HAMILTONIAN EVOLUTIONARY GAMES

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Abstract. We introduce a class of o.d.e.'s that generalizes to polymatrix games the replicator equations on symmetric and asymmetric games. We also introduce a new class of Poisson structures on the phase space of these systems, and characterize the corresponding subclass of Hamiltonian polymatrix replicator systems. This extends known results for symmetric and asymmetric replicator systems.

1. Introduction

State of the art. Evolutionary Game Theory (EGT) originated from the work of John Maynard Smith and George R. Price who applied the theory of strategic games developed by John von Neumann and Oskar Morgenstern to evolution problems in Biology. Unlike Game Theory, EGT investigates the dynamical processes of biological populations.

Independently A. Lotka and V. Volterra introduced the following class of o.d.e.'s

$$\frac{dx_i}{dt} = x_i \left(r_i + \sum_{j=1}^{n} a_{ij}\, x_j \right) \; (1 \leq i \leq n) \, ,$$

currently known as Lotka-Volterra (LV) systems, and usually taken as models for the time evolution of ecosystems in n species. Although historically this class of systems preceded EGT it plays an important role in this theory. The entries a_{ij} represent interactions between different species, while the coefficients r_i stand for the species' natural growth rates. In his studies [22] V. Volterra gave special attention to predator-prey systems and their generalization to food chain systems in n species, which fall in the category of dissipative and conservative LV systems. Denoting by $A = [\, a_{ij}]_{ij}$ its interaction matrix, a LV system is said to be *dissipative*, resp. *conservative*, if there exists a positive diagonal matrix D such that $AD + DA^t \leq 0$, resp. AD is skew symmetric. The entries d_i of the diagonal matrix D were interpreted by Volterra as some sort of normalization factors related with the average weights of the different species. If the LV system admits an equilibrium

Date: March 27, 2015.
Key words and phrases. Evolutionary games, Hamiltonian polymatrix replicator equation, Poisson Hamiltonian systems.

point $q \in \mathbb{R}^n$ the following function $H : \mathrm{int}(\mathbb{R}^n_+) \to \mathbb{R}$

$$H(x) = \sum_{j=1}^{n} d_j^{-1} \, (x_j - q_j \, \log x_j)$$

is either a decreasing Lyapunov function, if the system is dissipative, or else a constant of motion, if the system is conservative. Volterra proved that the dynamics of any n species conservative LV system can be embedded in a Hamiltonian system of dimension $2n$. More recently, in the 1980's, Redheffer et al. developed further the teory of dissipative LV systems, introducing and studying the class of stably dissipative systems [17, 18]. In [2] a re-interpretation was given for the Hamiltonian character of the dynamics of any conservative LV system: there is a Poisson structure on $\mathbb{R}^n_+$ which makes the system Hamiltonian. The Hamiltonian structures for Lotka-Volterra equations were also studied in [15]. Another interesting fact from [2], which stresses the importance of studying Hamiltonian LV systems, is that the limit dynamics of any stably dissipative LV system is described by a conservative LV system.

Another class of o.d.e.'s, which plays a central role in EGT, is the *replicator equation* defined on the simplex $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\}$ by

$$\frac{dx_i}{dt} = x_i \left(\sum_{j=1}^{n} a_{ij} \, x_j - \sum_{k,j=1}^{n} a_{kj} \, x_k \, x_j \right) \qquad (1 \leq i \leq n) \; .$$

The coefficients of this o.d.e. are stored in an $n \times n$ real matrix $A = [\, a_{ij}]_{ij}$, that is referred as the *payoff* matrix. A game theoretical interpretation for this equation is provided in section 3. Check [8] on the history of this equation. In [7] J. Hofbauer introduced a change of coordinates, mapping $\mathbb{R}^n_+$ to the simplex Δ^n minus one face, which conjugates any LV system in $\mathbb{R}^n_+$ to a time re-parametrization of a replicator system in Δ^n, and vice-versa. Thus when a LV system is conservative then the corresponding replicator system is orbit equivalent to a Hamiltonian system. On the other hand, any replicator system on Δ^{n-1} with skew symmetric payoff matrix extends to a LV system on $\mathbb{R}^n_+$ with $r_i = 0$, and hence can be viewed as a restriction of a Hamiltonian LV system on $\mathbb{R}^n_+$. The Hamiltonian character of this special subclass of replicator equations with skew-symmetric payoff matrices was studied by E. Akin and V. Losert in [1]. Other Hamiltonian replicator systems were identified by Plank [14, 15].

Asymmetric or bimatrix games lead to another fundamental class of models in EGT, the following system of o.d.e.'s whose coefficients are displayed in two payoff matrices, A of order $n \times m$ and B of order $m \times n$.

$$\frac{dx_i}{dt} = x_i \left(\sum_{j=1}^{m} a_{ij}\, y_j - \sum_{k=1}^{n} \sum_{j=1}^{m} a_{kj}\, x_k\, y_j \right) \qquad i = 1, .., n$$

$$\frac{dy_j}{dt} = y_j \left(\sum_{i=1}^{n} b_{ji}\, x_i - \sum_{k=1}^{m} \sum_{i=1}^{n} b_{ki}\, y_k\, x_i \right) \qquad j = 1, .., m$$

The phase space of this equation is the prism $\Delta^{n-1} \times \Delta^{m-1}$. A game theoretical interpretation is given in section 3. It was remarked by I. Eshel and E. Akin [4] that these systems preserve a certain volume form in the interior of the prism $\Delta^{n-1} \times \Delta^{m-1}$. For λ-zero-sum games ($\lambda < 0$) and λ-partnership games ($\lambda > 0$), with an interior equilibrium point in the prism $\Delta^{n-1} \times \Delta^{m-1}$, J. Hofbauer proved in [6] that this bimatrix system is Hamiltonian with respect to some Poisson structure in the interior of the prism.

The theory of equilibria for n-person games started with the work of J. Nash [11]. A subclass of n-person games, referred as *polymatrix games*, where the payoff of each player is the sum of the payoffs corresponding to simultaneous contests with the opponents, was studied by J. Howson [9] who attributes the concept to E. Yanovskaya (1968). The replicator equation for n-person games with multi-linear payoffs[1] was formulated first by Palm [12] and studied by Ritzberger, Weibull [19], Plank [16] among others.

Main results. We introduce a class of o.d.e's, referred as *polymatrix replicator equation*, that includes the symmetric and asymmetric replicator equations, as well as the subclass of the n-player replicator equation in [12, 16, 19] associated to the polymatrix games studied by Howson and Yanovskaya. However, the polymatrix replicator equation studied here should not be seen as the dynamical counterpart of a polymatrix game, which is only true for a subclass of equations where some block diagonal matrices vanish (i.e., $A^{\alpha,\alpha} = 0$ in (3.3)). More general versions of the n-player replicator equation, that include the mentioned subclass of our polymatrix replicator equation, have been studied before, see e.g. [20, exercise 3.3.5].

The phase space of these systems are finite products of simplexes. We introduce the concept of *conservative* polymatrix game, which in the case of bimatrix games extends the λ-zero-sum games ($\lambda < 0$) and the λ-partnership games ($\lambda > 0$). In Theorem 3.13 we introduce a class of stratified Poisson structures on finite products of simplexes. Then in Theorem 3.20 we show that any conservative polymatrix game determines a Hamiltonian polymatrix replicator equation. This work extends and unifies several known facts on Hamiltonian replicator o.d.e.'s. In the end of

[1] a multi-linear payoff is a function which is linear in each player's mixed strategy, when all other players' strategies are fixed.

