Birkhoff and Kingman's Theorems

Abstract

Proofs of Birkhoff and Kingman ergodic theorems based on the article [1] by Y. Katznelson and B. Weiss.

1 Notation

We will use the notation $f^+(x) := \max\{f(x), 0\}$ and $f^-(x) := \max\{-f(x), 0\}$, so that the following relations hold

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$

2 Birkhoffs Ergodic Theorem

Given $f : X \to \mathbb{R}$ we write

$$S_n(f)(x) := \sum_{j=0}^{n-1} f(T^j x).$$

This sum is called *Birkhoff's time average* of the observable f.

Theorem 1 (BET)**.** *Let $(T, X; \mathcal{F}, \mu)$ be a MPDS. Given $f \in L^1(X, \mu)$, the following limit exists for μ-a.e. $x \in X$*

$$f^*(x) = \lim_{n\to+\infty} \frac{1}{n} S_n(f)(x).$$

Moreover the limit function $f^* : X \to \mathbb{R}$ satisfies:

(a) $f^* \in L^1(X, \mu)$,

(b) $f^* \circ T = f^*$ *μ-a.e.,*

(c) $\int_X f^* \, d\mu = \int_X f \, d\mu$.

Exercise 1. *Show that it is enough to prove the BET for $f \geq 0$, $f \in L^1(X, \mu)$.*
Hint: *Using the decomposition $f = f^+ - f^-$, see that $f^* = (f^+)^* - (f^-)^*$ and also $S_n(f) = S_n(f^+) - S_n(f^-)$.*

Given $f : X \to \mathbb{R}$ we define $\underline{f}, \bar{f} : X \to [-\infty, +\infty]$,

$$\underline{f}(x) := \liminf_{n\to+\infty} \frac{1}{n} S_n(f)(x),$$
$$\bar{f}(x) := \limsup_{n\to+\infty} \frac{1}{n} S_n(f)(x).$$

Exercise 2. *Prove that for any measurable function $f : X \to \mathbb{R}$, the functions $\underline{f}$ and $\bar{f}$ are measurable and T-invariant.*

Remark 1. *Under the assumptions of the BET if one can prove that*

$$\int_X \bar{f}\, d\mu \leq \int_X f\, d\mu \leq \int_X \underline{f}\, d\mu \tag{1}$$

then all conclusions of the BET follow.

Given $M > 0$ we define the *M-truncation* of $f : X \to \mathbb{R}$ to be the function

$$f_M : X \to \mathbb{R}, \quad f_M(x) := \min\{f(x), M\}.$$

Exercise 3. *Prove that the following monotonic convergences hold for every $x \in X$*

1. $f_M(x) \nearrow f(x)$ *as* $M \to +\infty$,
2. $\overline{f_M}(x) \nearrow \bar{f}(x)$ *as* $M \to +\infty$,
3. $\underline{f_M}(x) \nearrow \underline{f}(x)$ *as* $M \to +\infty$.

Exercise 4. *Show it is enough to prove* (1) *for f non-negative and bounded measurable functions. Conclude it is enough to consider functions such that $0 \leq f \leq 1$.*
Hint: *Use exercise 3 and the monotone convergence theorem.*

Proof of the BET. Let $f : X \to \mathbb{R}$ be a measurable function such that $0 \leq f \leq 1$, and take $\varepsilon > 0$.

By Remark 1 it is enough to see that

(a) $\int \bar{f}\, d\mu \leq \int f\, d\mu + 3\varepsilon$, and

(b) $\int f\, d\mu \leq \int \underline{f}\, d\mu + 3\varepsilon$.

To prove (a) define $n\colon X \to \mathbb{N}$,

$$n(x) := \min\left\{ n \geq 1\colon \frac{1}{n} S_n(f)(x) \geq \bar{f}(x) - \varepsilon \right\}.$$

