iV_k$. According to our assumptions $X_k + iY_k \to 0$, and we wish to prove that $U_k + iV_k \to 0$.

Take $AK$ and $BK$ to be the cosets in $\mathrm{Sp}_{2n}\mathbb{R}/K$ corresponding to $Z$ and $W$ respectively, and consider furthermore the upper triangular form representatives for the sequences $\gamma_k AK$ and $\gamma_k BK$, as in equation (2.4):

$$A_k = \begin{pmatrix} Y_k^{\frac{1}{2}} & X_k Y_k^{-\frac{1}{2}} \\ 0 & Y_k^{-\frac{1}{2}} \end{pmatrix}, \ B_k = \begin{pmatrix} V_k^{\frac{1}{2}} & U_k V_k^{-\frac{1}{2}} \\ 0 & V_k^{-\frac{1}{2}} \end{pmatrix},$$

and we have

$$\gamma_k AK = A_k K \text{ and } \gamma_k BK = B_k K.$$

Now, since for any $C \in \mathrm{M}_{2m}\mathbb{R}$, its operator norm is given by

$$||C|| = \max_{||x||_2 = ||y||_2 = 1} |y^{\mathrm{T}} C x|,$$

we have

$$||C|| \geq \max_{l,m=1,2} ||C_{lm}||, \text{ for } C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \ C_{lm} \in \mathrm{M}_n\mathbb{R}, \ l, m = 1, 2. \tag{5.2}$$

Because the $\gamma_k$'s are isometries for all $d_p$'s, $p \in [1, \infty]$, we have

$$||B^{-1}A|| = \sigma_1(B^{-1}A) = \sigma_1(B_k^{-1}A_k) = ||B_k^{-1}A_k||.$$

Now, using the formula we have for the inverse of a symplectic matrix, we get

$$B_k^{-1}A_k = \begin{pmatrix} V_k^{-\frac{1}{2}} Y_k^{\frac{1}{2}} & V_k^{-\frac{1}{2}} X_k Y_k^{-\frac{1}{2}} - V_k^{-\frac{1}{2}} U_k Y_k^{-\frac{1}{2}} \\ 0 & V_k^{\frac{1}{2}} Y_k^{-\frac{1}{2}} \end{pmatrix},$$

In view of equation (5.2), and considering the $(2,2)$ block of matrix $B_k^{-1}A_k$ above,

$$||B^{-1}A|| \geq ||V_k^{\frac{1}{2}} Y_k^{-\frac{1}{2}}||.$$

Using the results in the previous lemma, we have

$$\begin{aligned} ||V_k^{\frac{1}{2}} Y_k^{-\frac{1}{2}}|| &= \sigma_1(V_k^{\frac{1}{2}} Y_k^{-\frac{1}{2}}) \\ &\geq \sigma_1(V_k^{\frac{1}{2}}) \sigma_n(Y_k^{-\frac{1}{2}}) \\ &= \sigma_1(V_k^{\frac{1}{2}}) (\sigma_n(Y_k)^{-1})^{\frac{1}{2}} \\ &= \sigma_1(V_k^{\frac{1}{2}}) \sigma_1(Y_k)^{-\frac{1}{2}} \\ &= ||V_k||^{\frac{1}{2}} ||Y_k||^{-\frac{1}{2}}. \end{aligned}$$

Part of the above inequality follows from the fact that for a symmetric positive definite matrix $Y$ the singular values are equal to the eigenvalues, and so $\sigma_n(Y^{\frac{1}{2}}) = \lambda_n(Y^{\frac{1}{2}}) = \lambda_n(Y)^{\frac{1}{2}} = \sigma_n(Y)^{\frac{1}{2}}$.