Section 3 we compare our results with known facts mentioned in the state of the art subsection.

The paper is organized as follows. In Section 2 we introduce the needed concepts from Poisson geometry. In Section 3 we state and prove the main results. In Section 4 we work out a couple of examples.

2. Generalities on Poisson Structures

In this section we will provide a short introduction to Poisson geometry focused on some dynamical aspects, see any standard textbook on Poisson manifolds and related topics, for example [3, 13].

Let M be an n-dimensional smooth manifold. We denote by $C^\infty(M)$ the space of smooth functions on M. A Poisson structure on M is an $\mathbb{R}$-bilinear bracket $\{.,.\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ which satisfies:

i) Anti-symmetry i.e. $\{f, g\} = -\{g, f\}$ for every $f, g \in C^\infty(M)$.
ii) Leibniz's rule i.e. $\{fg, h\} = f\{g, h\} + g\{f, h\}$ for every $f, g, h \in C^\infty(M)$.
iii) Jacobi identity i.e. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$

The Leibniz's rule says that for any smooth function $H : M \to \mathbb{R}$ the map $\{., H\} : f \mapsto \{f, H\}$ is a deriviation on $C^\infty(M)$ which in turn yields a vector X_H on M defined by the equality $\{f, H\} = \mathrm{d}f(X_H)$. The vector field X_H is called the *Hamiltonian vector field* associated to H on the Poisson manifold M.

The singular distribution $D(x) := \{X_f(x) \mid f \in C^\infty(M)\}$ is called the *characteristic distribution* of M. As a consequence of the Jacobi identity this distribution integrates to a singular foliation. Denote by S_x the leaf of this foliation through a point x. The Poisson structure induces a symplectic form on each leaf S_x, passing through arbitrary point $x \in M$, of this foliation defined by $\omega_{S_x}(X_f, X_h) = \{f, h\}$. The foliation $\mathcal{S} := \{S_x | x \in M\}$ is called the *symplectic foliation* of the Poisson manifold M.

Remark 2.1. *The following are well known properties of Poisson structures:*

1) *By (i),* $\mathrm{d}H(X_H) = \{H, H\} = -\{H, H\} = 0$*. Thus H is an integral of motion for the vector field X_H.*
2) *The dimension of the linear subspace $D(x)$ is called the* rank *of the Poisson structure at point x, which is equal to the dimension of the leaf S_x. Since this leaf is a symplectic manifold on its own it has even dimension.*
3) *The symplectic foliation $\mathcal{S} := \{(S_x, \omega_{S_x}) | x \in M\}$ completely determines the Poisson structure.*
4) *By definition, it is clear that every symplectic leaf S_x is an invariant submanifold for any Hamiltonian vector filed X_H. In fact, the restriction of X_H to S_x is Hamiltonian with respect to the symplectic structure ω_{s_x}.*

5) *Every symplectic manifold* (N, ω) *is a Poisson manifold with Poisson bracket defined by* $\{f, g\}_N := \omega(X_f, X_g)$, *where* X_f *and* X_g *are the Hamiltonian vector fields associated to* f *and* g *by symplectic structure.*
6) *A function* f *is called* Casimir *if* $\{., f\} = 0$. *Note that Casimirs are constants of motion for any Hamiltonian vector field. Furthermore, if* f_1, f_2 *are two Casimirs then* $\{f_1, f_2\}$ *is also a Casimir due to Jacobi identity.*

In a local coordinate chart $(U, x_1, .., x_n)$, or equivalently when $M = \mathbb{R}^n$, a Poisson bracket takes the form

$$\{f, g\}(x) = (\mathrm{d}_x f)^t \left[\, \pi_{ij}(x)\right]_{ij} \mathrm{d}_x g = \sum_{i<j} \pi_{ij}(x) \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right),$$

where $\pi(x) = \left[\, \pi_{ij}(x)\right]_{ij} = \left[\, \{x_i, x_j\}(x)\right]_{ij}$ is a skew symmetric matrix valued smooth function, and for every function f we write

$$\mathrm{d}_x f = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix} .$$

The Jacobi identity translates to:

$$\sum_{l=1}^{n} \frac{\partial \pi_{ij}}{\partial x_l} \pi_{lk} + \frac{\partial \pi_{jk}}{\partial x_l} \pi_{li} + \frac{\partial \pi_{ki}}{\partial x_l} \pi_{lj} = 0 \qquad \forall i, j, k \;, \tag{2.1}$$

or equivalently

$$\{\{x_i, x_j\}, x_k\} + \{\{x_j, x_k\}, x_i\} + \{\{x_j, x_k\}, x_j\} = 0 \qquad \forall i, j, k \;. \tag{2.2}$$

Clearly, every skew symmetric matrix valued function $\pi : \mathbb{R}^n \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ satisfying condition (2.1) defines a Poisson structure on $\mathbb{R}^n$. In the next section we shall introduce our Poisson structures through their associated skew symmetric matrix valued functions, referred as a bivectors $\pi : \mathbb{R}^n \to \mathrm{Mat}_{n\times n}(\mathbb{R})$. The term bivector means that $\pi(x)$ is as a linear operator $\pi(x) : (\mathbb{R}^n)^* \to \mathbb{R}^n$.

Remark 2.2. *Regarding the function* π *we have*

1) *For any function* H *the associated Hamiltonian vector field is defined by*
$$X_H = \pi \, \mathrm{d}H,$$
2) *The characteristic distribution* $D_\pi(x)$ *is the one generated by the columns of the matrix* $\pi(x)$.
3) *It transforms under a change of variable* $\psi : \mathbb{R}^n \to \mathbb{R}^n$ *by*
$$(\mathrm{d}_m\psi)\pi(m)(\mathrm{d}_m\psi)^t = \pi(\psi(m)), \tag{2.3}$$
4) *A function* $f \in C^\infty(\mathbb{R}^n)$ *is a Casimir if*
$$X_f = \pi \, \mathrm{d}f = 0.$$

Let $(M, \{,\}_M)$ and $(N, \{.,.\}_N)$ be two Poisson manifolds.

Definition 2.3. A smooth map $\psi : M \to N$ will be called a *Poisson map* if and only if

$$\{f \circ \psi, h \circ \psi\}_M = \{f, h\}_N \circ \psi \quad \forall f, h \in C^\infty(N).$$

In local coordinate, this condition reads as

$$(\mathrm{d}_m\psi)\pi_M(m)(\mathrm{d}_m\psi)^t = \pi_N(\psi(m)), \tag{2.4}$$

where π_M and π_N are skew symmetric matrix valued maps associated to Poisson structures of M and N, respectively, and $\mathrm{d}_m\psi$ is the Jacobian matrix of the map ψ at point m.

3. Polymatrix games

In this section we introduce the *evolutionary polymatrix games* to which our main result applies. This class of systems contains both the *replicator equations* and the *bimatrix replicator equations*.

