Note on Haar's Theorem

Abstract

The purpose of this note is to provide a proof for the uniqueness statement on Haar's Theorem for compact groups. We also prove in this compact setting that left invariant measures are right invariant.

A topological group is a group $(G, \cdot)$ with a Hausdorff topology which makes continuous the group operations $m : G \times G \to G$, $m(x, y) = xy$, and $i : G \to G$, $i(x) = x^{-1}$.

By compact group we mean a topological group which is a compact topological space. Likewise, a locally compact group is a topological group which is a locally compact topological space.

Every element $g \in G$ determines a left and a right translation $L_g, R_g : G \to G$ defined respectively by $L_g(x) := g\,x$ and $R_g(x) := x\,g$. Both these translations are homeomorphisms with obvious inverses, $(L_g)^{-1} = L_{g^{-1}}$ and $(R_g)^{-1} = R_{g^{-1}}$.

Let $\mathcal{F}$ be the Borel σ-algebra of a topological group G, and consider a positive measure $\mu : \mathcal{F} \to [0, +\infty[$. Denote by $L^\infty(G)$ the space of all bounded $\mathcal{F}$-measurable functions $\varphi : G \to \mathbb{R}$.

The measure μ is said to be **left-invariant** if it is defined on the Borel σ-algebra $\mathcal{F}$, and for all $g \in G$ any of the following equivalent statements holds:

1. $(L_g)_*\mu = \mu$,
2. $\mu((L_g)^{-1}A) = \mu(A)$, for all $A \in \mathcal{F}$,
3. $\int_G \varphi(g\,x)\, d\mu(x) = \int_G \varphi(x)\, d\mu(x)$, for every $\varphi \in L^\infty(G)$.

Similarly, the measure μ on $\mathcal{F}$ is said to be **right-invariant** if for all $g \in G$ any of the following equivalent statements holds:

1. $(R_g)_*\mu = \mu$,
2. $\mu((R_g)^{-1}A) = \mu(A)$, for all $A \in \mathcal{F}$,
3. $\int_G \varphi(x\,g)\, d\mu(x) = \int_G \varphi(x)\, d\mu(x)$, for every $\varphi \in L^\infty(G)$.

The following theorem is usually known as Haar's Theorem. Different versions were proven by A. Haar, J. von Neumann and A. Weyl.

Theorem 1 (Haar)**.** *Let G be a locally compact Hausdorff group. Then G admits left and right invariant measures on its Borel σ-algebra. Moreover, any two such left, resp. right, invariant measures are equal up to a positive scalar factor.*

When G is compact more can be said.

Theorem 2. *If G is compact then it admits a unique left invariant probability measure on its Borel σ-algebra. Moreover, this probability measure is also right invariant.*

Let G be a compact group with Borel σ-algebra $\mathcal{F}$. The unique probability measure on the Borel σ-algebra of G is called the **Haar measure** of G. The proofs below make a judicious use of the previous characterization of left invariant measures.

Proposition 1. *If μ is a left invariant probability measure on $\mathcal{F}$ then it is also a right invariant measure.*

Proof. Bounded $\mathcal{F}$-measurable functions are integrable functions w.r.t. any finite measure on $\mathcal{F}$. Given $\varphi \in L^\infty(G)$, since φ is bounded, all functions considered below are integrable in $(x, g) \in G \times G$. Applying Fubini's theorem,

$$\begin{aligned}
\int_G \varphi(x)\, d\mu(x) &= \int_G \int_G \varphi(g^{-1}x)\, d\mu(x)\, d\mu(g) \\
&= \int_G \int_G \varphi(g^{-1}x)\, d\mu(g)\, d\mu(x) \\
&= \int_G \int_G \varphi((x\, g)^{-1}x)\, d\mu(g)\, d\mu(x) \\
&= \int_G \int_G \varphi(g^{-1})\, d\mu(g)\, d\mu(x) = \int_G \varphi(g^{-1})\, d\mu(g).
\end{aligned}$$

This proves that $i_*\mu = \mu$.

On the other hand, for all $g, x \in G$,

$$(R_g \circ i)(x) = x^{-1}g = (g^{-1}x)^{-1} = (i \circ L_{g^{-1}})(x).$$

Therefore,

$$\begin{aligned}
(R_g)_*\mu &= (R_g)_* i_*\mu = (R_g \circ i)_*\mu \\
&= (i \circ L_{g^{-1}})_*\mu = i_*(L_{g^{-1}})_*\mu = i_*\mu = \mu,
\end{aligned}$$

which shows that μ is right invariant. □

Proposition 2. *If μ and ν are left invariant probability measures on $\mathcal{F}$ then $\mu = \nu$.*

Proof. Given $\varphi, \psi \in L^\infty(G)$, because $G \times G$ is compact and these functions are bounded, by Fubini's theorem we have

$$\begin{aligned}
\int \varphi(x)\,d\mu(x) \int \psi(y)\,d\nu(y) &= \int\int \varphi(x)\,\psi(y)\,d\mu(x)\,d\nu(y)\\
&= \int\int \varphi(y\,x)\,\psi(y)\,d\mu(x)\,d\nu(y)\\
&= \int\int \varphi(y\,x)\,\psi(y)\,d\nu(y)\,d\mu(x)\\
&= \int\int \varphi((yx^{-1})\,x)\,\psi(yx^{-1})\,d\nu(y)\,d\mu(x)\\
&= \int\int \varphi(y)\,\psi(yx^{-1})\,d\nu(y)\,d\mu(x)\\
&= \int\int \varphi(y)\,\psi(yx^{-1})\,d\mu(x)\,d\nu(y)\\
&= \int\int \varphi(y)\,\psi(x^{-1})\,d\mu(x)\,d\nu(y)\\
&= \int \varphi(y)\,d\nu(y) \int \psi(x^{-1})\,d\mu(x)\\
&= \int \varphi(y)\,d\nu(y) \int \psi(x)\,d\mu(x).
\end{aligned}$$

Thus

$$\int \varphi\,d\mu \int \psi\,d\nu = \int \varphi\,d\nu \int \psi\,d\mu.$$

Taking $\varphi = \mathbf{1}_B$ with $B \in \mathcal{F}$, and $\psi = 1$, we have $\mu(B) = \nu(B)$. Hence $\mu = \nu$. $\square$