Since $0 \leq \underline{f} \leq \bar{f} \leq 1$, the function $n(x)$ takes finite values everywhere. By definition

$$\bar{f}(x) \leq \frac{1}{n(x)} S_{n(x)}(f)(x) + \varepsilon. \tag{2}$$

By invariance of $\bar{f}$,

$$\bar{f}(T^j x) \leq \frac{1}{n(x)} S_{n(x)}(f)(x) + \varepsilon.$$

Adding up these inequalities in $j = 0, 1, \ldots, n(x) - 1$ we get

$$S_{n(x)}(\bar{f})(x) \leq S_{n(x)}(f)(x) + n(x)\varepsilon. \tag{3}$$

Consider now the sets $X_N := \{x \in X : n(x) \leq n\}$. Because $X = \cup_{N \geq 1} X_N \pmod 0$, for N large enough $\mu(X \setminus X_N) < \varepsilon$. Next we define the functions $\tilde{n} : X \to \mathbb{N}$

$$\tilde{n}(x) := \begin{cases} n(x) & \text{if} \quad x \in X_N \\ 1 & \text{if} \quad x \notin X_N \end{cases}$$

and $\tilde{f} : X \to \mathbb{R}$

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if} \quad x \in X_N \\ 1 & \text{if} \quad x \notin X_N \end{cases}$$

With this notation, (3) implies that

$$S_{\tilde{n}(x)}(\bar{f})(x) \leq S_{\tilde{n}(x)}(\tilde{f})(x) + \tilde{n}(x)\varepsilon. \tag{4}$$

Observe also that

$$\begin{aligned} \int \tilde{f}\, d\mu &\leq \int_{X_M} f\, d\mu + \int_{X \setminus X_M} 1\, d\mu \\ &\leq \int f\, d\mu + \mu(X \setminus X_M) \leq \int f\, d\mu + \varepsilon. \end{aligned} \tag{5}$$

The random variable $\tilde{n}(x)$ is referred as a *stopping time* in Probability Theory. Katznelson and Weiss idea is to split the orbit $\{T^n x\}_{n \geq 0}$ along the sequence of stopping times $\tilde{n}(x)$, $\tilde{n}(T^{n(x)} x)$, etc. By construction the distance between consecutive stopping times is bounded by N, while we have good bounds for the time averages of f between any two consecutive stopping times. More precisely, define recursively

$$\begin{cases} n_0(x) := 0 \\ n_k(x) := n_{k-1}(x) + \tilde{n}(T^{n_{k-1}(x)} x) \end{cases}.$$

Given $L > \frac{N}{\varepsilon}$, choose the largest $k = k(x) \in \mathbb{N}$ such that $n_k(x) \leq L$, so that in particular $L - n_k(x) < N$. From (4) we get

$$\begin{aligned} S_L(\bar{f})(x) &= \sum_{l=0}^{k-1} S_{\tilde{n}(T^{n_l} x)}(\bar{f})(T^{n_l} x) + S_{L - n_k}(\bar{f})(T^{n_k} x) \\ &\leq \sum_{l=0}^{k-1} S_{\tilde{n}(T^{n_l} x)}(\tilde{f})(T^{n_l} x) + S_{L - n_k}(\bar{f})(T^{n_k} x) + L\varepsilon \\ &\leq S_L(\tilde{f})(x) + N + L\varepsilon. \end{aligned}$$

Hence, dividing by L and integrating, from (5) we get

$$\begin{aligned}\int \bar{f}\, d\mu = \int S_L(\bar{f})\, d\mu &\leq \int S_L(\tilde{f})\, d\mu + \frac{N}{L} + \varepsilon. \\ &\leq \int \tilde{f}\, d\mu + \frac{N}{L} + \varepsilon \leq \int \tilde{f}\, d\mu + 2\varepsilon. \\ &\leq \int f\, d\mu + 3\varepsilon.\end{aligned}$$

This proves (a).

Exercise 5. *Prove claim (b) adapting the proof of (a).*

□

Exercise 6. *Prove the following extension of the BET: Given a measurable non-negative function $f\colon X \to [0, +\infty)$, the following limit exists for μ-a.e. $x \in X$*

$$f^*(x) = \lim_{n\to+\infty} \frac{1}{n} S_n(f)(x) \in [0, +\infty].$$

Moreover the limit function $f^* : X \to [0, +\infty]$ satisfies:

(a) $f^* \circ T = f^*$ *μ-a.e.,*

(b) $\int_X f^*\, d\mu = \int_X f\, d\mu$.

Hint: *Use exercise 3.*

Exercise 7. *Prove the following extension of the BET: Given a measurable function $f\colon X \to \mathbb{R}$ such that $f^+ \in L^1(X, \mu)$, the following limit exists for μ-a.e. $x \in X$*

$$f^*(x) = \lim_{n\to+\infty} \frac{1}{n} S_n(f)(x) \in [-\infty, +\infty).$$

Moreover the limit function $f^* : X \to [-\infty, +\infty)$ satisfies:

(a) $f^* \circ T = f^*$ *μ-a.e.,*

(b) $\int_X f^*\, d\mu = \int_X f\, d\mu$.