Combining the last two results, we get

$$||B^{-1}A|| \geq ||V_k||^{\frac{1}{2}}||Y_k||^{-\frac{1}{2}}$$

and thus

$$||V_k|| \leq ||B^{-1}A||^2||Y_k||.$$

As $||Y_k|| \to 0$ we deduce that $||V_k|| \to 0$. From the $(1,2)$ block of $(\gamma_k B)^{-1}(\gamma_k A)$, we get

$$\begin{aligned} ||B^{-1}A|| &\geq ||V_k^{-\frac{1}{2}}(X_k - U_k)Y_k^{-\frac{1}{2}}|| \\ &\geq ||V_k^{-\frac{1}{2}}(X_k - U_k)||\sigma_n(Y_k^{-\frac{1}{2}}) \\ &\geq \sigma_n(V_k^{-\frac{1}{2}})||X_k - U_k||\sigma_n(Y_k^{-\frac{1}{2}}) \\ &= \sigma_1(V_k)^{-\frac{1}{2}}||X_k - U_k||\sigma_1(Y_k)^{-\frac{1}{2}}. \end{aligned}$$

Thus

$$||X_k - U_k|| \leq ||B^{-1}A||.||Y_k||^{\frac{1}{2}}||V_k||^{\frac{1}{2}}, k = 1, \ldots, n,$$

and since $Y_k, V_k \to 0$, $X_k - U_k \to 0$. As $X_k \to 0$ we deduce $U_k \to 0$, thus proving the desired result. $\square$

In the 2-dimensionl upper half plane, all fixed points of a transformation $M$ would appear also as limit points for any orbit $<M>Z$. This, however, will not be the case in $\mathrm{SH}_n$. We give some examples.

In general, for a hyperbolic transformation $M = A \oplus A^{-\mathrm{T}}$, $|\lambda_n(A)| \geq 1$, and $\gamma = <M>$, there will be two accumulation points for the orbit $\{M^k(Z) : k \in \mathbb{Z}\}$, in spite of the other fixed points of the transformation. These two points will be $\lim_{k\to\infty} M^k(Z)$ and $\lim_{k\to-\infty} M^k(Z)$, for any $Z \in \mathrm{SH}_n$.

Let's study the case $M^{-k}$ first. Recall that $|\lambda_j(A)| > 1$, $1 \leq j \leq n$, and hence $|\lambda_j(A^{-1})| < 1$ and now it follows from the Jordan canonical form for $A$ that $||A^{-k}|| \to 0$ as $k \to \infty$.

The sequence $M^{-k}(Z)$ is

$$A^{-k}Z(A^{\mathrm{T}})^{-k} \text{ and } ||A^{-k}Z(A^{\mathrm{T}})^{-k}|| \leq ||A^{-k}||^2||Z|| \to 0,$$

since for every $m \in \mathbb{Z}$, $||A^m|| = ||(A^{\mathrm{T}})^m||$, so we must have $M^{-k}(Z) \to 0$ as $k \to \infty$.

For $M^k(Z)$, the limit point will be $\tilde{0} = J(0)$, the image of 0 under the map $Z \mapsto -Z^{-1}$, associated with the matrix $J$ used to define the symplectic group.

We have $J^{-1}MJ = A^{-\mathrm{T}} \oplus A$, $|\lambda_j(A^{-\mathrm{T}})| < 1$ for all $j$, and we can apply here the previous argument to conclude that $(J^{-1}MJ)^k(Z) = J^{-1}M^k(JZ) \to 0$, and hence $M^k(JZ) \to J(0)$, but since $Z$ was arbitrary, $M^k(Z) \to J(0)$.

We can see this more explicitly in the special case when $M = X_1 \odot \ldots \odot X_n$, with all $X_j$'s hyperbolic. We take them in diagonal form, with fixed points 0 and $\infty$ in $\partial\mathrm{H}$, and with $\infty$ attracting and 0 repelling, for each one of them. Take $p(0)$ and $p(\infty)$ to be the points in $\partial_n\mathrm{SH}_n$ corresponding to $(0\ldots,0)$ and $(\infty,\ldots,\infty)$ in $\partial(\mathrm{H}\times\ldots\times\mathrm{H})$. We have $p(0) = 0_n$ and $p(\infty) = \tilde{0}_n$.