Consider a population whose individuals interact with each other using one of n possible pure strategies. The state of the population is described by a probability vector $p = (p_1, \ldots, p_n)$, with the usage frequency of each pure strategy. This vector is a point in the $n-1$-*dimensional simplex*

$$\Delta^{n-1} = \{\, (x_1, \ldots, x_n) \in \mathbb{R}^n \,:\, x_1 + \ldots + x_n = 1,\, x_i \geq 0 \,\} \,.$$

A symmetric game is specified by a $n \times n$ payoff matrix $A = [\, a_{ij}]_{ij}$, where the entry a_{ij} represents the payoff of an individual using pure strategy i against another using pure strategy j. Given $x \in \Delta^{n-1}$, the value $(A\,x)_i = \sum_{j=1}^n a_{ij}\,x_j$ represents the average payoff of strategy i within a population at state x. Similarly, the value $x^t\,A\,x = \sum_{i,j=1}^n a_{ij}\,x_i\,x_j$ stands for the overall average of a population at state x, while the difference $(A\,x)_i - x^t\,A\,x$ measures the *relative fitness* of strategy i in the population x. The *replicator* model is the following o.d.e. on Δ^{n-1}

$$\frac{dx_i}{dt} = x_i\,\big((A\,x)_i - x^t\,A\,x\big) \quad 1 \leq i \leq n \tag{3.1}$$

which says that the logarithmic growth rate of each pure strategy's frequency equals its relative fitness. The flow of this o.d.e. is complete and leaves the simplex Δ^{n-1} invariant, as well as every of its faces.

Next we introduce the class of evolutionary asymmetric, or bimatrix, games, where two groups of individuals within a population (e.g. males and females), or two different populations, interact using different sets of strategies, say n strategies for the first group and m strategies for the second. The state of this model is a pair of probability vectors in the $n+m-2$-dimensional prism $\Gamma_{n,m} = \Delta^{n-1} \times \Delta^{m-1}$. There are no interactions within each group. The game is specified by two payoff matrices: a $n \times m$ matrix $A = [\, a_{ij}]_{ij}$, where a_{ij} is the payoff for a member of the first group using strategy i against an individual of the second group using strategy j, and a $m \times n$ matrix $B = [\, b_{ij}]_{ij}$ with the payoffs for the second group members. Assuming the first and second group states are x and y, respectively, the value

$(A\,y)_i$ is the average payoff for a first group individual using strategy i, the number $x^t\,A\,y$ is the overall average payoff for the first group members, and the difference $(A\,y)_i - x^t\,A\,y$ measures the *relative fitness* of the first group strategy i. Similarly, $(B\,x)_j - y^t\,B\,x$ measures the *relative fitness* of the second group strategy j when the group states are x and y. The bimatrix replicator equation is the following o.d.e. on the prism $\Gamma_{n,m}$

$$\begin{aligned} \frac{dx_i}{dt} &= x_i\,\big((A\,y)_i - x^t\,A\,y\big) \quad 1 \le i \le n \\ \frac{dy_j}{dt} &= y_j\,\big((B\,x)_j - y^t\,B\,x\big) \quad 1 \le j \le m \end{aligned} \tag{3.2}$$

which again says that the logarithmic growth rate of each strategy's frequency equals its relative fitness. The flow of this o.d.e. is complete and leaves the prism $\Gamma_{n,m}$ invariant, as well as every of its faces.

Finally we introduce the class of polymatrix replicator equations. Consider p different populations, or else a single population stratified in p groups. We shall use greek letters like α and β to denote these groups. Assume that for each group $\alpha \in \{1,\ldots,p\}$, there are n_α pure strategies for interacting with members of another group, including its own. Let us call *signature* of the game to the vector $\underline{n} = (n_1,\ldots,n_p)$. The total number of strategies is therefore $n = n_1 + \ldots + n_p$. The polymatrix game is specified by a single $n \times n$ matrix $A = \left[\,a_{ij}\right]_{ij}$ with the payoff a_{ij} for a user of strategy i, member of one group, against a user of strategy j, member of another group, possibly the same. The main difference between polymatrix games and the symmetric game, also specified by a single matrix A, is that in the polymatrix game competition is restricted to members of the same group. This means that the relative fitness of each strategy refers to the overall average payoff of strategies within the same group. To be more precise we need to introduce some notation. We decompose A in blocks, $A = \left[\,A^{\alpha,\beta}\right]_{\alpha,\beta}$, where each block $A^{\alpha,\beta} = \left[\,a^{\alpha,\beta}_{ij}\right]_{ij}$ is a $n_\alpha \times n_\beta$ matrix. Similarly we decompose each vector $x \in \mathbb{R}^n$ as $x = (x^\alpha)_\alpha$, where $x^\alpha \in \mathbb{R}^{n_\alpha}$. We say that a strategy i belongs to a group α, and write $i \in \alpha$, if and only if $n_1 + \ldots + n_{\alpha-1} < i \le n_1 + \ldots + n_\alpha$. Similarly we write $(i,j) \in \alpha \times \beta$ when $i \in \alpha$ and $j \in \beta$. With this notation we have

(a) $x^\alpha_i = x_i$ if $i \in \alpha$, and
(b) $a^{\alpha,\beta}_{ij} = a_{ij}$ if $(i,j) \in \alpha \times \beta$.

Hence the difference $(A\,x)_i - \sum_{\beta=1}^p (x^\alpha)^t A^{\alpha,\beta} x^\beta$ represents the *relative fitness* of a strategy $i \in \alpha$ within the group α. The polymatrix replicator equation is the o.d.e.

$$\frac{dx^\alpha_i}{dt} = x^\alpha_i \left((A\,x)_i - \sum_{\beta=1}^p (x^\alpha)^t A^{\alpha,\beta} x^\beta \right) \quad \forall\, i \in \alpha,\ \alpha \in \{1,\ldots,p\}\,, \tag{3.3}$$

which once more says that the logarithmic growth rate of each pure strategy's frequency equals its relative fitness. The flow of this o.d.e. is complete and leaves the prism $\Gamma_{\underline{n}} = \Delta^{n_1-1} \times \ldots \times \Delta^{n_p-1}$ invariant. The underlying vector field on $\Gamma_{\underline{n}}$ will be denoted by X_A. The pair $G = (\underline{n}, A)$ will be referred as a *polymatrix game*, and the dynamical system determined by $X_A = X_{(\underline{n},A)}$ as the associated *polymatrix replicator equation* on $\Gamma_{\underline{n}}$.

Remark 3.1. *When $p = 1$, $\Gamma_{\underline{n}} = \Delta^{n-1}$ and the evolutionary polymatrix game* (3.3) *coincides with the replicator o.d.e.* (3.1).

Remark 3.2. *When $p = 2$ and $A^{1,1} = 0$, $A^{2,2} = 0$ system* (3.3) *coincides with the bimatrix replicator equation* (3.2) *on $\Gamma_{\underline{n}} = \Delta^{n_1-1} \times \Delta^{n_2-1}$. This case with non-zero diagonals was considered by Schuster et al* [21].

The proofs of the following three propositions are easy exercises.

Proposition 3.3 (Identity)**.** *The correspondence $A \mapsto X_{(\underline{n},A)}$ is linear and its kernel is formed by matrices $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that the block matrix $A^{\alpha,\beta}$ has equal rows for all $\alpha, \beta = 1, \ldots, p$. Thus, two matrices $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ determine the same vector field $X_{(\underline{n},A)} = X_{(\underline{n},B)}$ on $\Gamma_{\underline{n}}$ iff the block matrix $A^{\alpha,\beta} - B^{\alpha,\beta}$ has equal rows for all $\alpha, \beta = 1, \ldots, p$.*

Definition 3.4. Given a signature $\underline{n} = (n_1, \ldots, n_p)$ and matrices $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, we say that the polymatrix games $(\underline{n}, A)$ and $(\underline{n}, B)$ are equivalent, and write $(\underline{n}, A) \sim (\underline{n}, B)$, iff $A^{\alpha,\beta} - B^{\alpha,\beta}$ has equal rows for all $\alpha, \beta = 1, \ldots, p$.