Hint: *Use exercise 6.*

3 Kingman's Ergodic Theorem

A sequence of numbers $\{a_n\}_{n\geq 0}$ in $[-\infty, +\infty)$ is called *sub-additive* if

$$a_{n+m} \leq a_n + a_m \quad \text{for all} \;\; n, m \geq 0 \;.$$

Lemma 1 (Fekete's Subadditive Lemma)**.** *Given a sub-additive sequence $\{a_n\}_{n\geq 0}$ the following limit converges*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\geq 1} \frac{a_n}{n} \in [-\infty, +\infty) \ .$$

Proof. If $a_p = -\infty$ for some $p \in \mathbb{N}$ then, by sub-additivity, $a_n = -\infty$ for all $n \geq p$. Assume now that $a_n > -\infty$ for all $n \geq 1$. Let $L = \inf_{n\geq 1} a_n/n \in [-\infty, \infty)$ and choose any number $L' > L$. Take $k \geq 1$ such that $a_k/k < L'$.

$n = q\,k + r$ with $0 \leq r < k$. Hence by sub-additivity

$$\frac{a_n}{n} \leq \frac{q\,a_k + a_r}{n} = \frac{n-r}{n}\,\frac{a_k}{k} + \frac{a_r}{n} \ .$$

Since $(n-r)/n$ converges to 1 and a_r/n converges to 0, as $n \to +\infty$, there exists $n_0 \in \mathbb{N}$ such that $a_n/n < L'$ for all $n \geq n_0$. This proves that

$$\lim_{n\to+\infty} \frac{a_n}{n} = L \ .$$

□ □

A random process $\{f_n\}_{n\geq 1}$ over the MPDS $(T, X, \mathcal{F}, \mu)$, i.e., a sequence of random variables $f_n \colon X \to \mathbb{R}$ on $(X, \mathcal{F}, \mu)$, is called *sub-additive* when for all $n, m \geq 1$,

$$f_{n+m} \leq f_n \circ T^m + f_m.$$

Theorem 2 (KET)**.** *Let $(T, X, \mathcal{F}, \mu)$ be MPDS. Given a sub-additive random process $\{f_n\}_{n\geq 1}$ such that $f_1^+ \in L^1(X,\mu)$ then the following limit exists for μ-a.e. $x \in X$*

$$\phi(x) = \lim_{n\to+\infty} \frac{1}{n} f_n(x) \in [-\infty, +\infty).$$

Moreover the limit function $\phi : X \to [-\infty, +\infty)$ satisfies:

(a) $\phi \circ T = \phi$ *μ-a.e.,*

(b) $\int_X \phi\, d\mu = \lim_{n\to+\infty} \frac{1}{n} \int_X f_n\, d\mu = \inf_{n\geq 1} \frac{1}{n} \int_X f_n\, d\mu \in [-\infty, +\infty)$.

Given the process $\{f_n\}_{n\geq 1}$ define $\underline{f}, \bar{f} : X \to [-\infty, +\infty)$,

$$\underline{f}(x) := \liminf_{n\to+\infty} \frac{1}{n} f_n(x),$$
$$\bar{f}(x) := \limsup_{n\to+\infty} \frac{1}{n} f_n(x).$$

From sub-additivity of $\{f_n\}_{n\geq 1}$ we get for all $j \geq 0$

$$f_{n+m}(T^j x) \leq f_n(T^{m+j} x) + f_m(T^j x).$$

Hence, adding up we have for all $L \in \mathbb{N}$ and $n, m \geq 1$

$$S_L(f_{n+m})(x) \leq S_L(f_n)(T^m x) + S_L(f_m)(x).$$

Dividing by L and taking the limit as $L \to +\infty$, for all $n, m \geq 1$

$$f^*_{n+m}(x) \leq f^*_n(x) + f^*_m(x).$$

By Fekete's lemma (Lemma 1) the following limit exists for all $x \in X$,

$$\phi(x) = \lim_{n \to +\infty} \frac{1}{n} f^*_n(x).$$

Exercise 8. *Prove that ϕ is T-invariant.*

Exercise 9. *Given a sub-additive process $\{f_n\}_{n\geq 1}$ such that $f_1^+ \in L^1(X,\mu)$, prove that for all $n \geq 1$, $f_n^+ \in L^1(X,\mu)$.*