By theorem 5.2.2, we can choose a point $Z$ in $\mathrm{H}\times\ldots\times\mathrm{H}$ and look at the accumulation points of its orbit—this will give us the accumulation points of any orbit, since these points will be in $\partial_n\mathrm{SH}_n$. As we can do the computations componentwise, we see that $p(0)$ and $p(\infty)$ will be the accumulation points, the first for $\{M^k(Z) : k \to -\infty\}$ and the second for $\{M^k(Z) : k \to \infty\}$. This agrees with the previous result.

Also, in the case $M = X_1 \odot \ldots \odot X_n$, with all $X_j$'s parabolic, all with fixed point 0, we can use a similar argument to see that the only point in of accumulation for the orbit will be $0_n = p(0)$, in spite of the other fixed points the transformation might have.

We can thus define several limit sets. We define them here as subsets of $\partial_n\mathrm{SD}_n$, but they can be obviously considered to be subsets of the Shilov boundary of any of the models.

**Definition 5.2.3** *Let $\Gamma < \mathrm{Sp}_{2n}\mathbb{R}$ be a discrete group. We define the following limit sets:*

- $\Lambda_h(\Gamma)$ *– the closure of the set of all fixed points of hyperbolic transformations in $M \in \Gamma$ that appear as accumulation points of $<M>Z$, $Z \in \mathrm{SD}_n$,*
- $\Lambda_l(\Gamma)$ *– the set of all limits of sequences of the type $\gamma_k Z$, $Z \in \mathrm{SD}_n$ that lie in $\partial_n\mathrm{SD}_n$,*
- $\Lambda_d(\Gamma)$ *– the complement in $\partial_n\mathrm{SD}_n$ of $\Omega$, the latter being the maximal open set of $\partial_n\mathrm{SD}_n$ on which the action of $\Gamma$ is properly discontinuous.*

In the 2-dimensional half plane we always have that $\Lambda_h(\Gamma) \subseteq \Lambda_l(\Gamma) \subseteq \Lambda_d(\Gamma)$, and if $\Gamma$ is non-elementary, then all three sets are equal (see the last proposition in the Introduction, or [Bea]).

In our case, it is evident that $\Lambda_h(\Gamma) \subseteq \Lambda_l(\Gamma)$ because all the accumulation points in the former are in the latter, and since $\Lambda_l$ is closed, we have the containment. Also, by the study of fixed points of hyperbolic transformations made above, we can see that every limit point for a hyperbolic transformation will also be an accumulation point for the orbit of an element in $\partial_n \mathrm{SH}_n$, as all the arguments will hold even for such a $Z$. This will imply that $\Lambda_h(\Gamma) \subseteq \Lambda_d(\Gamma)$.

If we take, instead of non-elementary subgroups, Zariski dense subgroups (as in [Alb]), we get an extra property for $\Lambda_l(\Gamma)$. The following result is due to Shmuel Friedland.

**Theorem 5.2.4** *Let $\Gamma$ be a Zariski dense subgroup of $\mathrm{Sp}_{2n}\mathbb{R}$. Then for any hyperbolic element $A \in \Gamma$, with $v$ being one of its fixed points that is a limit point, $\Lambda_h(\Gamma) = \mathrm{Cl}(\Gamma v)$ and this is the smallest closed $\Gamma$-invariant subset of $\partial_n \mathrm{SD}_n$. Moreover, $\Lambda_h(\Gamma)$ is a perfect set.*

**Proof.** Since $\Gamma$ is Zariski dense, it must have a hyperbolic element, this is a consequence of the results in [GM]. Let $A$ be such an element. Then, for $N = \binom{2n}{n}$, let $v \in \mathbb{C}^N$ be the eigenvector for the largest eigenvalue of $\wedge_n A$:

$$\wedge_n A v = \rho v, \quad \rho = \prod_{j=1}^{n} \lambda_j(A).$$

Now, given a point $w \in \mathbb{C}^N$, we know that $\lim_{m\to\infty}[\wedge_n(A^m)w] = [v]$ in $\mathbb{C}P^{N-1}$ if and only if $v^{\mathrm{T}} w \neq 0$ (again we use square brackets to denote equivalence classes).