Equivalent games determine the same polymatrix replicator equation on $\Gamma_{\underline{n}}$. In other words $(\underline{n}, A) \sim (\underline{n}, B)$ iff $X_{(\underline{n},A)} = X_{(\underline{n},B)}$. We denote by $\Gamma^\circ_{\underline{n}}$ the topological interior of $\Gamma_{\underline{n}}$ in the affine subspace of $\mathbb{R}^n$ spanned by $\Gamma_{\underline{n}}$.

Proposition 3.5 (Equilibria)**.** *A point $q \in \Gamma_{\underline{n}}$ is an equilibrium of $X_{(\underline{n},A)}$ if $(Aq)_i = (Aq)_j$, for all $\alpha = 1, \ldots, p$ and every $i, j \in \alpha$.*

Moreover, if $q \in \Gamma^\circ_{\underline{n}}$ is an equilibrium point then $(Aq)_i = (Aq)_j$, for all $\alpha = 1, \ldots, p$ and every $i, j \in \alpha$.

Definition 3.6. Given a signature $\underline{n} = (n_1, \ldots, n_p)$, we define the set

$$\mathcal{I}_{\underline{n}} := \{\, I \subset \{1, \ldots, n\} \,:\, \#(I \cap \alpha) \geq 1,\ \forall\, \alpha = 1, \ldots, p \,\}\,,$$

where $I \cap \alpha := I \cap [n_1 + \ldots + n_{\alpha-1} + 1, n_1 + \ldots + n_\alpha]$. A set $I \in \mathcal{I}_{\underline{n}}$ determines the face $\sigma_I := \{\, x \in \Gamma_{\underline{n}} \,:\, x_j = 0,\ \forall\, j \notin I \,\}$ of $\Gamma_{\underline{n}}$.

The correspondence between sets in $\mathcal{I}_{\underline{n}}$ and faces of $\Gamma_{\underline{n}}$ is bijective.

Definition 3.7. Consider a polymatrix game $G = (\underline{n}, A)$. Given a set $I \in \mathcal{I}_{\underline{n}}$ the pair $G|_I = (\underline{n}^I, A_I)$, where $\underline{n}^I = (n^I_1, \ldots, n^I_p)$ with $n^I_\alpha = \#(I \cap \alpha)$, and $A_I = [\, a_{ij}]_{i,j\in I}$, is called the *restriction* of the polymatrix game G to the face I.

The following proposition says that the restriction of a polymatrix replicator system to a face is another polymatrix replicator system.

Proposition 3.8 (Inheritance). *Consider the system* (3.3) *associated to the polymatrix game $G = (\underline{n}, A)$. Given $I \in \mathfrak{I}_{\underline{n}}$, the face σ_I of $\Gamma_{\underline{n}}$ is invariant under the flow of $X_{(\underline{n},A)}$ and the restriction of* (3.3) *to σ_I is the polymatrix replicator system associated to the restricted game $G|_I$.*

We set some notation in order to produce neater formulas. In any matrix equality the vectors in $\mathbb{R}^n$, or $\mathbb{R}^{n_\alpha}$, should be identified with column vectors. We set $\mathbb{1} = \mathbb{1}_n = (1, 1, .., 1)^t \in \mathbb{R}^n$ and will omit the subscript n whenever the dimension of this vector is clear from the context. Similarly, we write $I = I_n$ for the $n \times n$ identity matrix, and we omit the subscript n whenever its value is clear. Given $x \in \mathbb{R}^n$, we denote by D_x the $n \times n$ diagonal matrix $D_x = \mathrm{diag}(x_i)_i$. For each $\alpha \in \{1, \ldots, p\}$ we define the $n_\alpha \times n_\alpha$ matrix

$$T_x^\alpha := x^\alpha \, \mathbb{1}^t - I \ ,$$

and set T_x to be the $n \times n$ block diagonal matrix $T_x = \mathrm{diag}(T_x^\alpha)_\alpha$.

Given a polymatrix game $G = (\underline{n}, A)$, we define the matrix valued mapping $\pi_A : \mathbb{R}^n \to \mathrm{Mat}_{n\times n}(\mathbb{R})$

$$\pi_A(x) := (-1)\, T_x\, D_x\, A\, D_x\, T_x^t. \tag{3.4}$$

We have $D_x\, A\, D_x = \left[\, D_{x^\alpha}\, A^{\alpha,\beta}\, D_{x^\beta} \right]_{\alpha,\beta}$ where $D_{x^\alpha}\, A^{\alpha,\beta}\, D_{x^\beta} = \left[a_{ij}^{\alpha,\beta} x_i^\alpha x_j^\beta \right]_{i\in\alpha, j\in\beta}$. Simple calculations show that $\pi_A(x) = \left[\, \pi_{A,ij}(x) \right]_{i,j}$ where for all $(i,j) \in \alpha \times \beta$

$$\pi_{A,ij}(x) = x_i^\alpha x_j^\beta \left(-a_{ij}^{\alpha,\beta} + (A^{\alpha,\beta} x^\beta)_i + ((A^{\alpha,\beta})^t\, x^\alpha)_j - (x^\alpha)^t A^{\alpha,\beta} x^\beta \right). \tag{3.5}$$

These computations reduce to the simple case $p = 1,\ n_1 = n$ where

$$\pi_A(x) = (-1)\,(x\, \mathbb{1}^t - I)\, D_x\, A\, D_x\, (\mathbb{1}\, x^t - I) \ ,$$

and

$$\pi_{A,ij}(x) = x_i x_j (-a_{ij} + (Ax)_i + (A^t x)_j - x^t A x) \ .$$

Remark 3.9. *Notice that π_A is a skew symmetric matrix valued map whenever A is a skew symmetric matrix.*

Definition 3.10. A *formal equilibrium* of a polymatrix game $G = (\underline{n}, A)$ is any vector $q \in \mathbb{R}^n$ such that

(a) $(A\, q)_i = (A\, q)_j$ for all $i, j \in \alpha$, and all $\alpha = 1, \ldots, p$,
(b) $\sum_{j\in\alpha} q_j = 1$ for all $\alpha = 1, \ldots, p$.

Remark 3.11. A formal equilibrium of $G = (\underline{n}, A)$ is an equilibrium of the natural extension of $X_{(\underline{n},A)}$ to the affine subspace spanned by $\Gamma_{\underline{n}}$.

Next proposition says that the existence of a formal equilibrium is a sufficient condition for the vector field $X_{(\underline{n},A)}$ of system (3.3) to be a gradient of a simple function H with respect to π_A.