Exercise 10. *Given $f\colon X \to \mathbb{R}$ measurable, prove that:*

(a) $\liminf_{n\to+\infty} \frac{1}{n}\, |f(T^n x)| = 0$*, for μ-a.e. $x \in X$.*

(b) If $f \circ T - f \in L^1(X,\mu)$ then $\lim_{n\to+\infty} \frac{1}{n} f(T^n x) = 0$*, for μ-a.e. $x \in X$.*

(c) If $f \in L^1(X,\mu)$ then $\lim_{n\to+\infty} \frac{1}{n} f(T^n x) = 0$*, for μ-a.e. $x \in X$.*

(d) If $f^+ \in L^1(X,\mu)$ then $\limsup_{n\to+\infty} \frac{1}{n} f(T^n x) \leq 0$*, for μ-a.e. $x \in X$.*

The next step to KET is the following

Lemma 2. *Under the assumptions of the KET, $\bar{f}(x) \leq \phi(x)$ for μ-a.e. $x \in X$.*

Proof. Fix $N \in \mathbb{N}$ large and take $n \gg N$. For any $i = 0, 1, \ldots, N-1$, dividing $n - i$ by N there are integers m and $0 \leq k < N$ such that $n = i + mN + k$. By sub-additivity,

$$\begin{aligned} f_n(x) &\leq f_i(x) + f_{mN}(T^i x) + f_k(T^{i+mN} x) \\ &\leq f_i(x) + \sum_{l=0}^{m-1} f_N(T^{i+lN} x) + f_{n-i-mN}(T^{i+mN} x)\ . \end{aligned}$$

Adding up in $i = 0, 1, \ldots, N-1$ we get

$$\begin{aligned} N\, f_n(x) &\leq f_i(x) + f_{mN}(T^i x) + f_k(T^{i+mN} x) \\ &\leq \sum_{i=0}^{N-1} f_i(x) + \sum_{i=0}^{N-1}\sum_{l=0}^{m-1} f_N(T^{i+lN} x) + \sum_{i=0}^{N-1} f_{n-i-mN}(T^{i+mN} x) \\ &\leq \sum_{j=0}^{n-1} f_N(T^j x) + \sum_{i=0}^{N-1} \big(f_i(x) + f_{n-i-mN}(T^{i+mN} x)\big) \end{aligned}$$

and dividing by nN

$$\frac{1}{n}f_n(x) \leq \frac{1}{nN}\sum_{j=0}^{n-1} f_N(T^j x) + \frac{1}{nN}\sum_{i=0}^{N-1}\left(f_i(x) + f_{n-i-mN}(T^{i+mN}x)\right).$$

By exercise 10, the two terms on the right either converge to 0 or else have a limsup which is ≤ 0. Hence, using BET (exercise 7) and taking the limit as $n \to +\infty$

$$\bar{f}(x) \leq \frac{1}{N} f_N^*(x).$$

Finally, this implies

$$\bar{f}(x) \leq \phi(x) = \inf_{N\geq 1} \frac{1}{N} f_N^*(x).$$

□

Remark 2. *Under the assumptions of the KET if one can prove that*

$$\phi(x) = \lim_{n\to+\infty} \frac{1}{n} f_n(x) \quad \text{for } \mu\text{-a.e. } x \in X \tag{6}$$

then all conclusions of the BET follow.

Exercise 11. *Show it is enough to prove* (6) *when* ϕ *is bounded from below, i.e.,* $\phi \geq -M$.
Hint: *For each* $M > 0$, *the set* $X_M := \{x \in X : \phi(x) \geq -M\}$ *is* T*-invariant.*

Exercise 12. *Prove that if* $\{f_n\}_{n\geq 1}$ *is a sub-additive process then so is* $\{f_n + n\,M\}_{n\geq 1}$, *for any constant* M.

Use this fact to show that it is enough to prove (6) *when* $\phi > 0$ μ*-a.e..*
Hint: *If* $\phi \geq -M$ *consider the sub-additive process* $\{f_n + n\,(M+1)\}_{n\geq 1}$.

Proof of the KET. Let us assume that $\phi > 0$ μ-a.e.. By exercise 9, $f_n^+ \in L^1(X,\mu)$ for all $n \in \mathbb{N}$.