Now consider our projective model $\mathrm{SPH}_n$, and take $E$ to be any closed (non-empty) $\Gamma$-invariant subset of $\wedge_n(\partial_n \mathrm{SPH}_n) \subset \mathbb{C}P^{N-1}$. Take any point $w \in E$. Since $\Gamma$ is Zariski dense, there exists $Q \in \Gamma$ such that $v^{\mathrm{T}} \wedge_n (Q^{\mathrm{T}})w \neq 0$. Then consider the hyperbolic element $A_1 := QAQ^{-1} \in \Gamma$. By what we just said, $\wedge_n(A_1^m)w$ converges to $Qv \in \mathbb{C}P^{N-1}$. Since we chose $w \in E$, and $E$ was $\Gamma$-invariant, we have $Qv \in E$, and hence $\Gamma v \subseteq E$ and $E_v := \mathrm{Cl}(\Gamma v) \subseteq E$.

Clearly $E_v$ is also closed and $\Gamma$-invariant, so if we take another hyperbolic element $B \in \Gamma$ with the corresponding fixed point $u$, and repeat the previous argument with $E_v$ instead of $E$, we get $E_u := \mathrm{Cl}(\Gamma u) \subseteq E_v$, and finally, if we consider $E = E_u$, we conclude that $E_u = E_v$. This shows that, for any hyperbolic element $A$ with limit fixed point $v$,

$$\Lambda_h(\Gamma) = \mathrm{Cl}(\Gamma v)$$

and this is the smallest closed $\Gamma$-invariant subset of the $n$-th stratum of the boundary.

It is left to show that this is a perfect set. Let $w$ be a point in $\Lambda_h(\Gamma)$. If $w \notin \Gamma v$ then it has to be in its closure, and so it is an accummulation point of $\Gamma v \subseteq \Lambda_l(\Gamma)$. If $w \in \Gamma v$, then we can assume, without loss of generality that $v = w$. Since $\Gamma$ is Zariski dense, there exists a point $z \in \Gamma v$ such that $z \neq v$ and $v^{\mathrm{T}} z \neq 0$ and hence $\lim_{m\to\infty} \wedge_n(A^m)w = v$ and $v$ is an accumulation point of $\Lambda_h(\Gamma)$. $\square$

We now proceed to prove that $\Lambda_l(\Gamma) \subset \Lambda_d(\Gamma)$.

**Lemma 5.2.5** *Let $(M_k)$ be a sequence of elements in $\mathrm{Sp}_{2n}\mathbb{R}$. Then if $M_k(Z) \to Z_0$, $Z_0 \in \partial_n\mathrm{SH}_n$, then the largest $n$ singular values of $M_k$ go to $\infty$, and hence the other $n$ go to 0.*

**Proof.** Consider the singular value decomposition, with a slight modification: $M_k = Q_k D_k R_k$,

$$Q_k, R_k \in K,\ D_k = \Sigma_s(M_k)^{-1},$$
$$D_k = S_k \oplus S_k^{-1},\ S_k = \mathrm{diag}(\sigma_{2n}(M_k), \dots, \sigma_{n+1}(M_k)),$$

This decomposition exists because $\Sigma_s(M_k)^{-1} = J\Sigma_s(M_k)J^{\mathrm{T}}$. Assume, without loss of generality, that $Z_0 = 0$. Then consider the point $iI_n \in \mathrm{SH}_n$. We know, because of theorem 5.2.2, that $M_n(iI_n) \to 0$, $M_n(iI_n) = R_k D_k(iI_n)$, since $R_k(iI_n) = iI_n$. Then, $D_k(iI_n) = S_k iI_n S_k$. If not all the eigenvelues of $S_k$ go to 0, the limit matrix $\lim_k R_k(S_k iI_n S_k)$ will not be zero, so we must have $S_k \to 0$ as we wished. □

It would be possible, if a bit more cumbersome, to prove this result without using theorem 5.2.2, by considering the sequence $M_k\tilde{Z}K \in \mathrm{Sp}_{2n}\mathbb{R}/K$, with $\tilde{Z}K$ being the coset corresponding to $Z$, and looking at the $n$ smallest eigenvalues of matrices $M_k\tilde{Z}$, and concluding they have to go to 0. The result would then come from the Weyl inequalities.