Proposition 3.12. *Given $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, assume there exists a formal equilibrium $q \in \mathbb{R}^n$ of $G = (\underline{n}, A)$. Then, setting $H(x) = \sum_{i=1}^n q_i \log x_i$,*

$$X_{(\underline{n},A)}(x) = \pi_A(x)\, \mathrm{d}_x H \quad \textit{for every} \quad x \in \Gamma^\circ_{\underline{n}} \,.$$

Proof. Consider the vector field $Z = \pi_A \, \mathrm{d}H$. For any α, and $i \in \alpha$, denote by $Z_i^\alpha(x)$ the i-th component of $Z(x)$. Using that $\sum_{j\in\beta} q_j^\beta = 1$ we have

$$\begin{aligned}
Z_i^\alpha(x) &= \left(\sum_{\beta=1}^k \pi_A^{\alpha,\beta}(x)\,\frac{q^\beta}{x^\beta}\right)_i = \sum_{\beta=1}^k \left(\sum_{j\in\beta} \pi_{A,ij}^{\alpha,\beta}(x)\,\frac{q_j^\beta}{x_j^\beta}\right)_i \\
&= x_i^\alpha \sum_{\beta=1}^k \left[((A^{\alpha,\beta}x^\beta)_i - (x^\alpha)^t A^{\alpha,\beta}x^\beta)(\sum_{j\in\beta} q_j^\beta) + (-A^{\alpha,\beta}q^\beta)_i + (x^\alpha)^t A^{\alpha,\beta} q^\beta\right] \\
&= x_i^\alpha \left[(Ax)_i - \sum_{\beta=1}^k (x^\alpha)^t A^{\alpha,\beta} x^\beta + \overbrace{(-Aq)_i + \sum_{i\in\alpha} x_i^\alpha (Aq)_i}^{=0}\right] \\
&= x_i^\alpha \left[(Ax)_i - \sum_{\beta=1}^k (x^\alpha)^t A^{\alpha,\beta} x^\beta\right] = X_{A,ij}^\alpha(x)\,,
\end{aligned}$$

where the vanishing term follows from q being an equilibrium point and $x^\alpha \in \Delta^{n_\alpha - 1}$. This completes the proof. $\square$

For every $\alpha = 1, \ldots, p$ consider the $(n_\alpha - 1) \times n_\alpha$ matrix

$$E_\alpha := \begin{bmatrix} -1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix}$$

and set

$$\begin{aligned} E &:= \mathrm{diag}(E_1, \ldots, E_p), \\ B &:= (-1) E A E^t \,. \end{aligned} \tag{3.6}$$

Note that $E \in \mathrm{Mat}_{(n-p)\times n}(\mathbb{R})$ and $B \in \mathrm{Mat}_{(n-p)\times(n-p)}(\mathbb{R})$. Next we introduce a mapping $\phi : \mathbb{R}^{n-p} \to \Gamma^\circ_{\underline{n}}$. We write a vector $u \in \mathbb{R}^{n-p} = \mathbb{R}^{n_1 - 1} \times \ldots \times \mathbb{R}^{n_p - 1}$ as $u = (u^\alpha)_\alpha$, where $u^\alpha := (u_1^\alpha, \ldots, u_{n_\alpha - 1}^\alpha)$, and the components of ϕ as $\phi(u^\alpha)_\alpha :=$

$(\phi^\alpha(u^\alpha))_\alpha$, where each $\phi^\alpha : \mathbb{R}^{n_\alpha-1} \to (\Delta^{n_\alpha-1})^\circ$ is the map defined by

$$\phi^\alpha(u^\alpha) := \left(\frac{e^{u_1^\alpha}}{1+\sum_{i=1}^{n_\alpha-1} e^{u_i^\alpha}}, \ldots, \frac{e^{u_{n_\alpha-1}^\alpha}}{1+\sum_{i=1}^{n_\alpha-1} e^{u_i^\alpha}}, \frac{1}{1+\sum_{i=1}^{n_\alpha-1} e^{u_i^\alpha}} \right) .$$

The following is our main result. Consider a polymatrix game $G = (\underline{n}, A)$.

Theorem 3.13. *If A is skew symmetric then the mapping π_A in* (3.4) *defines a Poisson structure on $\Gamma_{\underline{n}}$. Moreover the mapping $\phi : \mathbb{R}^{n-1} \to \Gamma_{\underline{n}}^\circ$ is a Poisson diffeomorphism if we endow $\mathbb{R}^{n-p}$ with the constant Poisson structure associated to the skew symmetric matrix B defined in* (3.6).

Proof. The map $\phi : \mathbb{R}^{n-1} \to \Gamma_{\underline{n}}^\circ$ is a diffeomorphism whose inverse is easily computed. If A is skew symmetric then so is B. Hence this matrix induces a constant Poisson structure on $\mathbb{R}^{n-p}$. We want to prove that π_A determines a Poisson structure on $\Gamma_{\underline{n}}^\circ$ which makes ϕ a Poisson map. By (2.3) we just need to show that for every $u \in \mathbb{R}^{n-p}$ and $x = \phi(u)$,

$$(\mathrm{d}_u\phi)B(\mathrm{d}_u\phi)^t = (-1)T_x D_x A D_x T_x^t = \pi_A(x) . \tag{3.7}$$

The fact that π_A also determines a Poisson structure on $\Gamma_{\underline{n}}$, and on $\mathbb{R}^n$, will be proved later. See Remark 3.15. In order to prove (3.7), it is enough to see that for every $x = \phi(u)$

$$(\mathrm{d}_u\phi)E = T_x D_x .$$

Writting the components of ϕ^α as $\phi^\alpha(u^\alpha) = (\phi_1^\alpha(u^\alpha), \ldots, \phi_{n_\alpha}^\alpha(u^\alpha))$ we compute for every $i = 1, \ldots, n_\alpha$ and $j = 1, \ldots, n_\alpha - 1$,

$$\frac{\partial \phi_i^\alpha}{\partial u_j^\alpha} = \delta_{ij}\, \phi_i^\alpha - \phi_i^\alpha\, \phi_j^\alpha .$$

Hence if $x = \phi(u)$, the Jacobian of ϕ at the point u is

$$\mathrm{d}_u\phi = \mathrm{diag}(J_\alpha(x^\alpha))_\alpha,$$

where for every $\alpha = 1, \ldots, p$,

$$J_\alpha(x_1, \ldots, x_{n_\alpha}) := \begin{bmatrix} x_1 - x_1^2 & -x_1\, x_2 & -x_1\, x_3 & \ldots & -x_1\, x_{n_\alpha-1} \\ -x_2\, x_1 & x_2 - x_2^2 & -x_2\, x_3 & \ldots & -x_2\, x_{n_\alpha-1} \\ -x_3\, x_1 & -x_3\, x_2 & x_3 - x_3^2 & \ldots & -x_3\, x_{n_\alpha-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{n_\alpha-1}\, x_1 & -x_{n_\alpha-1}\, x_2 & -x_{n_\alpha-1}\, x_3 & \ldots & x_{n_\alpha-1} - x_{n_\alpha-1}^2 \\ -x_{n_\alpha}\, x_1 & -x_{n_\alpha}\, x_2 & -x_{n_\alpha}\, x_3 & \ldots & -x_{n_\alpha}\, x_{n_\alpha-1} \end{bmatrix}$$

A simple multiplication of matrices, using the relation $x_1 + \ldots + x_{n_\alpha} = 1$, shows that $J_\alpha(x^\alpha)E_\alpha = T_{x^\alpha} D_{x^\alpha}$ for every $\alpha = 1, \ldots, p$. Therefore

$$(\mathrm{d}_u\phi)E = \mathrm{diag}(J_\alpha(x^\alpha)E_\alpha)_\alpha = \mathrm{diag}(T_{x^\alpha} D_{x^\alpha})_\alpha = T_x D_x ,$$

which completes the proof. □

The next corollary gives a complete description of the symplectic foliation of $(\Gamma^\circ_{\underline{n}}, \pi_A)$. Two examples will be given in section 4.