Given $\varepsilon > 0$ define

$$n(x) := \min\left\{ n \geq 1 \colon \frac{1}{n} f_n(x) \leq \underline{f}^+(x) + \varepsilon \right\}.$$

By definition

$$\frac{1}{n(x)} f_{n(x)}(x) \leq \underline{f}^+(x) + \varepsilon. \tag{7}$$

By invariance of $\underline{f}^+$,

$$\frac{1}{n(x)} f_{n(x)}(x) \leq \underline{f}^+(T^j x) + \varepsilon.$$

Adding up these inequalities in $j = 0, 1, \ldots, n(x) - 1$ we get

$$f_{n(x)}(x) \leq S_{n(x)}(\underline{f}^+)(x) + n(x)\varepsilon. \tag{8}$$

Consider now the sets $X_N := \{x \in X : n(x) \leq n\}$. Because $X = \cup_{N \geq 1} X_N \pmod 0$, for N large enough $\int_X f_1^+ d\mu < \varepsilon$. Next we define the functions $\tilde{n} : X \to \mathbb{N}$

$$\tilde{n}(x) := \begin{cases} n(x) & \text{if} \quad x \in X_N \\ 1 & \text{if} \quad x \notin X_N \end{cases}$$

and $\tilde{f}^+ : X \to \mathbb{R}$

$$\tilde{f}^+(x) := \begin{cases} \underline{f}^+(x) & \text{if} \quad x \in X_N \\ f_1(x) & \text{if} \quad x \notin X_N \end{cases}$$

With this notation, (8) implies that

$$f_{\tilde{n}(x)}(x) \leq S_{\tilde{n}(x)}(\tilde{f}^+)(x) + \tilde{n}(x)\varepsilon. \tag{9}$$

Observe also that

$$\begin{aligned} \int \tilde{f}^+ \, d\mu &\leq \int_{X_M} \underline{f}^+ \, d\mu + \int_{X \setminus X_M} f_1 \, d\mu \\ &\leq \int \underline{f}^+ \, d\mu + \int f_1^+ d\mu \leq \int \underline{f}^+ d\mu + \varepsilon. \end{aligned} \tag{10}$$

Next we define recursively the sequence of stopping times

$$\begin{cases} n_0(x) := 0 \\ n_k(x) := n_{k-1}(x) + \tilde{n}(T^{n_{k-1}(x)} x) \end{cases} .$$

Given $L > \frac{N}{\varepsilon} \int f_1^+ d\mu$, choose the largest $k = k(x) \in \mathbb{N}$ such that $n_k(x) \leq L$, so that in particular $L - n_k(x) < N$. From (9) we get

$$\begin{aligned} f_L(x) &\leq \sum_{l=0}^{k-1} f_{\tilde{n}(T^{n_l} x)}(T^{n_l} x) + f_{L-n_k}(T^{n_k} x) \\ &\leq \sum_{l=0}^{k-1} S_{\tilde{n}(T^{n_l} x)}(\tilde{f}^+)(T^{n_l} x) + f_{L-n_k}(T^{n_k} x) + L\varepsilon \\ &\leq S_{n_k}(\tilde{f}^+)(x) + \sum_{j=n_k}^{L} f_1^+(T^j x) + L\varepsilon. \end{aligned}$$

Hence, dividing by L and integrating, from (10) we get

$$\begin{aligned} \int \phi d\mu &\leq \int \frac{1}{L} f_L^* \, d\mu = \int \frac{1}{L} f_L \, d\mu \\ &\leq \int S_L(\tilde{f}^+) \, d\mu + \frac{N}{L} \int f_1^+ d\mu + \varepsilon \\ &\leq \int \tilde{f}^+ d\mu + 2\varepsilon \leq \int \underline{f}^+ \, d\mu + 3\varepsilon. \end{aligned}$$

By definition $\underline{f} \leq \underline{f}^+$. On the other hand, by Lemma 2, $\underline{f} \leq \bar{f} \leq \phi$. Hence, since $\phi \geq 0$ we get $\underline{f}^+ \leq \phi$. Thus, because

$$\int (\underbrace{\underline{f}^+ - \phi}_{\leq 0}) d\mu \geq 0$$

we have $\phi = \underline{f}^+$ μ-a.e.. Finally, if $\underline{f}^+(x) \neq \underline{f}(x)$ then $\underline{f}^+(x) = 0$, and because $\phi > 0$ μ-a.e., this can only happen on a set with zero measure. Therefore $\underline{f} = \phi$ μ-a.e.. □

References

[1] Y. Katznelson, B. Weiss, *Simple proofs of some ergodic theorems*, Israel Journal of Mathematics,Vol. 42, No. 4, 1982