**Theorem 5.2.6** *Let $\Gamma < \mathrm{Sp}_{2n}\mathbb{R}$ be a discrete subgroup. Let $Z_0 \in \partial_n\mathrm{SH}_n$ such that for some sequence of elements $(M_k)$ in $\Gamma$, $M_k(Z) \to Z_0$ for some (and hence any) $Z \in \mathrm{SH}_n$. Then there exist $W \in \partial\mathrm{SH}_n$ and a subsequence of $(M_k)$, $M_{k_m}$, such that $M_{k_m}(W) \to Z_0$.*

**Proof.** Let $M_k = R_k\Sigma_s(M_k)^{-1}Q_k$ in the previous result. Now, since $K$ is compact, we can find subsequences of $Q_k$ and $R_k$ such that $Q_{k_m} \to Q$ and $R_{k_m} \to R$, $Q, R \in K$. Rename the subsequence, so that we can assume that $Q_k$ and $R_k$ converge. As usual, without loss of generality, we'll consider $Z_0 = 0$.

We are going to use our projective model $\mathrm{SPH}_n$. As a general result in the projective space $\mathbb{C}^{N-1}$, we know that if $v \in \mathbb{C}^N$, $v \neq 0$, and $X_k$ is a sequence of matrices in $\mathrm{M}_N\mathbb{C}$ such that $X_k.z_k \to X$ for some sequence of non-zero complex numbers $z_k$, and $Xv \neq 0$, then $\lim[X_k v] = [Xv]$, using as usual square brackets to denote equivalence classes.

Let's apply this last result to our case, with $X_k := \wedge_n(M_k)$ and $z_k := 1/||M_k||$. Take $\Sigma_s(M)^{-1} = \mathrm{diag}(d_{k,1}, \dots, d_{k,2n})$, a symplectic diagonal matrix whose first $n$

diagonal entries are the $n$ smallest singular values of $M_k$. For $1 \leq j \leq n$, we have

$$\begin{aligned}
\Sigma_s(M_k) &= \mathrm{diag}(\sigma_1(M_k)^{-1}, \ldots, \sigma_n(M_k)^{-1}, \sigma_{2n}(M_k)^{-1}, \ldots, \sigma_{n+1}(M_k)^{-1}) \\
&= \mathrm{diag}(\sigma_{2n}(M_k), \ldots, \sigma_{n+1}(M_k), \sigma_1(M_k), \ldots, \sigma_n(M_k)) \\
&= \mathrm{diag}(d_{k,1}, \ldots, d_{k,2n}),
\end{aligned}$$

and, by the previous result, $d_{k,j} \to 0$ for $1 \leq j \leq n$.

Notice that, given the way we chose the order of the singular values of $M_k$ in the diagonal of $D_k$, we have

$$\frac{\wedge_n(D_k)}{\prod_{j=n}^{2n} d_{k,j}} \to \wedge_n(0_n \oplus I_n) \text{ as } k \to \infty.$$

Thus we have:

$$\begin{aligned}
\frac{\wedge_n(M_k)}{|| \wedge_n (M_k)||} &= \frac{\wedge_n(Q_k) \wedge_n (D_k) \wedge_n (R_k)}{|| \wedge_n (D_k)||} \\
&= \wedge_n(Q_k)\frac{\wedge_n(D_k)}{\prod_{j=n}^{2n} d_{k,j}} \wedge_n (R_k) \\
&\to \wedge_n(Q) \wedge_n (0_n \oplus I_n) \wedge_n (R) \\
&= \wedge_n(Q(0_n \oplus I_n)R).
\end{aligned}$$