Corollary 3.14 (Symplectic Foliation)**.** *The symplectic leaves of* $(\Gamma^\circ_{\underline{n}}, \pi_A)$ *are the images of the symplectic leaves of* $(\mathbb{R}^{n-p}, B)$ *under the diffeomorphism* ϕ*. The symplectic leaf* S_u *of* $(\mathbb{R}^{n-p}, B)$ *is the (even dimensional) affine subspace through* u *parallel to the subspace generated by the columns of* B*.*

Remark 3.15. Given a face $I \in \mathfrak{I}_{\underline{n}}$ consider the payoff matrix A_I, see Definition 3.7. Applying Theorem 3.13 to any face σ_I of $\Gamma_{\underline{n}}$ we see that σ_I is a Poisson manifold on its own with the Poisson structure π_{A_I}. Moreover (σ_I, π_{A_I}) is the restriction of $(\Gamma_{\underline{n}}, \pi_A)$ in the sense that the inclusion map $i : \sigma_I \to \Gamma_{\underline{n}}$ is a Poisson map. Hence the interiors of the faces of $\Gamma_{\underline{n}}$, regarded as Poisson manifolds, give $(\Gamma_{\underline{n}}, \pi_A)$ the structure of a Poisson stratified space. The proof of this fact is a simple adaptation of example 2.5 in [5].

Proposition (3.12) together with Theorem (3.13) yields the following corollary.

Corollary 3.16. *If* A *is skew symmetric and* $q \in \mathbb{R}^n$ *is a formal equilibrium of* $G = (\underline{n}, A)$ *then* $X_{(\underline{n},A)}$ *is a Hamiltonian vector field, with Hamiltonian* $H(x) = \sum_{i=1}^n q_i \log x_i$*, w.r.t. the Poisson structure* π_A *in* $\Gamma^\circ_{\underline{n}}$*.*

Definition 3.17. A polymatrix game $G = (\underline{n}, A)$ is said to be *conservative* iff
- (a) G has a formal equilibrium,
- (b) there are matrices $A_0, D \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ such that
  - (i) $A \sim A_0 D$,
  - (ii) A_0 is a skew symmetric,
  - (iii) $D = \mathrm{diag}(\lambda_1 I_{n_1}, \ldots, \lambda_p I_{n_p})$ with $\lambda_\beta \neq 0$ for every $\beta = 1, \ldots, p$.

The matrix A_0 will be referred as a *skew symmetric model* for G, and $(\lambda_1, \ldots, \lambda_p) \in (\mathbb{R}^*)^p$ as a *scaling co-vector.*

Remark 3.18. Given a skew symmetric matrix $A_0 \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, a signature $\underline{n}$ and a point $\tilde{q} \in \mathbb{R}^n$ such that
- (a) $(A_0\,\tilde{q})_i = (A_0\,\tilde{q})_j$ for all $i, j \in \alpha$, and all $\alpha = 1, \ldots, p$,
- (b) $\sum_{j\in\alpha} \tilde{q}_j \neq 0$ for all $\alpha = 1, \ldots, p$,

then $G = (\underline{n}, A_0 D)$ is a conservative polymatrix game, where $D = \mathrm{diag}(\lambda_\alpha I_{n_\alpha})_\alpha$ with $\lambda_\alpha := \sum_{j\in\alpha} \tilde{q}_j$, and $q = D^{-1}\tilde{q}$ is a formal equilibrium of G.

It follows from the previous remark that any generic skew symmetric matrix can be taken as a model for a conservative polymatrix game. More precisely,

Proposition 3.19. *Given a signature* $\underline{n} = (n_1, \ldots, n_p)$ *with* $\sum_{\alpha=1}^p n_\alpha = n$*, the set of skew symmetric matrices* $A_0 \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ *such that* $G = (\underline{n}, A_0\,D)$ *is a conservative polymatrix game for some diagonal matrix* D *is an open and dense subset of the space of skew symmetric matrices.*

Next theorem basically says that the replicator system (3.3) is Hamiltonian for every conservative polymatrix game.

Theorem 3.20. *Consider a conservative polymatrix game $G = (\underline{n}, A)$ with formal equilibrium q, skew symmetric model A_0 and scaling co-vector $(\lambda_1, \ldots, \lambda_p)$. Then $X_{(\underline{n},A)}$ is Hamiltonian in the interior of the Poisson stratified space $(\Gamma_{\underline{n}}, \pi_{A_0})$, with Hamiltonian function*

$$H(x) = \sum_{\beta=1}^{p} \lambda_\beta \sum_{j\in\beta} q_j^\beta \log x_j^\beta \ . \tag{3.8}$$

Proof. In view of definition 3.4 we can assume that $A = A_0 D$. For every α, β,

$$T_x^\alpha\, D_{x^\alpha}\, A_0^{\alpha,\beta}\, D_{x^\beta}\, (T_x^\beta)^t \lambda_\beta\, \frac{q^\beta}{x^\beta} = T_x^\alpha\, D_{x^\alpha}\, A^{\alpha,\beta}\, D_{x^\beta}\, (T^\beta)^t\, \frac{q^\beta}{x^\beta} \ ,$$

where q^β / x^β stands for the componentwise division of the vectors. Adding up in β, and using Proposition 3.12, we get

$$\pi_{A_0}(x)\, \mathrm{d}_x H = \pi_A(x)\, \mathrm{d}_x \left(\sum_{j=1}^{n} q_j\, \log x_j \right) = X_{(\underline{n},A)}(x) \ .$$

□

In the next paragraphs we compare our results with previously known facts. Given a skew symmetric matrix $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, since $x^t\, A\, x = 0$ for all $x \in \mathbb{R}^n$, the replicator equation (3.1) reduces to a Lotka-Volterra equation with growth rates $r_i = 0$

$$\frac{dx_i}{dt} = x_i\, (A\, x)_i \quad 1 \leq i \leq n \ . \tag{3.9}$$

For any $q \in \mathbb{R}^n$ such that $A\, q = 0$ the function $H(x) = \sum_{j=1}^n x_j - q_j \log x_j$ is a constant of motion for (3.9). The Hamiltonian character of these models was first studied in [1] using symplectic formalism instead of Poisson structures. The authors prove that these systems are Hamiltonian in the symplectic leaves described in corollary 3.14, with respect to the symplectic structures induced by the Poisson structure π_A. A Poisson structure on $\mathbb{R}^n$ defined by the bivector $\hat{\pi}_A(x) = D_x\, A\, D_x$ was introduced in [2]. System (3.9) is Hamiltonian in the interior of $\mathbb{R}^n_+$ w.r.t. $\hat{\pi}_A$ having H as Hamiltonian function. Like $\hat{\pi}_A$ the Poisson structure π_A introduced here can be extended to $\mathbb{R}^n$, but unlike π_A the structure $\hat{\pi}_A$ does not restrict to a Poisson structure on the simplex Δ^{n-1}. Using the Poisson structure π_A we can now say, if there exists $q \in \mathbb{R}^n$ such that $A\, q = 0$ and $\sum_{j=1}^n q_j \neq 0$, that the system (3.9) is Hamiltonian in the interior of the simplex Δ^{n-1}. Furthermore, here we study the replicator equation itself, not topologically equivalent LV systems which have non-compact domains. Our approach makes it possible to extend the results regarding symmetric and asymmetric games to general polymatrix games.