Denote this limit matrix by $M$:

$$M := \wedge_n(Q(0_n \oplus I_n)R).$$

Now, take any point $W$ in $\mathrm{Cl}(\mathrm{SH}_n)$ of the form $R^{\mathrm{T}}W'$ where $W'$ admits a representative of the type $(W'_0, I_n)^{\mathrm{T}}$, $W'_0 \in \mathrm{Sym}_n\mathbb{C}$, $\mathrm{Im}(W'_0) \geq 0$; we have this if $W'$ is either inside $\mathrm{SH}_n$ or in the finite part of $\partial\mathrm{SH}_n$, so we can find such $W$'s in any stratum of the boundary. Take the following representative of $W$:

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} := R^{\mathrm{T}} \begin{pmatrix} W'_0 \\ I_n \end{pmatrix}.$$

Then we have

$$\begin{aligned}
M \wedge_n (W_1, W_2)^{\mathrm{T}} &= \wedge_n \left(Q(0_n \oplus I_n)(W'_0, I_n)^{\mathrm{T}}\right) \\
&= \wedge_n(Q(0_n, I_n)^{\mathrm{T}}) \neq 0,
\end{aligned}$$

since this corresponds to the image of $(0_n, I_n)^{\mathrm{T}} \in \partial_n \mathrm{SPH}_n$ under $Q$. This means we can write

$$\begin{aligned}
\lim \wedge_n M_k \wedge_n W &= \lim[\wedge_n(M_k) \wedge_n (W_1, W_2)^{\mathrm{T}}] \\
&= [M \wedge_n (W_1, W_2)^{\mathrm{T}}] \\
&= [\wedge_n(Q(0_n \oplus I_n) R R^{\mathrm{T}} (W_0', I_n)^{\mathrm{T}})] \qquad (5.3) \\
&= [\wedge_n(Q(0_n, I_n)^{\mathrm{T}}] \\
&= \wedge_n[Q(0_n, I_n)^{\mathrm{T}}].
\end{aligned}$$

Now consider our hypothesis that $M_n(Z) \to 0$. We must have the following: first, since obviously $Z = R^{\mathrm{T}} Z'$, with $Z' = RZ \in \mathrm{SH}_n$, by formula (5.3) above with $W = Z$, we get

$$\lim M_k[(Z, I_n)^{\mathrm{T}}] = \wedge_n[Q(0_n, I_n)^{\mathrm{T}}];$$

and second, since $M_n(Z) \to 0$,

$$\lim M_k[(Z, I_n)^{\mathrm{T}}] = \wedge_n[(0_n, I_n)^{\mathrm{T}}].$$

Combining the two equalities, we get $\wedge_n[Q(0_n, I_n)^{\mathrm{T}}] = \wedge_n[(0_n, I_n)^{\mathrm{T}}]$, and hence $Q \in \mathrm{Stab}(0) \cap K = \tilde{\mathrm{O}}_n$.

Finally, we get, from the equality above and equation (5.3),

$$\begin{aligned}
\lim_{k \to \infty} \wedge_n M_k \wedge_n W &= \wedge_n[Q(0_n, I_n)^{\mathrm{T}}] \\
&= \wedge_n[(0_n, I_n)^{\mathrm{T}}]
\end{aligned}$$

and $M_k(W) \to 0$, as we wished, since we could pick $W'$ from any of the boundary strata, in particular, from $\partial_n \mathrm{SH}_n$, which proves what we wished. $\square$

This last result would also provide a proof that every orbit under the action of $\Gamma$ has the same accumulation points in $\partial_n \mathrm{SH}_n$, with a result slightly weaker that the one in theorem 5.2.2, namely, that if a sequence $M_k Z$ converges to $Z_0 \in \partial \mathrm{SH}_n$, then there is a subsequence of $M_k$, $M_{k_m}$, such that $M_{k_m} Z \to Z_0$. This is because, as we have seen, any $Z \in \mathrm{SH}_n$ can be written as $R^{\mathrm{T}} Z'$, with $Z' = RZ$ having (obviously) the representative $(Z', I_n)^{\mathrm{T}}$ in $\mathrm{SPH}_n$.