Consider now a bimatrix game with signature (n_1, n_2) and matrix

$$A = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} .$$

If $\lambda > 0$, resp. $\lambda < 0$, the polymatrix game $((n_1, n_2), A)$ is conservative with scaling vector $(1, \lambda)$ if and only if it has a formal equilibrium and the bimatrix game (A_{12}, A_{21}) is λ-zero-sum game, resp. λ-partnership game, (see definitions in Section 11.2 of [8]). Theorem 3.20 generalizes the main result in [6, Section 5], which says that the evolutionary system (3.2) associated to a λ-zero-sum or λ-partnership game is Hamiltonian w.r.t. some Poisson structure in the interior of the prism. We obtain here the same constant of motion (3.8), and the same Poisson structure in the interior of $\Delta^{n_1-1} \times \Delta^{n_2-1}$ that was obtained in [6].

We finish this section with an extension of the class of Hamiltonian polymatrix replicator equations. Given p smooth functions $\lambda_\alpha : \Gamma_{\underline{n}} \to \mathbb{R}^*$, $\alpha = 1, \ldots, p$, consider the matrix valued smooth function $D : \Gamma_{\underline{n}} \to \mathrm{Mat}_{n\times n}(\mathbb{R})$, $D(x) = \mathrm{diag}(\lambda_\alpha(x) I_{n_\alpha})_\alpha$, and the system of o.d.e.'s

$$\frac{dx_i^\alpha}{dt} = x_i^\alpha \left((AD(x)\, x)_i - \sum_{\beta=1}^{p} \lambda_\beta(x)\, (x^\alpha)^t A^{\alpha,\beta} x^\beta \right) \quad \forall\, i \in \alpha,\ 1 \leq \alpha \leq p \qquad (3.10)$$

associated with the vector field $Y(x) = X_{(\underline{n}, AD(x))}(x)$ on $\Gamma_{\underline{n}}$.

Proposition 3.21. *Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ be a skew symmetric matrix, $q \in \mathbb{R}^n$ a formal equilibrium of $G = (\underline{n}, A)$, and consider the 1-form*

$$\xi(x) = \sum_{\alpha=1}^{p} \sum_{j\in\alpha} \lambda_\alpha(x)\, q_j^\alpha \, \frac{dx_j^\alpha}{x_j^\alpha} .$$

Then system (3.10) is the gradient of the 1-form ξ w.r.t. the Poisson structure π_A in the interior of $\Gamma_{\underline{n}}$, i.e.,

$$Y(x) = \pi_A(x)\, \xi(x) .$$

System (3.10) is Hamiltonian if the form ξ is exact, i.e., there exists a smooth function H such that $\xi = \mathrm{d}H$. But even if ξ is not exact, the dynamics of Y leaves invariant the symplectic foliation of $(\Gamma^\circ_{\underline{n}}, \pi_A)$.

Proof. The proof is similar to that of Theorem 3.20. □

The previous model (3.10) contains the following class of o.d.e.'s introduced by J. Maynard Smith as an extension of the asymmetric replicator equation (3.2).

$$\begin{aligned} \frac{dx_i}{dt} &= x_i \, \big((A_{12}\, y)_i - x^t\, A_{12}\, y \big)\; m_1(x,y) \quad 1 \leq i \leq n \\ \frac{dy_j}{dt} &= y_j \, \big((A_{21}\, x)_j - y^t\, A_{21}\, x \big)\; m_2(x,y) \quad 1 \leq j \leq m \end{aligned} \qquad (3.11)$$

See appendix J of [10], and system (9.1) in [6]. Taking

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix} \quad \text{and} \quad D(x) = \begin{bmatrix} m_2(x,y)\, I_m & 0 \\ 0 & m_1(x,y)\, I_n \end{bmatrix}$$

system (3.11) reduces to (3.10). Since system (3.11) has a dissipative character for certain choices of the functions $m_1(x,y)$ and $m_2(x,y)$ it would be interesting to investigate analogous properties of system (3.10).

4. Examples

It is possible to fully classify the dynamics of 2D and 3D conservative polymatrix replicator systems, but in this section we just briefly describe two examples of 3D polymatrix replicator systems.

First Example. Consider the signature $\underline{n} = (2,2,2)$, take the skew symmetric matrix

$$A_0 = \begin{bmatrix} 0 & -1 & 0 & \frac{1}{2} & 0 & 1 \\ 1 & 0 & 0 & -\frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -1 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 \end{bmatrix},$$

and the point $p = \left(\frac{7}{4}, \frac{3}{4}, \frac{5}{4}, 1, 1, 1\right)$ such that $A_0\, p = \left(\frac{3}{4}, \frac{3}{4}, -\frac{1}{2}, -\frac{1}{2}, -\frac{3}{8}, -\frac{3}{8}\right)$. Consider the matrix $A = A_0\, D$, where $D = \operatorname{diag}\left(\frac{5}{2}, \frac{5}{2}, \frac{9}{4}, \frac{9}{4}, 2, 2\right)$. This matrix is

$$A = \begin{bmatrix} 0 & -5/2 & 0 & 9/8 & 0 & 2 \\ 5/2 & 0 & 0 & -9/8 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ -5/4 & 5/4 & 0 & 0 & 0 & 0 \\ 0 & 5/2 & -9/8 & 0 & 0 & -1 \\ -5/2 & -5/4 & 9/4 & 0 & 1 & 0 \end{bmatrix}.$$

By remark 3.18 $((2,2,2), A)$ is a conservative polymatrix game. The phase space of the associated replicator system is the cube

$$\Gamma_{(2,2,2)} = \Delta^1 \times \Delta^1 \times \Delta^1 \equiv [0,1]^3 .$$

In the model $[0,1]^3$, the equilibrium point $q = D^{-1}p$ has coordinates $q = \left(\frac{7}{10}, \frac{5}{9}, \frac{1}{2}\right)$, and hence is an interior point. The line through q with direction $v = \left(\frac{6}{5}, -\frac{4}{9}, -1\right)$ intersects the cube $[0,1]^3$ along the set Σ of equilibria of this replicator system. This set Σ is a line segment joinning two points in the faces $\{x = 1\}$ and $\{z = 1\}$. To compute the symplectic foliation of $]0,1[^3$ consider the matrix

$$E = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

and define $B = -E\,A_0 E^t$. A simple calculation gives

$$B = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & 0 & -\frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & 0 \end{bmatrix} .$$