Since clearly the set $\partial_n \mathrm{SD}_n \setminus \Lambda_d(\Gamma)$ cannot have accumulation points for any orbit of an element of $\partial_n \mathrm{SD}_n$, we arrive at the desired result.

**Corollary 5.2.7** *For a discrete subgroup $\Gamma < \mathrm{Sp}_{2n}\mathbb{R}$, we have*

$$\Lambda_h(\Gamma) \subset \Lambda_l(\Gamma) \subset \Lambda_d(\Gamma).$$

To finish this chapter, we present discrete subgroups of $\mathrm{Sp}_4\mathbb{R}$, $\Gamma_1$ and $\Gamma_2$, such that $\Lambda_h(\Gamma_1) \neq \Lambda_l(\Gamma_1)$ and $\Lambda_l(\Gamma_2) \neq \Lambda_d(\Gamma_2)$. For $\Gamma_1$, we can take

$$\Gamma_1 := <X \odot X>, \text{ with } X = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The group will have only one limit point, $0 \in \partial_2\mathrm{SH}_2$, so $\Lambda_l(\Gamma_1) = \{0\}$, and since there are no hyperbolic maps in the group, $\Lambda_h(\Gamma_1) = \emptyset$.

For the other group, we can take

$$\Gamma_2 := <Y \odot Y>, \text{ with } Y = \mathrm{diag}(1/2, 2).$$

In this case, $\Lambda_l(\Gamma_2) = \{0, \tilde{0}\}$, where $\tilde{0}$ represents as usual the point $J(0) \in \partial_2\mathrm{SH}_2$. The set $\Lambda_d(\Gamma_2)$ will have to include at least the 1-dimensional variety of fixed points in $\partial_2\mathrm{SH}_2$ also, so we must have $\Lambda_l(\Gamma_2) \neq \Lambda_d(\Gamma_2)$.

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**Present situation**. Assistant at the *Universidade de Lisboa*, and a member of the *Departamento de Matemática, Faculdade de Ciencias, Universidade de Lisboa, 1749-016, Lisboa,* and of *Centro de Algebra da Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003, Lisboa* (CAUL)

**Publications**.

- Master's thesis: *Dimensões de espaços e álgebras de matrizes — um levantamento*, 1994 (in Portuguese).

- PhD. Thesis: *On the action of the symplectic group on the Siegel upper half plane*, 1999.

The papers from the PhD. thesis are currently in preparation.

**Meetings**.

- Participated in the 4th annual International Linear Algebra Society (ILAS) meeting in 1992.
- Presented a talk, as an invited speaker, with the title *On 'Möbius' transformations of the Siegel upper half plane* at the AMS meeting in Orlando, Florida, January 1996. This talk was included in a special session on Dynamics in several complex variables.
- Presented a talk with the title *On an action of the Symplectic group* at the 7th annual ILAS meeting in Madison, Wisconsin, June 1998.

**Talks**.

- *Path coverings of graphs and height characteristic of matrices*, 1993, Math Dept, Univ of Lisbon,
- *A generalization of the Hamilton-Cayley theorem and some consequences*, 1994, CAUL seminar,
- *Dimensions of spaces of nilpotent matrices: a combinatorial approach*, 1994, CAUL seminar.
- *On 'Möbius' transformations of the Siegel upper half plane*, 1996, AMS annual meeting (invited speaker),
- *On an action of the Symplectic group*, ILAS annual meeting, 1998.
- *The dynamics of an action of the symplectic group, I and II*, CAUL seminar, 1998.

**Institutions**. Member of the Center of Algebra of the University of Lisbon since 1992, of the International Linear Algebra Society since 1994, of the American Mathematical Society since 1996, and of the Department of mathematics of the University of Lisbon since 1998.

**Languages**. Written and spoken English, French, Spanish and Portuguese.

**Other credits**. Has been awarded both a Master's fellowship from the Junta Nacional de Investigação Científica e Tecnológica, and a PhD. fellowship from *Fun-*

*dação para a Ciência e Tecnologia.* Served as president of the Mathematics Graduate Students Association at the University of Illinois at Chicago during the 96/97 school year.