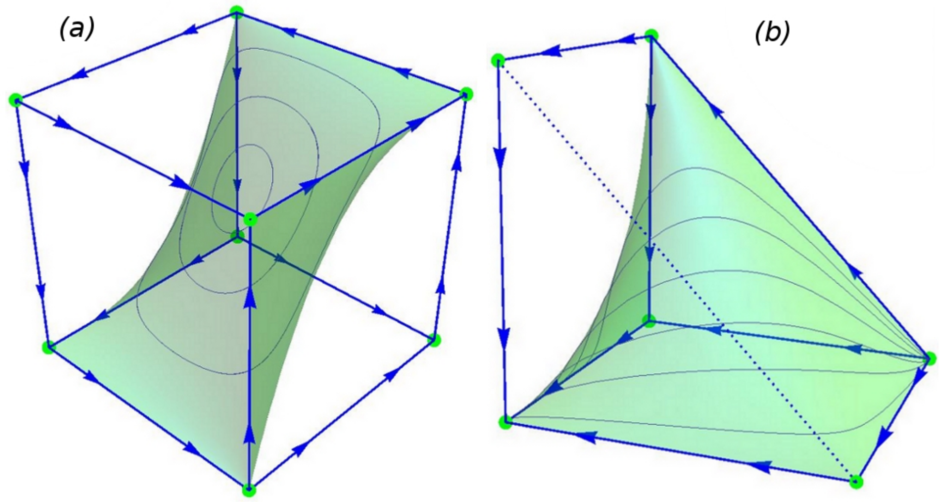

FIGURE 1. Phase portraits of 3D polymatrix replicator systems

The vector $w = \left(-\frac{3}{2}, \frac{1}{2}, 1\right)$ is orthogonal to the space spanned by the columns of B. The symplectic leaves of the constant Poisson structure on $\mathbb{R}^3$ defined by the skew symmetric matrix B are the planes orthogonal to w. Thus, if we consider the Poisson diffeomorphism $\phi : \mathbb{R}^3 \to]0,1[^3$,

$$\phi(u_1, u_2, u_3) = \left(\frac{e^{u_1}}{1+e^{u_1}}, \frac{e^{u_2}}{1+e^{u_2}}, \frac{e^{u_3}}{1+e^{u_3}} \right) ,$$

the symplectic leaves on $]0,1[^3$ are the ϕ images of these planes. Inverting the map ϕ, the symplectic leaves are given by the equations

$$\left(\frac{x}{1-x}\right)^{-3/2} \left(\frac{y}{1-y}\right)^{1/2} \left(\frac{z}{1-z}\right) = e^c$$
$$\Leftrightarrow \quad (1-x)^{3/2} y^{1/2} z = e^c x^{3/2} (1-y)^{1/2} (1-z) ,$$

with $c \in \mathbb{R}$. Let U_+, resp. U_-, be the union of the faces $\{x = 1\}$, $\{y = 0\}$, $\{z = 0\}$, resp. $\{x = 0\}$, $\{y = 1\}$, $\{z = 1\}$. On the interiors of these two open subsets of the cube's boundary the equation above is never satisfied. Therefore the closure of every symplectic leaf intersects the cube's boundary along the closed curve $C = \partial U_+ = \partial U_- \subset \partial[0,1]^3$. Because Σ intersects both U_- and U_+, it follows that every symplectic leaf must intersect Σ, hence having a unique equilibrium. The orbits of our polymatrix replicator system foliate each symplectic leaf into closed curves around that equilibrium point. We can also check that C is a heteroclinic cycle of the vector field $X_{(2,2,2),A}$. See Figure 1(a).

Second Example. Consider the signature $\underline{n} = (3,2)$, take the skew symmetric matrix

$$A_0 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -1 & 0 & 0 \\ 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix},$$

and the point $p = \left(\frac{9}{10}, -\frac{8}{5}, \frac{1}{2}, 0, 1\right)$ such that $A_0\, p = \left(-\frac{3}{4}, -\frac{3}{4}, -\frac{3}{4}, -\frac{3}{20}, -\frac{3}{20}\right)$. Consider the matrix $A = A_0\, D$, where $D = \operatorname{diag}\left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, 1, 1\right)$. This matrix is

$$A = \begin{bmatrix} 0 & 0 & -\frac{1}{10} & \frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{10} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{10} & -\frac{1}{10} & 0 & 1 & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{10} & \frac{1}{5} & 0 & 0 \\ -\frac{1}{5} & -\frac{1}{10} & \frac{1}{10} & 0 & 0 \end{bmatrix}.$$

By remark 3.18 $((3,2), A)$ is a conservative polymatrix game. The phase space of the associated replicator system is the prism

$$\Gamma_{(3,2)} = \Delta^2 \times \Delta^1 \equiv \{\, (x,y,z) \,:\, 0 \le x,y,z \le 1,\ x+y \le 1 \,\} =: P\,.$$

In the model $P \subset \mathbb{R}^3$ the equilibrium point $q = D^{-1}p$ has coordinates $q = \left(-\frac{9}{2}, 8, 0\right)$, and hence is not interior to P. The line of equilibria goes through q with direction $v = \left(-\frac{5}{2}, 5, -1\right)$ and does not intersect the prism P . To compute the symplectic foliation of P° consider the matrix

$$E = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

and define $B = -E\, A_0 E^t$. A simple calculation gives

$$B = \begin{bmatrix} 0 & 1 & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & 1 & 0 \end{bmatrix}.$$

The vector $w = \left(-1, \frac{1}{2}, 1\right)$ is orthogonal to the space spanned by the columns of B. The symplectic leaves of the constant Poisson structure on $\mathbb{R}^3$ defined by the skew symmetric matrix B are the planes orthogonal to w. Thus, if we consider the Poisson diffeomorphism $\phi : \mathbb{R}^3 \to P^\circ$,

$$\phi(u_1,u_2,u_3) = \left(\frac{e^{u_1}}{1+e^{u_1}+e^{u_2}}, \frac{e^{u_2}}{1+e^{u_1}+e^{u_2}}, \frac{e^{u_3}}{1+e^{u_3}} \right),$$

the symplectic leaves on P° are the ϕ images of these planes. Inverting the map ϕ, the symplectic leaves are given by the equations

$$\left(\frac{x}{1-x-y}\right)^{-1}\left(\frac{y}{1-x-y}\right)^{1/2}\left(\frac{z}{1-z}\right) = e^c$$
$$\Leftrightarrow \quad (1-x-y)^{1/2}y^{1/2}z = e^c x(1-z)\ ,$$

with $c \in \mathbb{R}$. Let U_+, resp. U_-, be the union of the faces $\{x+y=1\}$, $\{y=0\}$, $\{z=0\}$, resp. $\{x=0\}$, $\{z=1\}$. On the interiors of these two open subsets of the prism's boundary the equation above is never satisfied. Therefore the closure of every symplectic leaf intersects the prism's boundary along the closed curve $C = \partial U_+ = \partial U_- \subset \partial P$. The points $r = (1,0,0)$ and $s = (0,0,1)$ on C are respectively a global repeller and a global sink of the polymatrix replicator system, and every symplectic leaf is foliated into orbits flowing from the repeller r to the sink s. The closed curve C is also the union of two heteroclinic chains from r to s. See Figure 1(b). Note that this dynamical behaviour does not contradict the Hamiltonian character of the system because the area of each symplectic leaf is infinite.

Acknowledgements

Both authors would like to thank Rui Loja Fernandes for valuable suggestions and in particular for pointing them Example 2.5 in [5], and also Telmo Peixe for proof reading the manuscript.

The first author was supported by the Geometry and Mathematical Physics Project, FCT EXCL/MAT-GEO/0222/2012. The second author was supported by "Fundação para a Ciência e a Tecnologia" through the Program POCI 2010 and the Project "Randomness in Deterministic Dynamical Systems and Applications" (PTDC-MAT-105448-2008).

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